Decoherence limits to quantum computation using trapped ions

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Abstract

We investigate the problem of factorization of large numbers on a quantum computer which we imagine to be realized within a linear ion trap. We derive upper bounds on the size of the numbers that can be factorized on such a quantum computer. These upper bounds are independent of the power of the applied laser. We investigate two possible ways to implement qubits, in metastable optical transitions and in Zeeman sublevels of a stable ground state, and show that in both cases the numbers that can be factorized are not large enough to be of practical interest. We also investigate the effect of quantum error correction on our estimates and show that in realistic systems the impact of quantum error correction is much smaller than expected. Again no number of practical interest can be factorized.

1 Introduction

Since Shor's discovery (Shor, 1994; Ekert *et al*, 1996) of an algorithm that allows the factorization of a large number by a quantum computer in polynomial time instead of an exponential time as in classical computing, interest in the practical realization of a quantum computer has been much enhanced (Plenio *et al*, 1996). Recent advances in the preparation and manipulation of single ions as well as the engineering of pre-selected cavity light fields have made quantum optics that field of physics which promises the first experimental realization of a quantum computer. Several proposals (reviewed in (Barenco, 1996)) for possible experimental implementations have been made relying on nuclear spins, quantum dots (Barenco *et al*, 1995a), cavity QED (Sleator & Weinfurter, 1995) and on ions in linear traps (Cirac & Zoller, 1995).

The realization of a quantum computer in a linear trap in particular was regarded as very promising as it was thought that decoherence could be suppressed sufficiently to preserve the superpositions necessary for quantum computation. Indeed, a single quantum gate in such an ion trap was recently realized by Monroe et al (Monroe *et al*, 1995). Nevertheless, the error rate in this experiment was too high to allow the realization of extended quantum networks. This experiment was not solely limited by fundamental processes but rather by technical difficulties and one aim of future experiments is to reduce these technical problems to come closer to the fundamental limits, such that at least small networks could be realized.

However, there remains the question whether overcoming technical problems will be sufficient to realize practically useful computations such as factorization of big numbers on a quantum computer in a linear ion trap. In fact one has to investigate more thoroughly those limitations that are due to processes of more fundamental nature than for example laser linewidths, pulse lengths etc. General considerations revealed that decoherence will lead to an exponential decrease in the probability to obtain the correct result of a calculation (**?**). However, these considerations did not state how fast this exponential decay would be and therefore left open the question to whether factorization will be possible or not. An analysis of this problem was first carried out in refs. (Plenio & Knight, 1996b; Plenio & Knight, 1996a) where the impact of spontaneous emission on the problem of factorization of a large number by a quantum computer was considered. It was shown that for realistic laser powers, factorization is limited to small numbers (at least if no sophisticated error correction methods are implemented). Subsequent investigations have appeared to be less pessimistic, but we note that these did not consider spontaneous emission but instead concentrated on other effects such as phonon decoherence (Garg, 1996) or the influence of the ion separation (Hughes *et al*, 1996).

In refs. (Plenio & Knight, 1996b; Plenio & Knight, 1996a) limits were obtained that still depended on the power of the applied lasers and which therefore leave open the possibility of improving the maximal bitsize L of the largest factorizable number by using high power lasers. Extremely high intensities, however, will lead to ionization or, more importantly, to a breakdown of the two-level-approximation as was mentioned briefly in (Plenio & Knight, 1996b; Plenio & Knight, 1996a). In Section II we will use the breakdown of the two-level-approximation to derive new estimates for the bitsize L of the factorizable number. The new feature in these estimates is their *independence* on the power of the applied laser. We derive intensity independent limits of L not only for qubits stored in a metastable optical transition (see Fig. 1) but also for qubits stored in the Zeeman sublevels of an ion which are then manipulated using detuned Raman transitions (see Fig. 2). On the basis of experimental parameters for several ions we give estimates for L_{max} and we show that spontaneous emission already imposes strong limitations to L. We then conclude that factorization without the use of efficient error correction methods is limited to almost trivial numbers if spontaneous emission is present, which in reality is inevitable. In Section III we therefore proceed to investigate the degree to which such quantum error correction methods are able to improve on these results. It is generally believed that quantum error correction is able to increase the number of possible operations substantially, e.g. a single error correcting code should allow to perform a number of operations which is approximately the square of the number of allowed operations without error correction. We analyze critically this idea in real atomic systems, especially taking into account the fact that due to the breakdown of the two-level approximation spontaneous transitions may leak population out of those levels that represent the qubit in the ion. It will be difficult, if not impossible, to bring this population back into the qubit. This means that after such an event the quantum error correction code will fail to reconstruct the state, as the reconstruction procedure works only if the entire population remains in the qubit states. We take this possibility into account and obtain estimates that show that the efficiency of quantum error correction is smaller than expected, although the application of quantum error correction allows factorization of somewhat larger numbers than without the use of quantum error correction. In Section IV we will then discuss the prospects of factorization and other applications of quantum computers in view of the results of the present paper. We conclude that spontaneous emission considerations preclude many

applications within the present model of quantum computation.

2 Decoherence limits without error correction

2.1 The linear ion-trap model of quantum computation

Before we come to the derivation of limits imposed on quantum computation by spontaneous emission we briefly describe the ion trap implementation of the quantum computer (Cirac & Zoller, 1995) on which we base our considerations. Several ions of mass M are trapped in a linear ion trap such that they are lined up and are well separated, ie the next neighbour distance is many wavelengths of the lasers used to manipulate the ions. This is necessary to be able to address ions separately and can lead to further restrictions on quantum computation in this model (Hughes *et al*, 1996; James & Hughes, 1996). For a schematic picture see Fig. 3. The motional degrees of freedom of the ions, and especially their collective center-of-mass (COM) motion with frequency ν, are cooled to the ground state. In order to be able to implement two-bit gates in this scheme we use the COM mode as a bus which allows us to create entanglement between different ions. This is achieved with an interaction that creates a phonon in the COM mode when we deexcite an ion and annihilates a phonon in the COM mode when an ion is excited. This interaction is generated by a laser which is detuned from the transition frequency by $\Delta = -\nu$. In that case the Hamilton operator in the Lamb-Dicke limit, the RWA and a suitable interaction picture is given by (Cirac & Zoller, 1995)

$$H = \frac{\eta}{\sqrt{5L}} \frac{\Omega_{01}}{2} \left[|1\rangle\langle 0|a + |0\rangle\langle 1|a^\dagger \right] , \tag{1}$$

where $\eta = \frac{2\pi}{\lambda}\sqrt{\frac{\hbar}{2M\nu}}$ is the Lamb-Dicke parameter, Ω_{01} is the Rabi frequency on the $0 \leftrightarrow 1$ transition where levels 0 and 1 represent the corresponding logical values of the qubit. The denominator $\sqrt{5L}$ originates from the fact that we are considering the COM mode which has an effective mass $5LM$ because all $5L$ ions are oscillating in the trap potential. The trap has to contain $5L$ ions, as this is the number of ions that is required to implement Shor's algorithm to factorize an L bit number (Shor, 1994; Vedral *et al*, 1996) (actually the correct number is $5L + 2$ but for simplicity we drop the 2). It is further assumed that coupling to other vibrational levels can be neglected. This assumption is a reasonable first approximation as the closest lying vibrational mode has frequency $\sqrt{3}\nu$ independent of the number of ions in the trap. Nevertheless, this approximation breaks down when the Rabi frequency becomes too large, ie. if $\Omega_{01} \approx \nu$ (Hughes *et al*, 1996; James & Hughes, 1996). For our considerations of the influence of spontaneous decay we neglect this effect although it may well become important in longer calculations. Using the Hamilton operator eq. (1) it can be shown that it is possible to construct a CNOT gate with only four π-rotations, and to construct the more involved Toffoli gate (Barenco *et al*, 1995b) with six π-rotations (Cirac & Zoller, 1995). In the following we will define an elementary time step in terms of the computation time that is required to perform a CNOT gate. All other performance times can be reexpressed in units ofthat of the CNOT gate. The implementation of Shor's algorithm also requires a number of one bit operations, ie. operations that leave the COM mode unaffected during their performance. However, these operations can be performed much faster than two-bit gates, because $\frac{\eta}{\sqrt{5L}} \ll 1$ and also because fewer individual laser

pulses are required. Therefore their contribution to calculation time and decoherence is small and will be neglected in the following.

2.2 Qubits stored in two-level systems

After this short discussion of the ion trap model we will now derive the intensity independent estimates for L, first for a qubit which is stored in a metastable two-level transition (see Fig. 1). Later in this section we will also consider the case of a qubit stored in Zeeman sublevels manipulated by strongly detuned Raman pulses (see Fig. 2). First we assume that the two-level approximation holds and derive an expression for the computation time that is required to factorize an L bit number. As the next step we then include in our calculation possible extraneous levels and their spontaneous decay.

As stated above, a CNOT gate can be implemented by four π-pulses according to the Hamilton operator eq. (1). This requires an elapsed time

$$\tau_{el} = 4\,\frac{\pi\sqrt{5L}}{\eta\Omega_{01}}\;, \tag{2}$$

which we will call the elementary time step. It is known (Vedral $et\ al$, 1996) that $\epsilon L^3 + O(L^2)$ of these elementary time steps are required to implement Shor's algorithm using elementary gates such as one bit gates, CNOT gates, Toffoli gates and Fredkin gates. Therefore the total computation time to complete a factorization will be

$$T \cong \frac{4\,\pi\sqrt{5L}}{\eta\Omega_{01}}\epsilon L^3\;, \tag{3}$$

which assumes that the gates are executed one after the other with zero time delay between two gates. The value of the constant ϵ very much depends on the actual practical implementation of Shor's algorithm. The implementation given in (Vedral $et\ al$, 1996) requires $80\,L^3$ CNOT gates, $80\,L^3$ Toffoli gates and $8\,L^3$ Fredkin which result in $\epsilon = 80 + 1.5 \cdot 80 + 2 \cdot 80 = 216$. We neglect here all contributions of order L^2 which give significant corrections only for small L.

