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DECOMPOSABLE GRAPHS AND HYPERGRAPHS

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Abstract

We define and investigate the notion of a decomposable hypergraph, showing that such a hypergraph always is conformal, that is, can be viewed as the class of maximal cliques of a graph. We further show that the clique hypergraph of a graph is decomposable if and only if the graph is triangulated and characterise such graphs in terms of a combinatorial identity.

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Introduction

There are a number of areas in mathematics in which one needs to consider the combinatorial propeties of a non-void class $\mathcal{C}$ of pairwise incomparable subsets of a finite set C. In the theory of games, Vorob'ev (1967), the subsets are *coalitions*; in a measure theoretic problem considered by Kellerer (1964) and also Vorob'ev (1962), the subsets correspond to *prescribed marginals*; in the general theory of contingency tables, discussed by Haberman (1974), the subsets define the *permissible interactions*, in graph theory, the subsets are the *maximal cliques* of a graph with vertex set C; whilst in the theory of Markov fields over graphs, see Suomela (1976) and Vorob'ev (1963) the subsets also correspond to maximal cliques of a graph. Certain problems of interset in these fields have led to the definition of a family of such classes $\mathcal{C}$ which, following Haberman, we call decomposable classes, and the main aim of this paper is to unify and extend the combinatorial results known concerning these classes.

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Let us examine in more detail the problems which arise in the fields mentioned.

Problem 1. Let $\{(X_\gamma, \mathcal{X}_\gamma): \gamma \in C\}$ be a set of *measure spaces* indexed by C; that is, for each $\gamma \in C$, X_γ is a non-empty set and $\mathcal{X}_\gamma$ is a σ-field of subsets of X_γ. For each subset $c \subseteq C$ write $(X_c, \mathcal{X}_c)$ for the *product* measure space $\otimes_{\gamma\in C}(X_\gamma, \mathcal{X}_\gamma)$, and put $(X, \mathcal{X}) = (X_C, \mathcal{X}_C)$. We will take as *given* a class $\mathcal{C}$ of subsets of C, and for each $c \in \mathcal{C}$, a probability measure μ_c on $(X_c, \mathcal{X}_c)$, such that the system $\{\mu_c: c \in \mathcal{C}\}$ satisfies the following *consistency condition*: if $d \subseteq a \cap b$ for $a, b \in \mathcal{C}$, then the *images* $\mu_{a,d} = \mu_a \circ \pi_{a,d}^{-1}$ and $\mu_{b,d} = \mu_b \circ \pi_{b,d}^{-1}$ of μ_a and μ_b under the canonical projections $\pi_{a,d}: X_a \to X_d$ and $\pi_{b,d}: X_b \to X_d$, respectively, *coincide*. An obvious way to get such a system is to take a measure μ on $(X, \mathcal{X})$ and put $\mu_c = \mu \circ \pi_c^{-1}$ where $\pi_c: X \to X_c$ is the canonical projection. In this case the measure μ is said to be an *extension* of the system $\{\mu_c: c \in \mathcal{C}\}$.

The problem considered by Vorob′ev (1962) and Kellerer (1964) is the following: *for which classes $\mathcal{C}$ of subsets of C does every consistent system $\{\mu_c: c \in \mathcal{C}\}$ admit an extension*? It is not hard to show that for $\mathcal{C}_1 = \{\{1,2\},\{2,3\}\}$, every consistent system of measures *does* admit an extension, whilst for $\mathcal{C}_2 = \{\{1,2\},\{2,3\},\{3,1\}\}$, this is *not* the case.

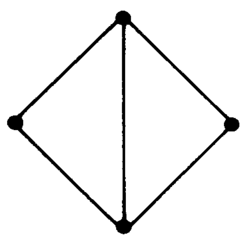
Problem 2. In this case we let X_γ denote the finite set of categories associated with a response γ from a set C of responses. The product set $X = \prod_{\gamma\in C} X_\gamma$ indexes the combinations of categories of responses, and we can consider $|C|$-dimensional contingency tables $\{n(x): x \in X\}$ over X. A *hierarchical log-linear model* for such a contingency table is uniquely specified by the (generating) class $\mathcal{C}$ of pairwise incomparable subsets of C, whose marginals $\{n_c: c \in \mathcal{C}\}$ constitute the minimal sufficient statistics for the model; see Haberman (1974) for background and further details.

A problem considered by Haberman (1974) is the following: *for which classes $\mathcal{C}$ does there exist an explicit formula for the maximum likelihood estimator $\hat{m}$, of $m = \mathbf{E}\{n\}$, under the model defined by $\mathcal{C}$*? For example, an explicit formula exists for $\mathcal{C}_3 = \{\{1,2\},\{2,3\},\{3,4\}\}$, but not for $\mathcal{C}_4 = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$.

