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DECODING NETS AND THE THEORY OF GRAPHS*

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Decoding nets are basic components of digital computers and other similar devices. The definition and the general theory of decoding nets are given in [3]. It is shown in this paper that the problem of minimizing decoding nets may be expressed in the language of the theory of graphs. (A decoding net may be assumed to be a graph; diodes are interpreted as branches, and wire connections between diodes as nodes of a graph). The necessary condition that a decoding net has the least number of diodes is given. In this paper, "decoding" has a little wider significance than in [3] and it will be defined exactly in what follows.

A graph G is a set $N(G)$ of *nodes* together with a set $B(G)$ of *branches* (the two sets having no common elements) such that:

a. With each branch $A \in B(G)$, the ordered pair of nodes $(a, b) \in N(G)$ is associated. (We say that A is *directed* from a to b, and a, b are called, respectively, the *beginning* and the *end* of the branch A.)
b. There is at most one branch between every pair of nodes $(a, b) \in N(G)$.
c. For all nodes $a \in N(G)$, there is a branch $A \in B(G)$ such that a is the end or the beginning of A.

In this paper, we restrict ourselves to *finite* graphs, i.e., graphs for which $N(G)$ and $B(G)$ are both finite.

We denote by $\varphi_N(G)$ and $\varphi_B(G)$ the number of elements of $N(G)$ and $B(G)$ respectively. In the following, $\varphi(A)$ denotes the number of elements of the set A. A sequence $a_0, A_1, a_1, A_2, \cdots, A_n, a_n$, $(n \geqq 1, a_i \in N(G), A_i \in B(G))$, will be called a *way* from a_0 to a_n in G, if for all i $(1 \leqq i \leqq n)$, a_{i-1} and a_i are the beginning and the end of A_i respectively.

If $(a, b) \in N(G)$, and there is exactly one way from a to b in G, we say that a is *connected* to b in G. (The case where there is more than one way from a to b in G will not be considered in this paper.)

If a is connected to b in G, we will write a Con b. Of course, the relation Con is not an equivalence relation.

A way which contains only two nodes is called *simple.*

We now introduce definitions of the relation Con between a set of nodes and a single node. These will be useful later. Let $M(G) \subset N(G)$ and $b \in N(G) - M(G)$. Then we define $M(G)$ Con b to mean that $a \in M(G)$ implies a Con b. Similarly, b Con $M(G)$ means that $a \in M(G)$ implies b Con a.

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A graph G is a *tree* if for all ordered pairs $(a, b) \in N(G)$ there is at most one way from a to b in G.

A graph G is *simple* if all ways in it are simple.

COROLLARY 1. *Each simple graph is a tree.*

If $a \in N(G)$, and there is no $b \in N(G)$ such that a Con b, then a is an *output* of G.

If $a \in N(G)$ and there is no $b \in N(G)$ such that b Con a, then a is an *input* of G.

Let $I(G)$ (or $O(G)$) be the set of all inputs (or outputs) of a graph G, partitioned into n classes $K_1(G), K_2(G), \cdots, K_n(G)$ (i.e., $I(G) = \bigcup_{i=1}^{n} K_i(G)$ and $\bigcap_{i=1}^{n} K_i(G) = 0$). The set $R_I(G)$ of n elements $r_1, r_2, \cdots, r_n$ such that $r_i \in K_i(G)$ $(i = 1, 2, \cdots, n)$ will be called a *representation* of a set $I(G)$. If $K_1, K_2, \cdots, K_n$ are nonempty classes, then the set $I(G)$ has $\prod_{i=1}^{n} \varphi(K_i(G))$ different representations. For example, if $\varphi(K_1) = 2$, $\varphi(K_2) = 2$, $\varphi(K_3) = 4$, $\varphi(K_4) = 8$, then $I(G)$ has $2 \cdot 2 \cdot 4 \cdot 8$ $(= 126)$ different representations.

A graph G is a *decoding net* (d.n.) if

(a) the set $O(G)$ of outputs is not empty,

(b) the set $I(G)$ can be partitioned into classes $K_1(G), K_2(G), \cdots, K_n(G)$, $n \geqq 2$ such that $\varphi(K_i(G)) \geqq 2$, and such that

(c) for all representations $R_I(G)$ there exists at most one $a \in O(G)$ such that $R_I(G)$ Con a.