If the quantum computation is to give a useful answer, no spontaneous emission is allowed to occur during the whole calculation time T, because an emission usually alters the wavefunction of the quantum computer completely. We will illustrate this point for a quantum computer which performs a discrete Fourier transformation (DFT) (Barenco, 1996). Qualitatively similar results were obtained in ref. (Barenco $et\ al$, 1996) for phase errors.

Spontaneous emission in fact leads to two sources of errors. One, quite obviously, arises from the actual spontaneous emissions in the quantum computer. The other one is less obvious and is due to the conditional time evolution when no spontaneous emission has taken place. This time evolution differs from the unit operation because the failure to detect a photon provides us with information about the system which is reflected in a change of the wavefunction (Plenio & Knight, 1996c).

First we illustrate the case of an unstable quantum computer which performs a DFT on a function which is evaluated at 32 points. The resulting square modulus of the wavefunction of the quantum computer is compared to the exact result obtained from a absolutely stable quantum computer. The function on which we perform the DFT is given by $f(n) = \delta_{8,(n\,mod\,10)}$ for $n = 0, 1, \ldots, 31$. We have implemented the Hamilton operators (in the Lamb-Dicke

limit) for all the necessary quantum gates in a linear ion trap (Cirac & Zoller, 1995) to realize this DFT. In addition to the coherent time evolution we also take into account possible spontaneous emissions from the upper levels of the ions but we neglect all other sources of loss. To calculate the time evolution of the quantum computer, we then use the quantum jump approach (Plenio & Knight, 1996c; Dalibard *et al*, 1992; Carmichael, 1993; Knight & Garraway, 1996; Hegerfeldt & Wilser, 1991) to simulate the time evolution of a *single* quantum computer. The simulation runs as follows: We generate a random number and compare this random number after each time step with the squared norm of the wavefunction of the quantum computer (The norm of the wavefunction decreases as it represents the probability that no photon has been emitted). If the squared norm of the wavefunction is smaller than the random number, an emission is deemed to have occured. We then generate another random number and continue our simulation. Two results of our simulations are shown in Figs. 5 and 6. In Fig. 5 one emission has taken place during the calculation time of the quantum computer. If we compare the resulting wavefunction with the correct wavefunction, we observe a marked difference between the two. In Fig. 6 we show the wavefunction of an unstable quantum computer which has not suffered a spontaneous emission during the calculation of the DFT. We clearly see that even when no spontaneous emission has taken place, the wavefunction of the quantum computer differs substantially from the correct result. This difference becomes stronger and stronger the larger the ratio between the computation time T and the spontaneous lifetime τ_{sp} of the quantum computer becomes. Therefore the wavefunction of the quantum computer will be sufficiently close to the correct result only if the whole computation is finished in a time T that is much shorter than the spontaneous lifetime τ_{sp} of the quantum computer.

Therefore we need to determine the spontaneous lifetime of the quantum computer. It is reasonable to assume that each ion has an average excitation of 0.5 during the whole calculation. The reason for this is that the quantum computer is in a superposition of very many states each of which represents a string of logical states 0 and 1. The probability to find an atom excited will on average be 0.5. In addition during the computation each ion will suffer many 2π rotations which again leads to an average population of 0.5. As the ions are separated by many wavelengths it is reasonable to assume that each ion decays independently of all others with a decay rate of $2\Gamma_{11}$. As the quantum computer consists of $5L$ ions the spontaneous lifetime of our quantum computer is

$$\tau_{sp} = \frac{1}{5L\Gamma_{11}} \ . \tag{4}$$

Therefore we obtain the condition

$$p_{em}^{(1)} = \frac{T}{\tau_{sp}} \ll 1 \ , \tag{5}$$

and from that

$$\frac{\Omega_{01}}{\Gamma_{11}} = \frac{20\pi\epsilon\sqrt{5L^9}}{\eta p_{em}^{(1)}} \ . \tag{6}$$

We can now insert eq. (6) into eq. (3) and obtain

$$T = \frac{400\pi^2\epsilon^2}{\eta^2 p_{em}^{(1)}} \frac{\Gamma_{11}}{\Omega_{01}^2} L^8 \ , \tag{7}$$

where we have isolated the fraction $\Gamma_{11}/\Omega_{01}^2$ because it can be expressed in a form that is independent of the actual type of transition we are considering (electric dipole, quadrupole,...). We find

$$\frac{\Omega_{01}^2}{\Gamma_{11}} = \frac{6\pi c^3 \epsilon_0}{\hbar \omega_{01}^3} E^2 \quad , \tag{8}$$

which can be derived along similar lines as in the usual quantum mechanical derivation of the relation between the Einstein A and B coefficient. If we want to reduce the estimated computation time T we have to have a ratio $\frac{\Omega_{01}^2}{\Gamma_{11}}$ which is as large as possible. As the transition frequency is given by the ion we can adjust only the field strength of the laser. However, there is an upper limit to this field strength which is approximately given by the field strength between an electron and the proton in a hydrogen atom in its ground state. The value is

$$E_{hyd} = \frac{e}{4\pi\epsilon_0 a_0^2} = 5.52 \, 10^{11} \frac{V}{m} \quad , \tag{9}$$

where $a_0 = \frac{4\pi\epsilon_0 \hbar^2}{e^2 m_0}$. This would yield a maximum value of $1.46 \, 10^{24}$ for eq. (8). However, one should note that this field strength is sufficient to destroy the ion within one optical cycle by tunnel ionization (Augst *et al*, 1989) and represents therefore a very hypothetical upper limit. In fact we will see later in this section that, not surprisingly, other approximations such as the two-level approximation break down much earlier.

We may use eq. (7) to obtain a first estimate for an upper limit on L by requiring that T has to be smaller than the decoherence time τ_{aux} due to all other decohering processes that may occur, such as collisions, stray fields, phonon losses, laser phase fluctuations, ... Nevertheless, from a fundamental point of view the situation is not yet entirely satisfying, as in principle higher laser powers and technical improvements concerning the auxiliary decoherence effects could change the estimates quite drastically. Therefore we will now proceed to show that the inclusion of spontaneous emission from auxiliary levels into the dynamics will lead to an estimate for the upper limit of L which is entirely *intensity independent*.

To model the influence of extraneous levels on the decoherence rate due to spontaneous emissions we assume that there is one other level 2 present which couples to both levels 0 and 1 by the same laser that drives the $0 \leftrightarrow 1$ transition. As the contribution of even further detuned levels decreases rapidly with the detuning (it can be shown that the infinitely many states in the atom give only a finite contribution) we restrict our treatment to one additional level. As the laser is strongly detuned from the $i \leftrightarrow 2$ transitions the population in level 2 will be very small and we can safely assume that the contributions from levels 0 and 1 to the population of level 2 add up. The population in level 2 will then be

$$\rho_{22} = \frac{1}{2} \left\{ \frac{\Omega_{02,eff}^2}{4\Delta_{02}^2} + \frac{\Omega_{12,eff}^2}{4\Delta_{12}^2} \right\} \quad , \tag{10}$$

where $\Omega_{i2,eff}$ is the Rabi frequency on the $i \leftrightarrow 2$ transition and Δ_{i2} the detuning on that transition. In eq. (10) we have not taken into account that for very large detuning the RWA is not very good any more. However, the contribution of all possible atomic levels should be approximated well by eq. (10). We have used the notation $\Omega_{12,eff}$ because the Rabi frequency on the $i \leftrightarrow 2$ transition now depends on the type of the $i \leftrightarrow 0$ transition. There are two cases.

a) If the $0 \leftrightarrow 1$ transition is an electric quadrupole transition (E2) then for the excitation of phonons it is necessary to place the ion at the antinode of the electric field of the

laser. Then the same laser on an electric dipole transition (E1) will, in leading order, leave the COM mode untouched. This implies that $\Omega_{i2,eff} = \Omega_{i2}$

b) If the $0 \leftrightarrow 1$ transition is an electric octupole transition (E3) then for the excitation of phonons it is necessary to place the ion at the node of the electric field of the laser. Then the same laser on an electric dipole transition (E1) will also excite phonons in the COM mode. This implies that $\Omega_{i2,eff} = \frac{\eta' \Omega_{i2}}{\sqrt{5}L}$ where η' is the Lamb-Dicke parameter on the $i \leftrightarrow 2$ transition.