Problem 3. Let $\mathbf{C} = (C, E(\mathbf{C}))$ be a simple graph with vertex set C and edge set $E(\mathbf{C})$, and suppose (continuing the notation of Problem 1) that for each $\gamma \in C$ we have a measure space $(X_\gamma, \mathcal{X}_\gamma)$. An *X-valued random field* over $\mathbf{C}$ consists of a random variable $\boldsymbol{\xi}: \Omega \to X$ defined over some probability space $(\Omega, \mathfrak{A}, \mathbf{P})$, and we write $\boldsymbol{\xi} = (\xi_\gamma: \gamma \in C)$. For each $c \subseteq C$ we let $\boldsymbol{\xi}_c: \Omega \to X_c$ be the *c-marginal* random field and let P_c be the *distribution* $\mathbf{P} \circ \boldsymbol{\xi}_c^{-1}$ of $\boldsymbol{\xi}_c$ on $(X_c, \mathcal{X}_c)$.

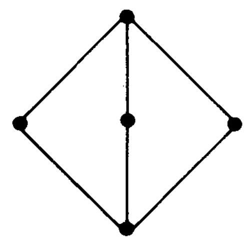
The random field $\boldsymbol{\xi}$ is said to be $\mathbf{C}$-*Markov* if for any three pairwise disjoint subsets a, b and d of C with d separating a from b, we have $\boldsymbol{\xi}_a$ and $\boldsymbol{\xi}_b$ *conditionally* *independent* given ξ_d. A question of great interest, particularly for those who wish to simulate such Markov random fields, is the following: *for which graphs* $\mathbf{C}$ *does there exist a closed-form expression for the distribution* $P = \mathbf{P} \circ \xi^{-1}$ *of any X-valued* $\mathbf{C}$-*Markov random field* ξ? It is not hard to show that this problem can be reduced to a discussion of the interrelations between the distributions $\{P_c\colon c \in \mathcal{C}_{\mathbf{C}}\}$ of the random fields $\{\xi_c\colon c \in \mathcal{C}_{\mathbf{C}}\}$, where $\mathcal{C}_{\mathbf{C}}$ is the hypergraph of all (maximal) cliques of the graph $\mathbf{C}$; see Suomela (1976) and Vorob'ev (1963) for further details.

We remark at this point that there does exist a simple expression for the distribution P when we are considering $\mathbf{C}_5$ below, but not for $\mathbf{C}_6$.



$\mathbf{C}_5$

FIGURE 1



$\mathbf{C}_6$

FIGURE 2

For further information on the relation between these problems and the work which follows, we refer to Speed (1978). (Note that the formula displayed on page 303 of this reference is wrong and should involve a conditional probability distribution.)

A pair $(C, \mathcal{C})$ of the type described above is a hypergraph in the sense of Berge (1973), as long as the union of all the members of $\mathcal{C}$ coincides with C, indeed a hypergraph in which no edge is contained in any other edge. Any such hypergraph may be associated with a graph, its 2-section, and we will discuss the relation between the decomposability of C and properties of this graph. We find, for example, that the family of all decomposable classes C may be identified with the family of graphs called *triangulated* by Berge (1973), page 368, *rigid circuit* (Dirac, 1961) or *chordal graphs* (Gavril, 1972), a class of graphs apparently first investigated by Hajnal and Suranyi (1958). In our list of references we cite many other papers which discuss this class of graphs. The majority refer to their role in describing systems of linear equations which can be solved by elimination in an efficient manner, see for example Parter (1961), Rose (1970, 1972), and the associated computation problems (Gavril (1975), Lueker (1975), Ohtsuki (1976), Ohtsuki *et al.* (1976), Rose *et al.* (1975)), whilst others give further graph-theoretic results (Berge (1967), Fulkerson and Gross (1965), Gavril (1972, 1974),

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Lekkerkerker and Boland (1972)). An exposition of much of the work just referred to can be found in Chapter 4 of Golumbic (1980).

We turn now to an outline of the contents of this paper. In Section 2 we organize the main set-theoretic facts concerning decomposability and in the process prove the equivalence between Haberman's definition and that of Vorob'ev and Kellerer. Also included is a brief discussion of algorithms for checking decomposability. Apart from the definition of decomposability, the material in this section is independent of the rest of the paper. Our main work begins in Section 3 where we consider the 2-section of any decomposable hypergraph, showing that it is conformal and thereby reducing the discussion to graph theory. A further simplification allows us to consider only connected graphs. In Section 4 we explore the properties of complete articulation sets, and also give a general graph-theoretic analogue of an index defined by Haberman (1974). After obtaining some properties of this index, we are in a position to draw these ideas together and prove the equivalence of the following properties of a connected graph: (D) the associated clique hypergraph is decomposable; (I) the index satisfies an extremal condition; and (T) the graph is triangulated, that is, no subset of the vertex set generates a cycle Z_n with $n > 3$.