It is easy to prove the following.

THEOREM 1. *Let the directions of all branches in* the d.n. G *be reversed. Call the resulting graph* G^*. *Then the set of outputs* $O(G^*)$ *can be partitioned into classes* $K_1(G^*), K_2(G^*), \cdots, K_n(G^*)$, *and for all representations* $R_O(G)$, *there exists at most one* $a \in I(G^*)$ *such that* a Con $R_O(G^*)$.

Such a graph G^* will be called a *reverse decoding net*, with respect to the d.n. G. In the following we will consider d.n.'s only; however all results concerning d.n.'s hold also for reverse decoding nets.

From the definition of a d.n. follows

COROLLARY 2. *Each* d.n. *is a tree.*

A d.n. is *complete* if for all $R_I(G)$ there exists exactly one $a \in O(G)$ such that $R_I(G)$ Con a and for all $a \in O(G)$ there exists exactly one $R_I(G)$ such that $R_I(G)$ Con a. A complete d.n. is abbreviated c.d.n.

THEOREM 2. *For an arbitrary* c.d.n. G,

$$\varphi(O(G)) = \prod_{i=1}^{n} \varphi(K_i(G)).$$

Let G and H be c.d.n.'s. Then the d.n. $G \cup H$ formed so that $O(G) =$

$K_i(H)$ will be called a *superposition* of the d.n. G into a d.n. H and will be denoted by Sup $K_i(H)/G$. We remark that a superposition of G into H does not always exist.

From the definition of a superposition, we have

COROLLARY 3. *If G and H are* c.d.n.'s, *then* Sup $K_i(H)/G$ *is also a* c.d.n.

We now give a recursive definition of decomposable c.d.n.'s.

a. Simple c.d.n.'s are decomposable (into themselves).

b. If G and H are decomposable c.d.n.'s, then the c.d.n.'s Sup $K_i(H)/G$ are also decomposable (if Sup $K_i(H)/G$ exists).

In the remainder of this paper, by c.d.n. we will understand a decomposable c.d.n. only.

Let A_n ($n \geqq 2$) denote a sequence of K integers a_i separated by "periods" such that $a_i \geqq 2$, $i = 1, 2, \cdots, n$. A_n is a *formula* if it contains parentheses "(", ")". For example 2.8.(5.)3) is a formula. A formula is *well-formed* if it contains properly paired† parentheses. The recursive definition of well-formed formulas (w.f.f.'s) is given below.

a. (A_n) is a w.f.f. (for example—(2.2), (2.2.2), (3.8.4)).

b. If α and β are w.f.f.'s, then replacing a_i in β by α or in symbols

$\mathrm{Sup}\begin{pmatrix}\beta\\ a_i\end{pmatrix}\alpha$ is also a w.f.f. For example, we have

$$\mathrm{Sup}\begin{pmatrix}(2.4)\\ 4\end{pmatrix}(2.2) = (2.(2.2)).$$

A w.f.f. with k pairs of parentheses will be denoted by A_n^k. In particular, $A_n' = (A_n^0)$.

From the definitions of a decomposable c.d.n. and a w.f.f. we have

LEMMA 1. *The set of all decomposable* c.d.n.'s *and the set of all* w.f.f.'s *are isomorphic.*

Thus w.f.f.'s may be assumed to be the names of c.d.n.'s. The w.f.f.'s are associated with c.d.n.'s in the following way.

a. If G is a simple c.d.n., with inputs partitioned into classes $K_1(G)$, $K_2(G), \cdots, K_n(G)$, then the w.f.f. associated with it has the form

$$(\varphi(K_1(G)).\varphi(K_2(G)).\ \cdots\ .\varphi(K_n(G))).$$

b Let $F(G)$ and $F(H)$ denote w.f.f.'s associated with c.d.n.'s G and H respectively. With the c.d.n. Sup $K_i(H)/G$ we associate the w.f.f. $\mathrm{Sup}\begin{pmatrix}F(H)\\ a_i\end{pmatrix}F(G)$ where $a_i = \varphi(K_i(G))$. c.d.n.'s A and B are *equivalent* if and only if the w.f.f.'s A and B differ at most in

(1) the order of factors,

† Parentheses occurring in a formula are said to be properly paired if there are as many left-hand parentheses as there are right-hand parentheses.