We will have to distinguish between the two possibilities a) and b), the first of which occurs for example in Ba^+,Ca^+ and Hg^+ while the second one occurs in Yb^+ (an ion which might allow factorization of very large numbers according to estimates which do not include spontaneous emission (Hughes *et al*, 1996)) . We first deal with case a) and then state the result for case b). In case a) we obtain

$$\rho_{22} = \frac{1}{2} \left\{ \frac{\Omega_{02}^2}{4\Delta_{02}^2} + \frac{\Omega_{12}^2}{4\Delta_{12}^2} \right\} \quad . \tag{11}$$

As in the pure two-level case, we want to be sure that no spontaneous emission will take place during the whole calculation, neither from level 1 nor from level 2. This implies the condition

$$2\Gamma_{22}\rho_{22}T = p_{em}^{(2)} \ll 1 \quad . \tag{12}$$

We now use eq. (7) and

$$\frac{\Gamma_{11\to00}}{\Omega_{10}^2} \frac{\Omega_{02}^2}{\Gamma_{22\to00}} = \left(\frac{\omega_{10}}{\omega_{21}}\right)^3 \tag{13}$$

which derives from eq. (8) and where $2\Gamma_{ii\to00}$ is the decay rate of level 2 on the $i \leftrightarrow 0$ transition and ω_{i0} the corresponding transition frequency. Inserting eqs. (7) and (13) into eq. (11) we obtain

$$L = \left\{ \frac{\eta^2 p_{em}^{(1)} p_{em}^{(2)}}{100\pi^2 \epsilon^2 \Gamma_{22}^2} \frac{1}{\frac{\Gamma_{22\to00}}{\Delta_{20}^2 \Gamma_{22}} \left(\frac{\omega_{10}}{\omega_{02}}\right)^3 + \frac{\Gamma_{22\to11}}{\Delta_{21}^2 \Gamma_{22}} \left(\frac{\omega_{10}}{\omega_{21}}\right)^3} \right\}^{1/8} , \tag{14}$$

where we have assumed that $\Gamma_{11} = \Gamma_{11\to00}$, ie. level 1 only decays back into state 0. For case b) we obtain a slightly different result which nevertheless has a very similar structure. We find

$$L = \left\{ \frac{p_{em}^{(1)} p_{em}^{(2)}}{20\pi^2 \epsilon^2 \Gamma_{22}^2} \frac{1}{\frac{\Gamma_{22\to00}}{\Delta_{20}^2 \Gamma_{22}} \left(\frac{\omega_{10}}{\omega_{02}}\right) + \frac{\Gamma_{22\to11}}{\Delta_{21}^2 \Gamma_{22}} \left(\frac{\omega_{10}}{\omega_{21}}\right)} \right\}^{1/7} , \tag{15}$$

where we have used the fact that the Lamb-Dicke parameter depends on the transition frequency so that

$$\frac{\eta_{10}}{\eta_{20}} = \frac{\omega_{10}}{\omega_{02}} \quad . \tag{16}$$

It is important to note that the expressions eqs. (14) and (15) are *independent* of the laser power as long as $\Omega_{i2} \ll \Delta_{i2}$. This independence has its origin in the fact that increasing the laser power gives rise to two competing effects. It decreases the computation time T given in eq. (7) and therefore decreases the probability for an emission from level 2. On

the other hand an increased laser power increases the population in level 2 which increases the probability for a spontaneous emission. Both effects cancel, leading to the intensity independent results eqs. (14) and (15).

In table 1 we give values for the bounds eq. (14) and (15) for realistic ions. The values for Ba, Hg and Ca were calculated using eq. (14) as the qubit transition is quadrupole allowed, while Yb with an octupole allowed qubit transition is an example of case b) and is therefore calculated according to eq. (15). We calculate the values for L assuming $p_{em}^{(1)} = p_{em}^{(2)} = 1$, which is the most optimistic choice, and for two values of the Lamb-Dicke parameter: an optimistic $\eta = 1$ and the more realistic $\eta = 0.01$ (James & Hughes, 1996). We see that the numbers that may be factorized on a quantum computer using metastable optical transitions are very small even for the optimistic choice $\eta = 1$. Only the Yb ion gives results from which one may hope to factorize at least small numbers. However, it should be realized that it is extremely difficult to drive the hyperstable qubit transition in Yb sufficiently quickly to finish the calculation in a reasonable time. If we assume a ratio $\Omega_{01}^2/\Gamma_{11} = 10^{16}$ where $\Gamma_{11} = 3.77 \cdot 10^{-9}$ then from eq. (3) we find for $\eta = 1$ the value $T = 126s$ for a 4 bit number. This is so long that it is very unlikely that one can isolate the system during that time from all other decoherence sources. Therefore the practical limit to L for Yb is probably smaller than the one given in table 1.

2.3 Qubits as Zeeman sublevels of stable ground state

So far we have dealt with the case where the qubit is stored in a metastable optical transition. This method certainly has the disadvantage that at practically all times the quantum computer has a mean excitation of about 0.5 per qubit. Therefore, even if a qubit does not take part in a quantum gate operation, the qubit will decohere due to spontaneous emissions from the excited state of the qubit. To circumvent this problem one would like to store the qubits in Zeeman sublevels of a stable ground state. In that case stored qubits do not suffer any decoherence due to spontaneous emissions. However, how do we manipulate the qubit? It is unfortunately not practical to drive the qubit with a microwave field directly because the very long wavelength of this radiation does not allow us to address single ions as they cannot be separated far enough in a linear trap. Therefore we have to use a different method of manipulating the qubit. This method uses the presence of another level 2 (see Fig. 2) which is coupled to the two qubit levels 0 and 1 by two lasers of Rabi frequency Ω_{02} and Ω_{12}. Each laser is strongly detuned from its transition, and its Rabi frequency is much smaller than the detuning. However, it is assumed that both detunings are equal, ie. the two photon detuning vanishes while the one photon detuning Δ_2 is large. The advantage of this method is that due to the strong detuning Δ_2 the population in level 2 is small, so that spontaneous emissions from that level are rare. A Hamilton operator that implements an interaction analogous to the one generated by eq. (1), ie the qubit is excited while a phonon in the COM mode is deexcited and vice versa, is given by

$$H = -\hbar\Delta_2 |2\rangle\langle 2| + \frac{\hbar\Omega_{02}}{2} \left[|2\rangle\langle 0| + |0\rangle\langle 2|\right] + \frac{\hbar\eta\Omega_{12}}{2\sqrt{5L}} \left[|2\rangle\langle 1|a + |1\rangle\langle 2|a^\dagger\right] , \qquad (17)$$

where again the Lamb-Dicke limit, the RWA and a suitable interaction picture is used. To make sure that the Raman transitions have the effect as full Rabi oscillations between levels 0 and 1 the condition

$$\Omega_{02} = \frac{\eta\Omega_{12}}{\sqrt{5L}} \qquad (18)$$

has to be satisfied which we will assume in the following. The condition eq. (18) can be obtained from the solution of the time evolution

$$|\langle 0|\psi(t)\rangle|^2 = \left| \frac{\Omega_{02}^2 e^{-i(\Omega_{02}^2 + \eta^2 \Omega_{12}^2/5L)t/4\Delta_2} + \eta^2 \Omega_{12}^2/5L}{\Omega_{02}^2 + \Omega_{12}^2} \right|^2 \qquad (19)$$

$$|\langle 1|\psi(t)\rangle|^2 = 1 - |\langle 0|\psi(t)\rangle|^2 \ . \qquad (20)$$

$$|\langle 2|\psi(t)\rangle|^2 = \frac{\Omega_{02}^2}{2\Delta_2^2}(1 + \cos\Delta_2 t) \qquad (21)$$

which is valid for $\Gamma_{22}t \ll 1$ and for an initial state $|\psi(t)\rangle = |0\rangle$. To obtain $|\langle 0|\psi(t)\rangle|^2 = 0$ for some t eq. (18) has to be satisfied. Before we continue we have to realize that it is indeed enough to consider the time evolution eqs. (19-21) and the corresponding time evolution for the initial condition $|\psi(t)\rangle = |1\rangle$. To see this we realize that to a good approximation the coherences between levels 0 and 1 are uniformly distributed. Averaging the initial state over this distribution then yields a mixed state as an initial state in which both states 0 and 1 have equal weight. Therefore calculating the time evolution for both possible initial states separately and averaging over the two results leads to the final result.

Again in the Raman pulse implementation of qubits a CNOT gate can be implemented with four π pulses. From the solution eqs. (19-21) we see that the effective Rabi frequency of the qubit transition is given by

$$\Omega_{eff} = \frac{\Omega_{02}^2}{2\Delta_2} \ . \qquad (22)$$

Therefore the performance of a CNOT gate requires the time

$$\tau_{el} = \frac{8\pi\Delta_2}{\Omega_{02}^2} \ . \qquad (23)$$

Again the probability for a spontaneous emission from level 2 has to be small, ie.

$$2\Gamma_{22}\rho_{22}\tau_{el}\epsilon L^3 = p_{em}^{(2)} \ll 1 \ . \qquad (24)$$

This condition immediately yields

$$L = \left(\frac{\Delta_2 p_{em}^{(2)}}{8\pi\epsilon\Gamma_{22}} \right)^{1/3} \ . \qquad (25)$$

which is already intensity independent. From eq. (25) one could conclude that one is able to factorize gigantic numbers by making the ratio Δ_2/Γ_{22} sufficiently large, eg. $\Delta_2 = 10^{13}s^{-1}, \Gamma_{22} = 1s^{-1}$ and $\epsilon = 216$ would lead to $L = 1225$. But again this reasoning is invalid because we have neglected the fact that the two-level approximation breaks down because in the ion other levels, which we model by a level 3, also couple to levels 0 and 1 via the Raman pulses. This additional coupling is very important because it potentially involves levels that may have $\Gamma_{33} \cong 10^8 s^{-1}$. In addition the presence of level 3 changes the effective Rabi frequency in the qubit and may therefore lead to additional errors. Finally there is a limit to the size of Δ_2 from the nature of the qubit quantum numbers, as usually states 0 and 1 are hyperfine levels which require a nuclear spin flip which is precluded for very large Δ_2.