Our set-theoretic notation is fairly rigorously restricted to the following: elements of base sets are denoted by α, β, γ and δ; sets of elements, that is, subsets of base sets by a, b, c, e, f and g; and classes of such sets by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}$ and $\mathcal{F}$. Furthermore the unions of all the sets in the classes, that is, the base sets, are denoted by A, B, C, E and F, the corresponding upper-case roman letter: that is, $A = \bigcup\{a\colon a \in \mathcal{A}\}$. The letter d will be reserved for special use. It will frequently be necessary to sub- or superscript the foregoing symbols with asterisks, primes and so on. Whilst we use the usual notation $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ for unions, intersections and differences of classes, we will abbreviate $a \cap b$ and $A \cap B$ by ab and AB when referring to sets. We write $|A|$ for the cardinality of A and denote the empty set by $\varnothing$. Finally we emphasise that all graphs in this paper are *undirected*, with no loops and multiple edges; more formally we will speak of a graph $\mathbf{G} = (V(\mathbf{G}), E(\mathbf{G}))$ consisting of a set $V(\mathbf{G}) = G$ of vertices, and a set $E(\mathbf{G})$ of unordered pairs of elements of G termed edges. All objects in this paper: sets, graphs, hypergraphs and so on are finite.

2. Decomposable hypergraphs

In this section we give an account of the main set-theoretic properties of decomposable hypergraphs. The results are an integration of those of Haberman (1974), Chapter 4, whose terminology we follow, the set-theoretic parts of Kellerer (1964), and of Vorob'ev (1962).

All of the hypergraphs which we consider in this paper will be a class of pairwise incomparable subsets of a (finite) set, and as such a class is called a generating class by Haberman (1974), we will call such a hypergraph a generating class hypergraph. More formally

DEFINITION 1. A *generating class* (*abbrev. g.c.*) *hypergraph* is a pair $(C, \mathcal{C})$ consisting of a finite set C together with a class $\mathcal{C}$ of pairwise incomparable subsets of C whose union coincides with C.

Where no confusion can result we will denote $(C, \mathcal{C})$ more simply by $\mathcal{C}$ (since $\bigcup \mathcal{C} = C$ this should cause no problems). For two g.c. hypergraphs $(A, \mathcal{A})$ and $(B, \mathcal{B})$ we write $(A, \mathcal{A}) \leqslant (B, \mathcal{B})$ if $A \subseteq B$, and if for every $a \in \mathcal{A}$ there exists $b \in \mathcal{B}$ with $a \subseteq b$. It is easy to see that this relation is a partial order, and if we denote by $\mathcal{H}$ the family of all g.c. hypergraphs, we have the lattice operations $\vee$ and $\wedge$ defined as follows:

$$(A, \mathcal{A}) \vee (B, \mathcal{B}) = (C, \mathcal{C}),$$

where $C = A \cup B$ and $\mathcal{C}$ is the class of maximal elements of $\mathcal{A} \cup \mathcal{B}$. Further

$$(A, \mathcal{A}) \wedge (B, \mathcal{B}) = (E, \mathcal{E}),$$

where $E = A \cap B$ and $\mathcal{E}$ is the class of maximal elements of $\{ab;\, a \in \mathcal{A},\, b \in \mathcal{B}\}$.

We observe that

LEMMA 1. *With the lattice operations defined above, the partially ordered set* $(\mathcal{H}, \leqslant)$ *is a distributive lattice with zero* $(\varnothing, \{\varnothing\})$.

The proof is elementary and omitted.

With this preliminary observation, we return to the basic definition of the paper. It is convenient for a later purpose to formulate it somewhat more generally than in Haberman (1974).

DEFINITION 2. The g.c. hypergraph $\mathcal{C}$ is said to be *decomposed into* $\{\mathcal{C}_i\colon i \in I\}$ *relative to* $d \subseteq C$ if $\mathcal{C} = \vee\{\mathcal{C}_i\colon i \in I\}$, and if for every pair i, j of distinct elements of I we have $\mathcal{C}_i \wedge \mathcal{C}_j = \{d\}$.

COROLLARY 1. *If* $\mathcal{C}$ *is decomposed into* $\{\mathcal{C}_i\colon i \in I\}$ *relative to* d, *then for any ordering* $i_1, i_2, \ldots, i_m$ *of* I $(m = |I|)$, *we have*

$$\mathcal{C} = \Big(\cdots\Big(\big(\mathcal{C}_{i_1} \vee \mathcal{C}_{i_2}\big) \vee \mathcal{C}_{i_3}\Big) \vee \cdots\Big) \vee \mathcal{C}_{i_m},$$

each join being a decomposition relative to d.