(2) the number of parentheses, and

(3) the location of parentheses.

Thus each set of integers $a_1, a_2, \cdots, a_n$, $n \geqq 2$, $a_i \geqq 2$ defines a class of equivalent c.d.n.'s.

From the definition of a simple c.d.n. we have

LEMMA 2. $\varphi_B(A_n^1) = n \prod_{i=1}^{n} a_i$.

LEMMA 3. $\varphi_N(A_n^1) = \prod_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i$.

LEMMA 4. *A c.d.n. A_n^k is decomposable into k simple c.d.n.'s.*

From Lemmas 2 and 4 follows

THEOREM 3. $\varphi_B(A_n^k) = \sum_{i=1}^{n} p_r \prod_r a_i$ *where* $\prod_r a_i$ *denotes the product of factors joined by the rth pair of parentheses, and p_r the number of factors in the rth pair of parentheses.*

We assume that each pair of parentheses in the w.f.f. A_n^k is numbered with the numbers $1, 2, \cdots, k$; however the manner of numeration is in our case not important. From Lemmas 3 and 4 we have

THEOREM 4. $\varphi_N(A_n^k) = \sum_{i=1}^{k} \prod_r a_i + \sum_{i=1}^{n} a_i$.

THEOREM 5. *A necessary and sufficient condition for $\varphi_N(A_n^k)$ to be minimal for a given $a_1, \cdots, a_n$ is that $k = 1$.*

THEOREM 6. *A necessary condition for $\varphi_B(A_n^k)$ to be minimal for a given $a_1, \cdots, a_n$ is that $k = n - 1$, i.e., each pair of parentheses encloses two factors.*

Proof. This theorem is equivalent to the inequality

$$\varphi_B(A_n^k) \geqq \varphi_B(A_n^{k+1}) \tag{1}$$

for $k = 1, 2, \cdots, n - 2$. Let A_n^k have the form

$$(x_1, x_2, \cdots, x_t) \qquad (t < n), \tag{2}$$

where each x_i is either a single term a_i or a product with u_i terms. We may assume that each x_i has $u_i - 1$ parentheses (and the whole formula has k parentheses). Thus from (1), by Lemma 2 and Theorem 3, we obtain

$$\sum_{i=1}^{t} \varphi_B(x_i) + t \prod_{i=1}^{t} x_i \geqq \sum_{i=1}^{t} \varphi_B(x_i) + (t - p + 1) \prod_{i=1}^{t} x_i + p \prod\nolimits_{k+1} x_i, \tag{3}$$

where p is the number of factors contained in the $(k + 1)$th pair of parentheses.

From (3), we have

$$\rho \leqq 1 - \frac{1}{p}, \tag{4}$$

where $\rho = 1/\prod_{\overline{k+1}} x_i$ and $\prod_{\overline{k+1}} x_i$ denotes the product of all factors in (2), with the exception of those contained in the $(k + 1)$th pair of parentheses. Because $\prod_{\overline{k+1}} x_i \geqq 2$ and $p \geqq 2$, the theorem is proved.

The condition is not necessary when $\prod_{\overline{k+1}} x_i = 2$ and $p = 2$; for example, if $A_3^1 = (2.2.2)$, and $A_3^2 = (2.(2.2))$, then $\varphi_B(A_3^1) = 3 \cdot 8 = 24$, $\varphi_B(A_3^2) = 2 \cdot 4 + 2 \cdot 8 = 24$; or if $B_3^1 = (2.3.4)$, $B_3^2 = (2.(3.4))$, then $\varphi_B(B_3^1) = 3 \cdot 2 \cdot 3 \cdot 4 = 72$, $\varphi_B(B_3^2) = 2 \cdot 3 \cdot 4 + 2 \cdot 2 \cdot 3 \cdot 4 = 72$. But we can say that the minimum case can always be attained with $k - 1$ parentheses.

It would be interesting to give a necessary condition that $\varphi_B(A_n^{n-1})$ be minimal. However, in the general case it seems to be rather difficult.

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