To achieve a more restrictive, but again intensity independent, upper bound for L than the one given in eq. (25) we now take into account the presence of one more level 3 which

acquires population during the execution of the quantum gate. One can give a complete analysis of the full four level system; however, this is very tedious, although not complicated in principle. In this paper we will restrict ourselves to two limiting cases where the analysis is simpler and already reveals the basic physics. We avoid an intermediate regime which is also not of practical interest as the precise control of the qubits will be difficult. If we denote the Rabi frequencies on the $i \leftrightarrow 3$ transition by Ω_{i3} and the one photon detuning as Δ_3 we can distinguish two cases, one of which reduces to the analysis given above, the other one requiring further investigation. First consider the case where

$$\frac{\Omega_{03}^2}{\Delta_3} \gg \frac{\Omega_{02}^2}{\Delta_2} \quad . \tag{26}$$

In that case the analysis given above, neglecting level 3, was not correct as the influence of level 3 on the dynamics of the qubit is actually larger than that of level 2. For example the proper effective Rabi flopping frequency of the qubit is then given by $\Omega_{03}^2/2\Delta_3$ instead of $\Omega_{02}^2/2\Delta_2$. Due to relation eq. (8) we can conclude from eq. (26) that

$$\frac{\Gamma_{33\to00}}{\Delta_3} \gg \frac{\Gamma_{22\to00}}{\Delta_2} \tag{27}$$

and therefore

$$\left(\frac{\Delta_3 p_{em}^{(3)}}{8\pi\epsilon\Gamma_{33}}\right)^{1/3} \ll \left(\frac{\Delta_2 p_{em}^{(2)}}{8\pi\epsilon\Gamma_{22}}\right)^{1/3} \quad . \tag{28}$$

In fact what one has to do is to redo the analysis leading to eq. (25) but replacing level 2 by level 3. The other case is given by

$$\frac{\Omega_{03}^2}{\Delta_3} \ll \frac{\Omega_{02}^2}{\Delta_2} \quad , \tag{29}$$

which means that indeed level 2 determines the dynamics of the system. In that case we immediately find

$$L = \left(\frac{\Delta_2 p_{em}^{(3)}}{8\pi\epsilon\Gamma_{22}}\right)^{1/3} \ll \left(\frac{\Delta_3 p_{em}^{(2)}}{8\pi\epsilon\Gamma_{33}}\right)^{1/3} \quad , \tag{30}$$

where Δ_3 and Γ_{33} is more or less determined because the lasers are extremely far detuned from the $i \leftrightarrow 3$ transitions. This could already serve as a strong limit for L, however, it is of a different structure compared to the intensity independent limit eqs. (14-15) for the two-level case. It is interesting to note that it is possible to construct a limit which has a form completely analogous to eqs. (14-15) only with slightly different exponents. For this derivation we assume that level 2 influences the qubit dynamics the most, ie. eq. (29) is satisfied. This assumption itself together with $\Omega_2 \ll \Delta_2$ (detuned Raman pulses) yields a lower limit to the computation time

$$T = \frac{8\pi\Delta_2}{\Omega_{02}^2}\epsilon L^3 \gg \frac{8\pi\epsilon L^3}{\Delta_2} \gg \frac{8\pi\epsilon L^3\Gamma_{33\to00}}{\Gamma_{22\to00}\Delta_3} \quad . \tag{31}$$

We will see from this expression for the computation time that any system with an exceedingly high lifetime of level 2 is practically useless for quantum computation because the time required to perform a computation is extremely high.

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Again of course it is important to distinguish between the cases a) and b) for the different types of qubit transitions (electric quadrupole, octupole, ..) as we have done when we investigated the case of a two-level system as a qubit. We will treat case a) explicitly in the following while we only state the result for case b).

To calculate the influence of level 3 to the decoherence we first have to find out the average population in that level. For this let us realize that during the implementation of a CNOT a full 4π-rotation will be performed. That means that on average the population in both levels 0 and 1 is equal. Therefore the much slower coupling to level 3 sees an averaged population in both levels with no average coherence. We can therefore average over two contributions to ρ_{33} which originate from the initial states $|0\rangle$ and $|1\rangle$ and we obtain

$$\rho_{33} = \frac{1}{2} \frac{\Omega_{13}^2 + \eta'^2 \Omega_{03}^2/5L}{4\Delta_3^2} \quad , \tag{32}$$

where η' is the Lamb-Dicke parameter for the $0 \leftrightarrow 3$ transition. EQ. (32) contains an additional factor $1/2$ compared to eq. (21). This is due to the fact that level 3 is strongly decaying and that therefore $\Gamma_{33}t \gg 1$. Note also that the roles of the lasers on the $i \leftrightarrow 3$ transitions has been reversed compared to their role on the $i \leftrightarrow 2$ transitions; the phonon is now destroyed by an excitation from level 0 and not from level 1 anymore. This is due to the fact that $i \leftrightarrow 3$ transitions are dipole allowed (E1) while the $i \leftrightarrow 2$ transition is quadrupole allowed (E2). Again an emission from level 3 should not occur during the whole calculation which means

$$
\begin{aligned}
p_{em}^{(3)} &= 2\Gamma_{33}\rho_{33}\tau_{el}\epsilon L^3 \\
&= \frac{16\pi^2\epsilon^2\Gamma_{22}\Gamma_{33}(\Omega_{13}^2 + \eta'^2\Omega_{03}^2/5L)}{\Delta_3^2\Omega_{02}^2 p_{em}^{(2)}}L^6 \quad .
\end{aligned}
\tag{33}
$$

Now we have to find out which term in the brackets of eq. (33) is dominant. Because we need $\Omega_{02} = \eta\Omega_{12}/\sqrt{5L}$, see eq. (18), and because we are in the Lamb-Dicke limit, ie. $\eta/\sqrt{5L} \ll 1$, the laser on the $0 \leftrightarrow 2$ transition has to be weaker than the one on the $1 \leftrightarrow 2$ transition. But again $\eta'/\sqrt{5L} \ll 1$ and therefore $\Omega_{13} \ll \eta'\Omega_{03}/\sqrt{5L}$. This implies

$$
\begin{aligned}
p_{em}^{(3)} &= \frac{16\pi^2\epsilon^2}{\Delta_3^2 p_{em}^{(2)}} \frac{\Gamma_{22}\Gamma_{33}\Omega_{13}^2}{\Omega_{02}^2}L^6 \\
&= \frac{16\pi^2\epsilon^2\Gamma_{22}\Gamma_{33}}{\Delta_3^2 p_{em}^{(2)}} \frac{\Gamma_{22\to00}}{\Omega_{02}^2} \frac{\Omega_{13}^2}{\Gamma_{33\to11}} \frac{\Gamma_{33\to11}}{\Gamma_{22\to00}}L^6
\end{aligned}
\tag{34}
$$

where we have inserted the decay coefficients $\Gamma_{ii\to jj}$ from level i to level j to be able to apply eq. (8) which then yields

$$p_{em}^{(3)} = \frac{16\pi^2\epsilon^2\Gamma_{22}\Gamma_{33}}{\Delta_3^2 p_{em}^{(2)}} \left(\frac{E_{12}}{E_{02}}\right)^2 \left(\frac{\omega_{02}}{\omega_{31}}\right)^3 \frac{\Gamma_{33\to11}}{\Gamma_{22\to00}}L^6 \quad , \tag{35}$$

where E_{i2} is the electric field of the laser on the $i \leftrightarrow 2$ transition. We now define the constant

$$\alpha = \frac{\Gamma_{33}}{\Gamma_{22}} \frac{\Gamma_{22\to00}}{\Gamma_{33\to11}} \left(\frac{E_{02}}{E_{12}}\right)^2 \tag{36}$$

and resolve eq. (35) for L to obtain

$$L = \left\{ \frac{\Delta_3^2 p_{em}^{(2)} p_{em}^{(3)} \alpha}{16\pi^2 \epsilon^2 \Gamma_{33}^2} \left(\frac{\omega_{13}}{\omega_{02}} \right)^3 \right\}^{1/6} . \tag{37}$$

Note that $E_{02} \ll E_{12}$ as argued below eq. (33) and that the ratio of the decay coefficients in α will be of the order of 1. Therefore α is much smaller than 1. In fact, because of $\Omega_{02} = \eta \Omega_{12}/\sqrt{5L}$, is a good approximation to assume $\alpha = \beta \eta^2/5L$ (with β defined below eq. (39)) which then yields

$$L = \left\{ \frac{\Delta_3^2 p_{em}^{(2)} p_{em}^{(3)} \eta^2 \alpha}{80\pi^2 \epsilon^2 \Gamma_{33}^2} \left(\frac{\omega_{13}}{\omega_{02}} \right)^3 \right\}^{1/7} . \tag{38}$$

Following very similar lines we obtain for case b)

$$L = \left\{ \frac{\Delta_3^2 p_{em}^{(2)} p_{em}^{(3)} \beta}{16\pi^2 \epsilon^2 \Gamma_{33}^2} \left(\frac{\omega_{13}}{\omega_{02}} \right)^3 \right\}^{1/6} . \tag{39}$$

where now $\beta = \Gamma_{33}\Gamma_{22\to00}/\Gamma_{22}\Gamma_{33\to00}$ is of the order of 1. Both limits eqs. (37-39) are analogous in their form to the estimates eqs. (14) and (15) if we neglect in eqs. (14 - 15) the contribution from the upper level 1 coupling to level 3.