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PROOF. This is an immediate consequence of the associativity of the join $\vee$ and of the distributivity (Lemma 1) of $\wedge$ over $\vee$.

Thus we can suppose where convenient, that our decompositions are sequences of decompositions into two pieces. The simplest kind of g.c. hypergraphs are those of the form $(c, \{c\})$, and following Haberman (1974) we give:

DEFINITION 3. A g.c. hypergraph $\mathcal{C}$ is said to be *decomposable* if either $|\mathcal{C}| = 1$, or if there exists a decomposition $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$ of $\mathcal{C}$ relative to some $d \subseteq C$, with $\mathcal{A} \vee \mathcal{B}$ both decomposable and $|\mathcal{A}| < |\mathcal{C}|$, $|\mathcal{B}| < |\mathcal{C}|$.

It is readily seen that this definition implies that the class of decomposable g.c. hypergraph is exactly the smallest class of g.c. hypergraphs that contains the simplest ones, that is, those with $|\mathcal{C}| = 1$, and is closed under joins that are decompositions.

By restricting the base set C of a hypergraph $(C, \mathcal{C})$ to a proper subset $E \subset C$, and taking the maximal elements of $\{cE: c \in \mathcal{C}\}$, we obtain the g.c. *subhypergraph* $\mathcal{C}^E$ of $\mathcal{C}$ generated by E, see Berge (1973), page 390. The family of all decomposable g.c. hypergraphs is closed under this operation, as the next lemma shows.

LEMMA 2. *Let $(C, \mathcal{C})$ be a decomposable g.c. hypergraph. Then for any $E \subset C$ the g.c. subhypergraph $(E, \mathcal{C}^E)$ is also decomposable.*

PROOF. The proof is by induction on $|\mathcal{C}|$. If $|\mathcal{C}| = 1$, the result is certainly true, whilst a decomposable g.c. hypergraph $\mathcal{C}$ with $|\mathcal{C}| > 1$ is (by definition) decomposable as $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$ relative to some $d \subseteq C$, with $\mathcal{A}$ and $\mathcal{B}$ both decomposable and $|\mathcal{A}| < |\mathcal{C}|$, $|\mathcal{B}| < |\mathcal{C}|$. It is easy to see that $\mathcal{C}^E$ is then decomposed into $\mathcal{A}^{AE} \vee \mathcal{B}^{BE}$ relative to dE, and so the inductive step can be proven, and the result follows.

In proving the equivalence of different set-theoretic formulations of decomposability, it is convenient to abstract the following notion, see Vorob'ev (1962).

DEFINITION 4. An edge $c^* \in \mathcal{C}$ is called *extremal in* $\mathcal{C}$ if $\mathcal{C}$ may be decomposed into $\{c^*\} \vee (\mathcal{C} \setminus \{c^*\})$ relative to $d^* = c^* \cap \bigcup \mathcal{C} \setminus \{c^*\}$; equivalently if there exists $c^{**} \in \mathcal{C} \setminus \{c^*\}$ such that $cc^* \subseteq c^{**}c^*$ for every $c \in \mathcal{C} \setminus \{c^*\}$.

COROLLARY 2. *If c^* is an extremal edge of the decomposable g.c. hypergraph $\mathcal{C}$, then $\mathcal{C} \setminus \{c^*\}$ is again decomposable.*

PROOF. We will see that $\mathcal{C}\backslash\{c^*\}$ is just the restriction $\mathcal{C}^E$ of $\mathcal{C}$ to $E = C\backslash(c^*\backslash d^*) = (C\backslash c^*) \cup d^*$, and the result will follow from Lemma 2. But this is clear, since none of the edges of $\mathcal{C}\backslash\{c^*\}$ intersect $c^*\backslash d^*$ in other than the empty set, and so they all remain pairwise incomparable, whilst $d^* \subset c^{**}$.

LEMMA 3. *Let $\mathcal{C}$ be a decomposable g.c. hypergraph with $|\mathcal{C}| \geqslant 2$. Then there exist at least two extremal edges of $\mathcal{C}$.*

REMARK. With a different (but equivalent) form of decomposability, Vorob′ev (1962) proved this result as a lemma in Section 1.51.

PROOF. Again the proof is by induction on $|\mathcal{C}|$. All hypergraphs with two incomparable edges are decomposable, and in this case both edges are trivially extremal.