In table 2 we present some values for L calculated from eqs. (38) and (39) for the same ions as in table 1. Again we assume that $p_{em}^{(2)} = p_{em}^{(3)} = 1$ (an optimistic choice) and observe that the resulting limits for the numbers that can be factorized on a quantum computer are very small, especially if $\eta = 0.01$. Nevertheless one could think that transitions in Ba^+ and Yb^+ are promising. However, two effects appear which lead to further limitations. One is the fact that eqs. (38) and (39) are not necessarily more restrictive than the equivalent of eq. (25) for level 3 as we have assumed eq. (30) which does not need to be true. In fact simply applying eq. (25) where Γ_{22} is replaced by Γ_{33} yields for Hg^+ $L = 11$, for Ba^+ $L = 20.5$ and for Yb^+ $L = 17.3$ which are somewhat smaller than the values for those ions in table 2. Secondly and more importantly the promising results for Ba^+ and Yb^+ are illusionary because a very stable transition is also very slow. From eq. (31) we can easily calculate that the factorization time using Ba^+ and assuming $\eta = 1$ is

$$T = 0.013L^3 \tag{40}$$

which gives for $L = 10$ a value of $T = 13s$ during which no decohering event of any kind is allowed to happen. This is a very long time and it will be difficult to isolate the quantum computer from the environment. For Yb^+ the numbers are even more devastating. We obtain

$$T = 51186L^3 \tag{41}$$

which indicates for $L = 4$ that $T = 3.2 \cdot 10^6 \approx 38days$, a completely unrealistic number. This again shows very clearly that the solution to the problem of decoherence cannot be to employ extremely stable transitions for qubits but to find methods that enable the quantum computer to cope with a certain level of decoherence, ie we need a form of quantum error correction. In principle it is indeed possible to implement quantum error correction in quantum computers but it remains the question inasmuch these methods will in fact improve the prospects of quantum computation. In the next section we will address this problem and derive limitations to quantum factoring including the use of quantum error correction methods.

3 Bounds on L including quantum error correction

In the preceding section we derived intensity independent upper bounds for the numbers that can be factorized on a quantum computer. The results that we obtained are not very promising as they rule out the possibility to factorize large numbers (numbers with $L \leq 400$ can be factorized on a classical computer) on a quantum computer. However, in these estimates we have yet to include the possibility of quantum error correction (Shor, 1995; Steane, 1996a; Steane, 1996b; Calderbank & Shor, 1996; Ekert & Machiavello, 1996; Knill & Laflamme, 1996; Cirac *et al*, 1996) and fault-tolerant quantum computation (Shor, 1996; DiVincenzo & Shor, 1996; Plenio *et al*, 1996). These methods have the ability to change drastically the probability that the calculation becomes corrupted by an emission. For example, consider a quantum error correction code that can correct a single general error. If the probability to suffer an error is $p \ll 1$ then the probability to suffer two errors, an event that cannot be corrected by the code, is p^2. Therefore, we have reduced the probability for a fatal error from p to p^2 and it is then possible to perform quadratically as many computational steps as possible without quantum error correction. Higher order correcting codes promise even higher gains in possible computational steps. However, this simple argument neglects two effects that counteract the expected benefits of quantum error correction codes. One reason is the quite obvious fact that the implementation of quantum error correction and fault-tolerant quantum computation requires substantial overheads in additional qubits for encoding the qubits and additional quantum gate operations to implement quantum logic operations on the encoded quantum bits. This overhead can be quite substantial even for simple quantum error correction codes.

The other effect is less obvious but can have an even stronger impact on the practical efficiency of quantum error correction codes. The problem is that quantum error correction codes loose their ability to correct for decohering events if population leaves the qubit as a result of the decohering event. Therefore we have to hope that all population will remain within the system representing the qubit. However, as we have seen in the analysis of section II this will not be the case for a realistic implementation of a qubit. The reason is simply that the two-level approximation is never perfectly satisfied. If we excite an extraneous level then an emission from this level does not necessarily bring the population back into the qubit. An error of this kind cannot be dealt with by a quantum error correction code. One would have to find ways to repump the lost population back into the qubit. This, however, is quite difficult if not impossible because of the sheer multitude of possible levels where the spontaneous emission may lead the population to.

In this section we will first consider the case of quantum computation using two-level systems to represent the qubits. Subsequently we will then consider the representation of qubits in Zeeman sublevels using far detuned Raman pulses for their manipulation.

Before we start with the discussion of the two-level system case, we first define two constants, q and c, describing the necessary overheads in the number of qubits and the number of additional quantum gate operations.

- The number of required qubits rises by a factor of q as all qubits will have to be encoded and will be kept encoded during the calculation. Repeated decoding and encoding would raise the number of operations substantially and lead to unnecessary errors.

- The number of operations required to implement one quantum logical operation on

encoded qubits rises by a factor of c because we have to perform more than one quantum gate to implement one quantum logic operation on the encoded qubits. in addition we also have to perform quantum error corrections periodically.

The value of the factors q and c depends heavily on the quantum error correction code that is used. If one wants to protect a single qubit against one general error, then it is known that $q = 5$ is the minimal possible value (Knill & Laflamme, 1996). The number c for this code is uncertain in general for the most optimal codes but is at least $c = 5$ for a CNOT gate (Shor, 1996) and probably much more for other gates such as a Toffoli gate.

3.1 Qubits as two-level systems

For the following analysis we assume an error correction code capable of correcting one error. A similar calculation will then yield the result for codes that can correct $k - 1$ errors and we simply state these results. First we need to find the time required to perform a CNOT operation on encoded quantum bits. The computer contains q times more qubits, ie. $5qL$ qubits, and c times more operations are required. Therefore

$$\tau_{el} = \frac{4\pi\sqrt{5qL}}{\eta\Omega_{01}} c \ . \tag{42}$$

The spontaneous lifetime of the quantum computer is then

$$\tau_{sp} = \frac{1}{5Lq\Gamma_{11}} \ . \tag{43}$$

As we now have the ability to correct for errors during the calculation, we no longer need to require that there is practically no spontaneous emission during the whole calculation. Instead we ask for the probability p_N to have one error during N logical operations of our algorithm to be small compared to 1. We find

$$p_N = \frac{4\pi\sqrt{5qL}}{\eta\Omega_{01}} 5cqL\Gamma_{11}N \ . \tag{44}$$

If we perform a fault-tolerant error correction after these N operations then this correction will fail with probability p_N^2, the probability to have suffered two errors, because the code is not designed to correct for two errors. Therefore the probability that the whole computation fails is given by

$$p_{fail} = p_N^2 \frac{\epsilon L^3}{N} \ . \tag{45}$$

This immediately implies that $N = 1$ is the optimal choice unless we take the N dependence of c into account which is however quite difficult as c is not known very well. For the following we will assume $N = 1$ and we obtain

$$L = \left(\frac{\eta^2 p_{fail}}{2000\pi^2 q^3 c^2 \epsilon} \left(\frac{\Omega_{01}}{\Gamma_{11}} \right)^2 \right)^{1/6} \ . \tag{46}$$

From eq. (46) we easily deduce the total computation time

$$T = \frac{400\pi^2 q^2 c^2 \epsilon^{3/2}}{\eta^2 p_{fail}^{1/2}} \frac{\Gamma_{11}}{\Omega_{01}^2} L^{6.5} \ . \tag{47}$$

Again eqs. (46) and (47) are dependent on the laser power driving the system. To eliminate this intensity dependence we now have to take other levels into account as we did in section II. Again we have to distinguish the two cases a) (qubit transition is quadrupole allowed) and b) (qubit transition is octupole allowed) depending on the nature of the qubit transition. We first treat case a) and then state the result for case b).

To keep the resulting expressions more transparent we neglect the contribution of the upper qubit level 1 to the violation of the two-level approximation. The generalization to this case is easy. We then find for the population in the extraneous level 2

$$\rho_{22} = \frac{\Omega_{02}^2}{8\Delta_2^2} \quad . \tag{48}$$

Now we want to know the probability p_{out} that an emission from level 2 which leads to population *outside* the qubit transition occurs during the whole computation. It is very important to note that it is this probability p_{out} that has to be much smaller than unity because even a single emission leading out of the qubit cannot be corrected by the quantum error correction code. We find

$$p_{out} = 2\Gamma_{22}^{out} \frac{\Gamma_{22\to00}\Omega_{02}^2}{8\Delta_2^2\Gamma_{22\to00}} \frac{400\pi^2q^2c^2\epsilon^{3/2}}{\eta^2 p_{fail}^{1/2}} \frac{\Gamma_{11}}{\Omega_{01}^2} L^{6.5} \quad . \tag{49}$$

This can be solved for L and with eq. (8) we obtain

$$L = \left\{ \frac{\eta^2\Delta_2^2 p_{fail}^{1/2} p_{out}}{100\pi^2q^2c^2\epsilon^{3/2}\Gamma_{22\to00}\Gamma_{22}^{out}} \left(\frac{\omega_{02}}{\omega_{01}}\right)^3 \right\}^{\frac{2}{13}} \quad . \tag{50}$$

With an analogous calculation we obtain for case b)

$$L = \left\{ \frac{\Delta_2^2 p_{fail}^{1/2} p_{out}}{20\pi^2q^2c^2\epsilon^{3/2}\Gamma_{22\to00}\Gamma_{22}^{out}} \frac{\omega_{02}}{\omega_{01}} \right\}^{\frac{2}{11}} \quad . \tag{51}$$

The estimates eqs. (50) and (51) are valid for a quantum error correction code that is able to correct for one error. One can quite easily extend these results to quantum error correction codes that are able to correct for $k-1$ errors. We state only the main results for the cases a) and b) as they can be derived easily along the lines that lead us to eqs. (50) and (51). For case a) we obtain the computation time

$$T = \frac{400\pi^2c^2q^2\epsilon}{\eta^2} \left(\frac{\epsilon}{p_{fail}}\right)^{\frac{1}{k}} \frac{\Gamma_{11}}{\Omega_{01}^2} L^{5+\frac{3}{k}} \quad . \tag{52}$$