Let $\mathcal{C}$ be a decomposable g.c. hypergraph with $|\mathcal{C}| > 2$ edges, and suppose the assertion of the lemma is true for all decomposable g.c. hypergraphs with fewer edges. By definition $\mathcal{C}$ may be decomposed into $\mathcal{A} \vee \mathcal{B}$ relative to some $d \subseteq C$, with $\mathcal{A}$ and $\mathcal{B}$ both decomposable and having fewer edges than $\mathcal{C}$. At least one of them must have two or more edges, say $\mathcal{A}$. Then if we write $d = a^*b^*$, the inductive hypothesis implies that $\mathcal{A}$ contains an extremal edge, a' say, distinct from a^*, and we will see that a' is extremal in $\mathcal{C}$. For if $b \in \mathcal{B}$, then

$$a'b = a'(a'b) \subseteq a'd = a'a^*b^* \subseteq a'a^* \subseteq a'a'',$$

where $a'' \in \mathcal{A}\backslash\{a'\}$ is such that $a'a \subseteq a'a''$ for all $a \in \mathcal{A}\backslash\{a'\}$ (see Definition 4). Since the same result is true with b in the above line of inclusions replaced by any $a \in \mathcal{A}\backslash\{a'\}$, we have proved that a' is extremal in $\mathcal{C}$. If $|\mathcal{B}| \geqslant 2$, a similar argument proves the existence of an element $b' \in \mathcal{B}$ distinct from b^* which is extremal in $\mathcal{C}$, whilst if $|\mathcal{B}| = 1$ the edge b^* is itself extremal in $\mathcal{C}$. In either case we have found at least two extremal edges of $\mathcal{C}$ and the inductive step is proved.

We now have the preliminary results necessary for our first theorem. Part of this theorem is an algorithm which we formulate separately as follows. (i) For a g.c. hypergraph $\mathcal{C}$ we choose and fix an edge $\bar{c} \in \mathcal{C}$. (ii) If $|\mathcal{C}| = n$ we let c_n be any extremal edge of $\mathcal{C}$ other that $\bar{c}$, if such exists; otherwise we put $c_n = \bar{c}$. (iii) If $c_n, \ldots, c_{m+1}$ have been determined, $1 < m < n$, we let c_m be any extremal edge of $\mathcal{C}\backslash\{c_n, \ldots, c_{m+1}\}$ if such exists, otherwise we put $c_m = \bar{c}$. This defines a sequence of edges of $\mathcal{C}$.

THEOREM 1. *The following are equivalent for a g.c. hypergraph $\mathcal{C}$ with n edges.*
(a) *$\mathcal{C}$ is decomposable.*
(b) *The algorithm described above has $c_m \neq \bar{c}$, $1 < m \leqslant n$, $c_1 = \bar{c}$.*

(c) *There exists an ordering of* $\mathcal{C}$ *as* $\{c_1, c_2, \ldots, c_n\}$ *such that for all* $m = 1, 2, \ldots, n$ *there exists* $m^* < m$ *such that for all* $l < m$, $c_l c_m \subseteq c_{m^*} c_m$.

REMARK. The equivalence between (a) and (b) above was essentially proved by Haberman (1974), and links his approach with that of Vorob'ev (1962), whilst (c) is the form preferred by Kellerer (1964), see Satz 3.5.

PROOF. (a) implies (b). This implication follows by successively applying Lemma 3 and Corollary 2, each time choosing an extremal element other then $\bar{c}$, until $m = 1$.

(b) implies (c). If c is not chosen until $m = 1$, we know that for all m, $1 < m \leqslant n$, c_m is extremal in $\mathcal{C} \setminus \{c_n, \ldots, c_{m+1}\} = \{c_1, c_2, \ldots, c_m\}$. By definition this means that $\{c_m\} \wedge \{c_1, \ldots, c_{m-1}\} = \{d_m\}$, that is, that $c_l c_m \subseteq d_m$ for all $l < m$, and also that $d_m = c_{m^*} c_m$ for some $m^* < m$.

(c) implies (a). It is always true that $\{c_1, c_2\}$ is decomposable. Suppose we have proved that for some m between 2 and n in the ordering given by (c), $\{c_1, \ldots, c_{m-1}\}$ is decomposable. Then there is a decomposition $\{c_1, \ldots, c_m\} = \{c_m\} \vee \{c_1, \ldots, c_{m-1}\}$ relative to $c_{m^*} c_m$ into decomposable hypergraphs, and so $\{c_1, \ldots, c_m\}$ is decomposable. Continuing until $m = n$ we prove that $\mathcal{C}$ is decomposable.

We close this section with some remarks concerning algorithms to check decomposability. The procedure given prior to Theorem 1 certainly gives an algorithm which works, but this one is not particularly convenient in practice as it requires searching for an extremal edge, a task which involves repeatedly computing and comparing many edge intersections. (A hypergraph would normally be stored in a computer as an incidence matrix with rows corresponding to edges and columns corresponding to the elements of the base set.)

An alternative algorithm was originally introduced by Goodman in the context of contingency tables, see Bishop *et al.* (1975), Goodman (1971), and Jensen (1978), pages 49–50. To motivate this algorithm we note that for any extremal edge c^* of a g.c. hypergraph $\mathcal{C}$, the elements of $c^* \setminus d^*$, where $d^* = c^* \cap \bigcup(\mathcal{C} \setminus \{c^*\})$ $[= c^* c^{**}$ for some $c^{**} \in \mathcal{C} \setminus \{c^*\}]$ belong to precisely one edge of $\mathcal{C}$, namely the extremal edge c^*. The converse to this observation: "an edge containing elements belonging to no other edge is extremal" is false in general, but is is near enough to true for a simple algorithm checking decomposability to exist. An example which rules out the possible converse is $\mathcal{C} = \{\{1, 2\}, \{2, 3, 4\}, \{4, 5\}\}$, in which 3 belongs only to the non-extremal edge $\{2, 3, 4\}$. However, such elements are always associated with a decomposition which may, in turn be associated with a restriction (see the proof of Corollary 1). We formulate the idea as follows:

PROPOSITION 1. *Let h be a subset of an edge c^* of a g.c. hypergraph $\mathcal{C}$ consisting of elements belonging to precisely one edge of $\mathcal{C}$. Put $d = c^* \backslash h$, $\bar{\mathcal{C}} = \{d\} \vee (\mathcal{C} \backslash \{c^*\})$ and $\bar{C} = \bigcup \bar{\mathcal{C}}$. Then $\mathcal{C}$ may be decomposed into $\{c^*\} \vee \bar{\mathcal{C}}$ relative to d, and $\mathcal{C}^{\bar{C}} = \bar{\mathcal{C}}$.*

PROOF. It is easy to see that $\{c^*\} \vee \bar{\mathcal{C}} = \mathcal{C}$, and if $c \in \mathcal{C} \backslash \{c^*\}$, $cc^* \subseteq d$, whilst $c^*d = d$. As in the proof of Corollary 1, distinct elements of $\mathcal{C} \backslash \{c^*\}$ remain incomparable when restricted to $\bar{c}$, because they do not intersect $h = c \backslash \bar{c}$.

COROLLARY 3. *If $\mathcal{C}$ is decomposable, then so also is $\bar{\mathcal{C}}$.*

Thus we may check $\mathcal{C}$ for decomposability by searching for one [or more] element[s] belonging to exactly one edge of $\mathcal{C}$ and suppressing that element [those elements], in the sense that we form $\bar{\mathcal{C}}$ as above. We then repeat the procedure. If $\mathcal{C}$ is decomposable, this will continue until no elements are left, whilst it cannot do so if $\mathcal{C}$ is not decomposable.

This concludes our general set-theoretic discussion of decomposability.

3. Conformal hypergraphs

The aim of this section is to reduce the study of decomposable g.c. hypergraphs to the study of certain connected graphs. We do this by discussing the graph known as the 2-*section* of a g.c. hypergraph $(C, \mathcal{C})$, here denoted by $\mathbf{C}_{\mathcal{C}}$, which, following Berge (1973), page 396, is defined to be the graph which has vertex set C, and as edges the set of all unordered pairs $\{\alpha, \beta\}$ for which there exists an element $c \in \mathcal{C}$ with $\{\alpha, \beta\} \subseteq c$.

For a general subclass $\mathcal{A} \leq \mathcal{C}$ we need to consider the 2-section $\mathbf{A}_{\mathcal{A}}$ of the hypergraph $(A, \mathcal{A})$, and ask about its relation to the subgraph of $\mathbf{C}_{\mathcal{C}}$ generated by, Berge (1973), page 7, equivalently, induced by, Harary (1969), page 11, $A \subseteq C$, here denoted by $\langle A \rangle$. In general $\langle A \rangle$ need not coincide with $\mathbf{A}_{\mathcal{A}}$, but there is an important special case in which it does so.

LEMMA 4. *If we have a decomposition $\mathcal{C} = \dot{\bigvee}_{i \in I} \mathcal{C}_i$ relative to $d \subseteq C$, then the subgraphs $\langle C_i \rangle$ of the 2-section $\mathbf{C}_{\mathcal{C}}$ generated by the subsets $C_i = \bigcup \mathcal{C}_i$ coincide with the 2-sections $\mathbf{C}_{i_{\mathcal{C}_i}}$.*

PROOF. By Corollary 1 we only have to prove the result for pairwise decompositions $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$. It is clear that the vertices of $\langle A \rangle$ and $\mathbf{A}_{\mathcal{A}}$ coincide, and it is equally clear that if α and α' are adjacent in $\mathbf{A}_{\mathcal{A}}$, that is, if $\{\alpha, \alpha'\} \subseteq a$ for some $a \in \mathcal{A}$, then α and α' are adjacent in $\mathbf{C}_{\mathcal{C}}$ and hence in $\langle A \rangle$.