Now we take into account other levels outside the qubit transition. A spontaneous emission that leads from these levels out of the qubit system cannot be corrected for by the quantum error correction code. Therefore we obtain

$$L = \left\{ \frac{\Delta_2^2\eta^2 p_{out}}{100\pi^2c^2q^2\epsilon\,\Gamma_{22}^{out}\Gamma_{22\to00}} \left(\frac{\omega_{02}}{\omega_{01}}\right)^3 \left(\frac{p_{fail}}{\epsilon}\right)^{\frac{1}{k}} \right\}^{\frac{k}{5k+3}} \quad . \tag{53}$$

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For case b) we obtain the result

$$L = \left\{ \frac{\Delta_3^2 p_{out}}{20\pi^2 c^2 q^2 \epsilon \, \Gamma_{22}^{out}\Gamma_{22\to00}} \frac{\omega_{02}}{\omega_{01}} \left(\frac{p_{fail}}{\epsilon} \right)^{\frac{1}{k}} \right\}^{\frac{k}{4k+3}} . \tag{54}$$

Again we can now discuss numerical values for eqs. (50) and (51) obtained from the atomic data of real ions. In table 3 we give the results for eqs. (50) and (51). we see that although the estimates have improved they still restrict factorization to very small numbers especially for a Lamb-Dicke parameter $\eta = 0.01$ where we obtain upper limits that restrict the numbers that are possible to factorize to trivial sizes. Again Yb^+ seems to be very promising but again we should note that it is very hard to actually drive the this transition sufficiently quickly so that the computation times quickly reach astronomical values. For $\Omega_{01}^2/\Gamma_{11} = 10^{16}$ we obtain for $L = 4$ a value of $T = 1400s$, a value which is extremely high. For Ba^+ we obtain for $L = 4$, $\Omega_{01}^2/\Gamma_{11} = 10^{16}$ and $\eta = 1$ the value $T = 0.84s$.

3.2 Qubits in Zeeman sublevels

Now we perform the same analysis including quantum error correction codes for the case where the qubits is stored in Zeeman sublevels. Of course the analysis here runs along similar lines as the one for the two-level system and that of section II. Therefore we only state the final result for the cases a) and b). We present the results for an error correcting code which is able to correct $k - 1$ errors. We only treat the case where the extraneous levels give a small correction to the time evolution of the system, ie we assume the regime of eq. (29). For the computation time we obtain the lower limit

$$T \gg \frac{8\pi\epsilon L^3 \Gamma_{33\to00}}{\Gamma_{22\to00}\Delta_3} c \tag{55}$$

for both cases a) and b). For case a) we obtain the estimate

$$L = \left\{ \frac{\alpha \, p_{em}^{(3)} \Delta_3^2}{16\pi^2 c^2 \epsilon \Gamma_{33}^{out}\Gamma_{33\to11}} \left(\frac{\omega_{13}}{\omega_{02}} \right)^3 \left(\frac{p_{fail}}{\epsilon} \right)^{\frac{1}{k}} \right\}^{\frac{k}{3k+3}} , \tag{56}$$

where

$$\alpha = \frac{\Gamma_{22\to00}}{\Gamma_{22}} \left(\frac{E_{02}}{E_{12}} \right)^2 \ll 1 . \tag{57}$$

E_{i2} denotes the electric field strength of the laser on the $i \leftrightarrow 2$ transition. Note that again we have to take into account that α contains a hidden dependence on η, L and q. This dependence is due to the fact that we have to satisfy the condition

$$\Omega_{02} = \frac{\eta\Omega_{12}}{\sqrt{5Lq}} . \tag{58}$$

Therefore $\alpha \cong \beta\eta^2/5Lq$ (β is defined in (61)) and we obtain

$$L = \left\{ \frac{\beta \, p_{em}^{(3)} \Delta_3^2}{80\pi^2 c^2 \epsilon \Gamma_{33}^{out}\Gamma_{33\to11}} \left(\frac{\omega_{13}}{\omega_{02}} \right)^3 \left(\frac{p_{fail}}{\epsilon} \right)^{\frac{1}{k}} \right\}^{\frac{k}{4k+3}} . \tag{59}$$

For case b) we obtain a very similar result. We find

$$L = \left\{ \frac{\beta\, p_{em}^{(3)}\, \Delta_3^2}{32\pi^2 c^2 \epsilon\, \Gamma_{33}^{out} \Gamma_{33\to 11}} \left(\frac{\omega_{03}}{\omega_{02}}\right)^3 \left(\frac{p_{fail}}{\epsilon}\right)^{\frac{1}{k}} \right\}^{\frac{k}{3k+3}} \quad , \tag{60}$$

where

$$\beta = \frac{\Gamma_{22\to 00}}{\Gamma_{22}} \approx \frac{1}{2} \quad . \tag{61}$$

Assuming that $p_{em}^{(3)}\beta = 1$ we then obtain the values given in table 4 which again seem to be promising if we assume $\eta = 1$. For the more realistic values of $\eta = 0.01$, however, the estimate reduces substantially. In the discussion of table 2 we already pointed out that high values for the estimates on L alone do not suffice to allow a practical implementation of quantum factoring. Important also is that the computation time be sufficiently short. This restriction again excludes Yb^+ immediately because for $\Omega_{02}^2/\Gamma_{22\to 00} = 10^{16}, \eta = 1$ and $L = 4$ we obtain $T = 189$days. It also makes Ba^+ a very unlikely candidate for successful factorization as for $L = 10$ we obtain with the same parameters as for Yb^+ the value $T = 65s$.

4   Conclusions

In this paper we have discussed the constraints imposed by spontaneous emission onto factorization of big numbers using a quantum computer. In section II we have investigated the possibility of factorization without the use of quantum error correction. We were able to derive limits to the bitsize of the numbers that can be factorized on a quantum computer. These limits are *independent* of the intensity of the laser that is used to implement quantum gates. This intensity independence was achieved by taking into account the failure of the two-level approximation even for modest laser powers. The result of these limits is that without the use of quantum error correction the factorization even of small numbers will not be possible. Therefore we then investigated in section III whether the implementation of quantum error correction can improve the prospects of factorization on a quantum computer. It turned out that quantum error correction in realistic atomic systems is much less effective than in the ideal case of closed two-level systems because the failure of the two-level approximation can lead to a leakage of population out of the system which can not be corrected by a quantum error correction code. Therefore even high order quantum error correction codes are susceptible to single errors that lead to loss of population out of the system which corrupts their performance. Nevertheless the upper limits in this case are higher than without quantum error correction. However, the total computation time using quantum error correction, and even in the case without quantum error correction, tends to be very large because a very stable transition cannot be driven very fast. We find lower bounds for the computation times and conclude that even for moderate numbers it will be difficult to isolate the system from the environment such that no decohering event, other than a spontaneous emission, can take place. At this point other decoherence effects have to be taken into account (see eg. (Garg, 1996; Hughes *et al*, 1996; James & Hughes, 1996)). However, we think that even the sole inclusion of spontaneous emission as a decohering effect already shows that the practical application of quantum computers in factorization of big numbers will be highly unlikely within the present models of quantum computation. The considerations of this paper also apply to other possible algorithms for quantum computers if they also require $O(L^3)$ operations.

Although the conclusions of this paper are rather pessimistic with regard to the practical application of quantum computers for actual computations we would like to stress that there are applications such as for example in precision spectroscopy (Wineland *et al*, 1992) which require much less operations, $O(L)$, and for which therefore the above presented estimates will be much more optimistic. We believe that in this area interesting applications of the concept of quantum computing will be found. However, unrealistic hopes that have been raised in the past about the practicality of quantum computers have to be moderated.

5   Acknowledgements

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Fig. 1 :   A two-level system representing a qubit. The levels 0 and 1 represent the two possible logical values. Unlike in classical computing coherent superpositions between the logical values 0 and 1 are possible. The upper level may decay spontaneously to the ground state with a decay rate $2\Gamma_{11}$ and the system can be driven by a laser with Rabi frequency Ω_{01}.

Fig. 2 :   The two lower levels 0 and 1 of the Λ system, are Zeeman sublevels of a stable ground state and represent the logical values 0 and 1 of the qubit. They are coupled via Raman pulses that are strongly detuned from the intermediate level 2. The two-photon detuning is assumed to be zero while the one photon detuning Δ_2 is much larger than the associated Rabi frequencies on the $i \leftrightarrow 2$ transition is Ω_{i2}.

Fig. 3 :   Schematic picture of the excitation of several ions in a linear ion trap. The translational degrees of freedom of the ions are assumed to be cooled to their respective ground states. To implement quantum gates, standing wave fields interact with the ions and thereby change the inner state of the ions as well as the state of the center-of-mass mode (which leads to entanglement).

Fig. 4 :   Schematic level scheme envisaged for quantum computation, in which the $0 \leftrightarrow 1$ transition represents the qubit. It is driven by a laser of Rabi frequency Ω_{01}. Level 1 has a spontaneous decay rate $2\Gamma_{11}$ which is small. The laser which is resonant with the $0 \leftrightarrow 1$ transition inevitably couples level 0 also to other non-resonant levels as for example level 2. The Rabi frequency on that transition is then Ω_{02} and the decay rate is $2\Gamma_{22}$ is usually much larger than Γ_{11}. The effective Rabi frequency on the $0 \leftrightarrow 2$ transition is very small as the laser is detuned by $\Delta_{02} \gg \Omega_{02}$.

Fig. 5 :   Results of a discrete Fourier transform (DFT) of a function $f(n) = \delta_{8,(nmod10)}$ with $n = 0, 1, \ldots, 31$. The solid line is the result for a quantum computer with stable qubits and represents the correct result. The dashed line shows the result of the same computation using a quantum computer with unstable qubits, one of which has suffered a spontaneous emission during the calculation. The results clearly differ and show the impact of a single spontaneous emission on a quantum computation. For the parameters chosen on average the quantum computer will suffer one emission per DFT, ie $\tau_{sp} = T$ in this case.