On the other hand, if $\{\alpha, \alpha'\}$ is an edge in $\langle A \rangle$, then $\{\alpha, \alpha'\} \subseteq c$ for some $c \in \mathcal{C}$. If $c \in \mathcal{A}$, then we have shown that $\{\alpha, \alpha'\}$ is an edge of $\mathbf{A}_{\mathcal{A}}$, whilst $c \in \mathcal{B}$, then $\{\alpha, \alpha'\} = AB \subseteq d$, and so $\{\alpha, \alpha'\}$ is still an edge of $\mathbf{A}_{\mathcal{A}}$.

With this lemma proved we can turn to the main result of this section. Recall that a *clique* in a simple graph is a maximal complete subgraph Harary (1969), page 20, although some writers including Berge (1973) do not require maximality, and hence speak of maximal cliques. Further, a g.c. hypergraph $\mathcal{C}$ is called *conformal*, Berge (1973), if the class of all cliques of the 2-section $\mathbf{C}_{\mathcal{C}}$ of $\mathcal{C}$ coincides with $\mathcal{C}$.

PROPOSITION 2. *Let the g.c. hypergraph $\mathcal{C}$ be decomposed into $\{\mathcal{C}_i : i \in I\}$ relative to $d \subseteq C$. Then $\mathcal{C}$ is conformal if and only if for all $i \in I$, $\mathcal{C}_i$ is conformal.*

PROOF. As before it is enough to consider pairwise decompositions $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are conformal and let c be a clique in $\mathbf{C}_{\mathcal{C}}$. We first note that we must have $c \subseteq A$ or $c \subseteq B$ for if this was not the case and $\alpha \in c \setminus B$, $\beta \in c \setminus A$ there must be a $c' \in \mathcal{C}$ with $\{\alpha, \beta\} \subseteq c'$. But $c' \in \mathcal{C}$ implies $c' \in \mathcal{A} \cup \mathcal{B}$ and $c' \in \mathcal{A}$ implies $\beta \in A$, $c' \in \mathcal{B}$ implies $\alpha \in B$, in both cases a contradiction. But if $c \subseteq A$, c is a clique in A by Lemma 4 and thus $c \in \mathcal{A}$ by assumption. The maximality of c implies $c \in \mathcal{C}$. Similarly, if $c \subseteq B$ we get $c \in \mathcal{C}$, which was to be proved.

Conversely suppose that $\mathcal{C}$ is conformal and let a be a clique in $\mathbf{A}_{\mathcal{A}} = \langle A \rangle$. We just have to show that there is an $a' \in \mathcal{A}$ such that $a \subseteq a'$ (the maximality of a will then imply $a = a'$). By Lemma 4, a is a complete subset of $\mathbf{C}$ and by the conformality of $\mathcal{C}$, there is a $c \in \mathcal{C}$ such that $a \subseteq c$. If $c \in \mathcal{B}$, $a = ac \subseteq AB = d$ and there is thus an $a' \in \mathcal{A}$ such that $a' \supseteq a$. If $c \in \mathcal{A}$, we can use c as a'.

The following result is essentially part (4°) of Theorem 2.2 of Vorob′ev and is Theorem 5 of Anderson (1974).

COROLLARY 4. *Every decomposable g.c. hypergraph is conformal.*

PROOF. Since the 2-section of a hypergraph $\mathcal{C}$ with $|\mathcal{C}| = 1$ is a complete graph, any such hypergraph is conformal. The corollary then follows directly from the definition and Proposition 2.

As a consequence of this proposition we need only discuss those decomposable hypergraphs $\mathcal{C}$ which consist of the class of all cliques of a graph $\mathbf{C}$. We will write $\mathcal{C}_{\mathbf{C}}$ for the hypergraph of all maximal cliques of the graph $\mathbf{C}$. The following

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discussion shows how we can, without loss of generality, restrict ourselves even further to consider only connected graphs.

For any pair a, b of edges of a hypergraph $(C, \mathcal{C})$ with 2-section $\mathbf{C}_{\mathcal{C}}$ write $a \equiv c$ if there exists a sequence $a = c_1, c_2, \ldots, c_m = b$ of edges such that $c_{k-1}c_k \neq \varnothing$, $1 < k \leqslant m$. This is easily seen to be an equivalence relation on $\mathcal{C}$ and we denote by $\{\mathcal{C}_t: t \in T\}$ the equivalence classes of $\mathcal{C}$ under $\equiv$. Put $C_t = \cup \mathcal{C}_t$ and let $\mathbf{C}_t$ denote the 2-section of the hypergraph $(C_t, \mathcal{C}_t)$, $t \in T$. In these terms we have:

Lemma 5. *The connected components of* $\mathbf{C}_{\mathcal{C}}$ *are precisely the graphs* $\{\mathbf{C}_t: t \in T\}$.