Fig. 6 :   The same quantum computation as in Fig. 5. The solid line again represents the result using a quantum computer with stable qubits, while the dashed line shows the result using a quantum computer with unstable qubits. This time, however, the unstable quantum computer does not suffer an emission during the whole calculation. Again the results differ illustrating the impact of the conditional time evolution between spontaneous emissions.

Table 1 :   For several possible systems the upper limit on the bitsize L of the number N that can be factorized on a quantum computer is calculated. A qubit is stored in a metastable optical transition. The atomic levels which are abbreviated in Fig. 1 by $0, 1$ and 2 are given. The atomic data are inserted into eqs. (14) and (15) and the result is given in the last row of the table. The atomic data are inserted into eqs. (14) and (15) and the result is given in the last row of the table. The atomic data are derived from (A) = (Bashkin & Stoner, 1978), (B) = (Bergquist *et al*, 1985), (C) = (Sauter *et al*, 1986a; Sauter *et al*, 1986b), (D) = calculated using(Bell *et al*, 1992) and (Gill *et al*, 1995), (E) = (Andersen & Sørensen, 1973), (F) = (Fawcett & Wilson, 1991),(G) = (Gosselin *et al*, 1988), (H) = (Eriksen & Poulsen, 1980), (I) = (Gallagher, 1967), (J) = (Wiese *et al*, 1969).

Table 2 :   For several possible systems the upper limit on the bitsize L of the number N that can be factorized on a quantum computer is calculated. A qubit is stored in two Zeeman sublevels of a stable ground state. The atomic levels which are abbreviated in Fig. 7 by $0, 1, 2$ and 3 are given. The atomic data are inserted into eq. (19) and the result is given in the last rows of the table.

Fig. 7 :   The Λ system as in Fig. 2 but now the coupling to an extraneous levels 3 is taken into account. The lasers performing the Raman pulses also couples the two lower states to level 3 which may subsequently decay spontaneously.

Table 3 :   For several possible systems the upper limit on the bitsize L of the number N that can be factorized on a quantum computer using quantum error correction is calculated. A qubit is stored in a metastable optical transition.The atomic levels which are abbreviated in Fig. 7 by $0, 1$ and 2 are given. The atomic data are inserted into eq. (19) and the result is given in the last rows of the table.

Table 4 :   For several possible systems the upper limit on the bitsize L of the number N that can be factorized on a quantum computer is calculated. A qubit is stored in two Zeeman sublevels of a stable ground state. The atomic levels which are abbreviated in Fig. 7 by $0, 1, 2$ and 3 are given. The atomic data are inserted into eq. (19) and the result is given in the last rows of the table.
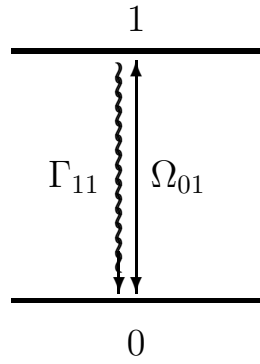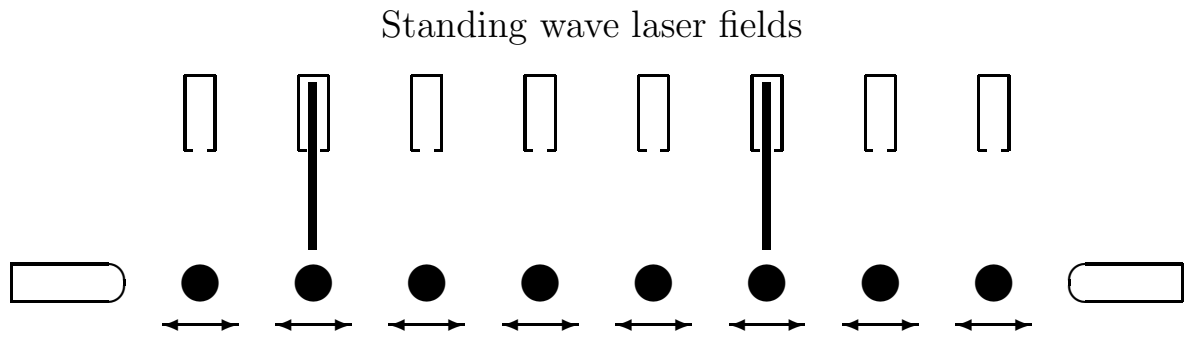
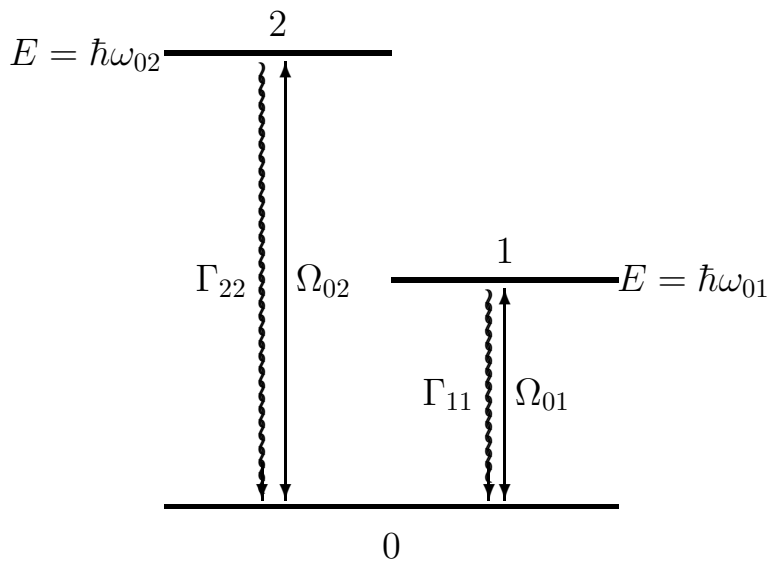Plenio and Knight FIGURE 1

Figure 1:
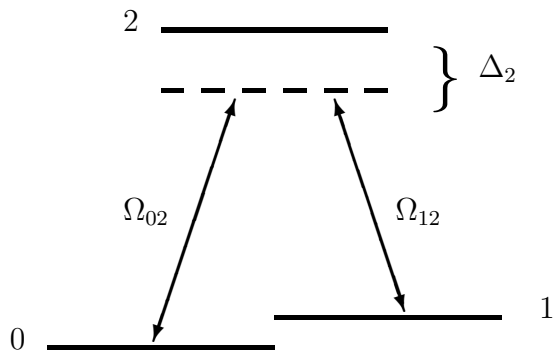
Plenio and Knight FIGURE 2

Figure 2:

Standing wave laser fields

Plenio and Knight FIGURE 3

Figure 3:

Plenio and Knight FIGURE 4

Figure 4:

Figure 5:

Plenio and KnightFIGURE 5

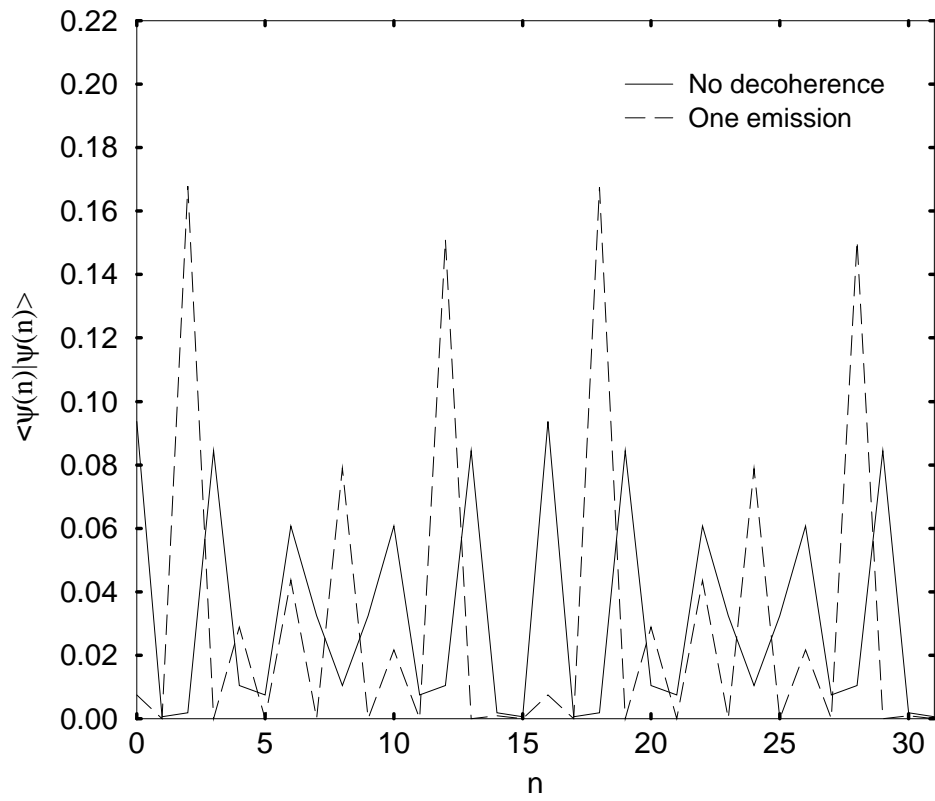Figure 6:

Plenio and Knight FIGURE 6
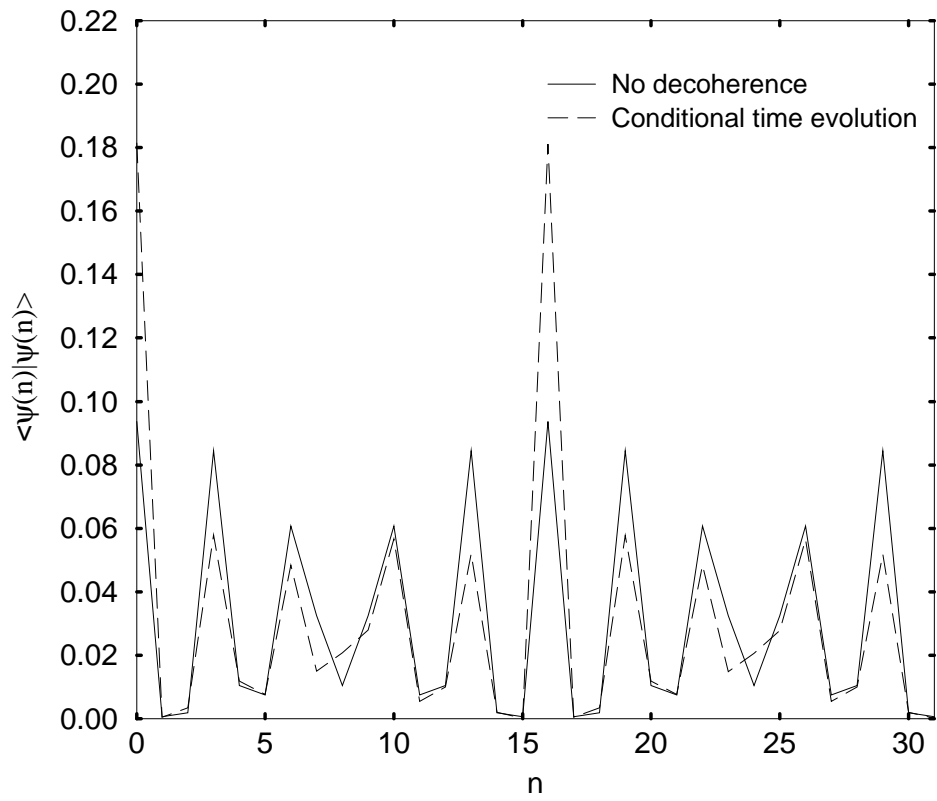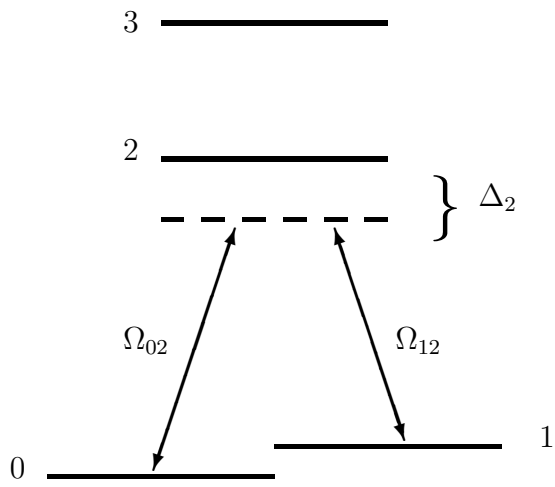
Ion	Ca^+	Hg^+	Ba^+	Yb^+
level 0	$4s\,^2S_{1/2}$	$5d^{10}6s^2\,^2S_{1/2}$	$6s\,^2S_{1/2}$	$4f^{14}6s^2\,^2S_{1/2}$
level 1	$3d\,^2D_{5/2}$	$5d^96s^2\,^2D_{5/2}$	$5d\,^2D_{5/2}$	$4f^{13}6s^2\,^2F_{7/2}$
level 2	$4s\,^2P_{3/2}$	$5d^{10}6p^2\,^2P_{1/2}$	$6s\,^2P_{3/2}$	$4f^{14}6p^2\,^2P_{3/2}$
$\omega_{01}\,[s^{-1}]$	$2.61\cdot10^{15}$ (A)	$6.7\cdot10^{15}$ (B)	$1.07\cdot10^{15}$ (C)	$4.04\cdot10^{15}$ (D)
$\omega_{02}\,[s^{-1}]$	$4.76\cdot10^{15}$ (A)	$11.4\cdot10^{15}$ (E)	$4.14\cdot10^{15}$ (C)	$5.7\cdot10^{15}$ (F)
$\Gamma_{22\to00}\,[s^{-1}]$	$67.5\cdot10^6$ (G)	$5.26\cdot10^8$ (H)	$58.8\cdot10^6$ (I)	$60.2\cdot10^6$ (F)
$\Gamma_{22\to11}\,[s^{-1}]$	$4.95\cdot10^6$ (J)	$1.25\cdot10^8$ (H)	$18.5\cdot10^6$ (I)	0.03 (F)
$L(\eta=1)$	6.9	4.9	14.2	14.3
$L(\eta=0.01)$	2.2	1.6	4.5	14.3

Plenio and Knight TABLE 1

Ion	Ca^+	Hg^+	Ba^+	Yb^+
level 0	$4s\,^2S_{1/2}$	$5d^{10}6s^2\,^2S_{1/2}$	$6s\,^2S_{1/2}$	$4f^{14}6s^2\,^2S_{1/2}$
level 2	$3d\,^2D_{5/2}$	$5d^96s^2\,^2D_{5/2}$	$5d\,^2D_{5/2}$	$4f^{13}6s^2\,^2F_{7/2}$
level 3	$4s\,^2P_{3/2}$	$5d^{10}6p^2\,^2P_{1/2}$	$6s\,^2P_{3/2}$	$4f^{14}6p^2\,^2P_{3/2}$
$\omega_{02}\,[s^{-1}]$	$2.61\cdot10^{15}$	$6.7\cdot10^{15}$	$1.07\cdot10^{15}$	$4.04\cdot10^{15}$
$\omega_{13}\,[s^{-1}]$	$4.76\cdot10^{15}$	$11.4\cdot10^{15}$	$4.14\cdot10^{15}$	$5.7\cdot10^{15}$
$\Gamma_{33}\,[s^{-1}]$	$73\cdot10^6$ (J)	$6.51\cdot10^8$ (H)	$79.7\cdot10^6$ (I)	$60.2\cdot10^6$ (F)
$L(\eta=1)$	14.0	4.2	24.4	26.0
$L(\eta=0.01)$	4.0	1.4	6.4	26.0

Plenio and Knight TABLE 2

Plenio and Knight FIGURE 7

Figure 7:

Ion	Ca^+	Hg^+	Ba^+	Yb^+
level 0	$4s\,^2S_{1/2}$	$5d^{10}6s^2\,^2S_{1/2}$	$6s\,^2S_{1/2}$	$4f^{14}6s^2\,^2S_{1/2}$
level 1	$3d\,^2D_{5/2}$	$5d^96s^2\,^2D_{5/2}$	$5d\,^2D_{5/2}$	$4f^{13}6s^2\,^2F_{7/2}$
level 2	$4s\,^2P_{3/2}$	$5d^{10}6p^2\,^2P_{1/2}$	$6s\,^2P_{3/2}$	$4f^{14}6p^2\,^2P_{3/2}$
$\omega_{01}\,[s^{-1}]$	$2.61\cdot10^{15}$	$6.7\cdot10^{15}$	$1.07\cdot10^{15}$	$4.04\cdot10^{15}$
$\omega_{02}\,[s^{-1}]$	$4.76\cdot10^{15}$	$11.4\cdot10^{15}$	$4.14\cdot10^{15}$	$5.7\cdot10^{15}$
$\Gamma_{out}\,[s^{-1}]$	$4.95\cdot10^6$ (H)	$1.5\cdot10^8$ (H)	$16.6\cdot10^6$ (I)	$0.9\cdot10^6$ (F)
$\Gamma_{33\rightarrow00}\,[s^{-1}]$	$67.5\cdot10^6$ (G)	$5.26\cdot10^8$ (H)	$45.5\cdot10^6$ (I)	$60.2\cdot10^6$ (F)
$L(\eta=1)$	16.0	15.0	21.0	32.0
$L(\eta=0.01)$	3.7	3.9	5.1	32.0

Plenio and Knight TABLE 3

Ion	Ca^+	Hg^+	Ba^+	Yb^+
level 0	$4s\,^2S_{1/2}$	$5d^{10}6s^2\,^2S_{1/2}$	$6s\,^2S_{1/2}$	$4f^{14}6s^2\,^2S_{1/2}$
level 2	$3d\,^2D_{5/2}$	$5d^96s^2\,^2D_{5/2}$	$5d\,^2D_{5/2}$	$4f^{13}6s^2\,^2F_{7/2}$
level 3	$4s\,^2P_{3/2}$	$5d^{10}6p^2\,^2P_{1/2}$	$6s\,^2P_{3/2}$	$4f^{14}6p^2\,^2P_{3/2}$
$\omega_{02}\,[s^{-1}]$	$2.61\cdot10^{15}$	$6.7\cdot10^{15}$	$1.07\cdot10^{15}$	$4.04\cdot10^{15}$
$\omega_{13}\,[s^{-1}]$	$4.76\cdot10^{15}$	$11.4\cdot10^{15}$	$4.14\cdot10^{15}$	$5.7\cdot10^{15}$
$\Gamma_{out}\,[s^{-1}]$	$4.95\cdot10^6$ (J)	$1.5\cdot10^8$ (H)	$16.6\cdot10^6$ (I)	$0.9\cdot10^6$ (F)
$\Gamma_{33\to00}\,[s^{-1}]$	$67.5\cdot10^6$ (G)	$5.26\cdot10^8$ (H)	$45.5\cdot10^6$ (I)	$60.2\cdot10^6$ (F)
$L(\eta=1)$	27.0	26	38.0	73.0
$L(\eta=0.01)$	5.0	4.9	7.2	73.0

Plenio and Knight TABLE 4