Proof. We begin by noting that each graph $\mathbf{C}_t$ is connected. If $\alpha, \beta \in C_t$ with $\alpha \in a$ and $\beta \in b$ say, then there should exist a chain $a = c_1, c_2, \ldots, c_m = b$ with $c_{k-1}c_k \neq \varnothing$, $1 < k \leqslant m$. Choosing $\lambda_k \in c_{k-1}c_k$, $1 < k \leqslant m$ we see that $\alpha = \lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1} = \beta$ is a chain in $\mathbf{C}_t$, thus proving that $\mathbf{C}_t$ is connected.

If α and β are connected in $\mathbf{C}_{\mathcal{C}}$ there exists a chain $\alpha = \lambda_1, \lambda_2, \ldots, \lambda_n = \beta$ such that $\{\lambda_{k-1}, \lambda_k\} \subseteq c_k \in \mathcal{C}$, $1 < k \leqslant n$. But this means that $c_1 \equiv c_n$ and so there exists $t \in T$ with $\{\alpha, \beta\} \subseteq C_t$. Thus the $\mathbf{C}_t$ are connected components of $\mathbf{C}_{\mathcal{C}}$ and since $\cup_{t \in T} C_t = C$, $\cup_{t \in T} \mathcal{C}_t = \mathcal{C}$ (union of classes), we have described all of the connected components and the proof is complete.

Our next lemma shows that the non-trivial decompositions of clique hypergraphs are associated with *complete articulation* sets, where a subset $d \subseteq C$ of a connected graph $\mathbf{C} = (C, E(\mathbf{C}))$ is called an articulation set if $\langle C \setminus d \rangle$ is disconnected, Berge (1973), page 8, and complete means here that all vertices in d are adjacent.

Lemma 6. *Let* $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$ *be a decomposition of the clique hypergraph* $\mathcal{C}$ *of the connected graph* $\mathbf{C}$ *relative to* $d \subseteq C$. *Then every path from* $A \setminus B$ *to* B *must contain an element of* d.

Proof. Now suppose that $\alpha \in A \setminus B$ and $\beta \in B$. If $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_m = \beta$ is a connecting path, then for each i, $1 \leqslant i \leqslant m$, there exists $c_i \in \mathcal{C}$ with $\{\gamma_{i-1}, \gamma_i\} \subseteq c_i$. Since $\alpha \in A \setminus B$ we must have $c_1 \in \mathcal{A}$, and so $k = \min(i: c_i \in \mathcal{B})$ must satisfy $1 < k \leqslant m$. But then $c_{k-1} \in \mathcal{A}$ and so $\gamma_{k-1} \in c_{k-1}c_k \subseteq d$.

Corollary 5. *Any cycle in* C *which intersects both* $B \setminus A$ *and* $A \setminus B$ *must contain two non-consecutive elements of* d.

Proof. Let the cycle contain $\alpha \in A \setminus B$ and $\beta \in B \setminus A$. By arguing as above in (say) the clockwise direction, we get an element $\delta_1 \in d$, and by arguing in the counter clockwise direction we find an element $\delta_2 \in d$. These elements cannot be consecutive in the cycle for α and β separate them.

COROLLARY 6. *The graphs* $\mathbf{A}_{\mathcal{A}} = \langle A \rangle$ *and* $\mathbf{B}_{\mathcal{B}} = \langle B \rangle$ *are connected subgraphs of* $\mathbf{C}$ *with clique hypergraphs* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively.*

PROOF. Let α, α' be distinct elements of A. Since $\mathbf{C}$ is connected there is a path in $\mathbf{C}$ between α and α', and we will see that any such path of shortest length must lie entirely within A. For if this was not the case, it would have to meet $B \setminus A$ and so pass through d twice. But then the two elements of d could be joined (within A) thereby shortening the path. Thus A is a connected subset. The remaining assertions are consequences of the foregoing Lemma 4 and Proposition 2.

We close this section with some remarks on the relation between our notion of decomposition applied to the clique hypergraph of a connected graph, and to the separation into pieces of such a graph relative to a complete articulation set (Berge (1973), page 329). Let d be a complete articulation set of a connected graph $\mathbf{C}$, that is, d is complete and $\langle C \setminus d \rangle$ is disconnected, and suppose that $\langle C \setminus d \rangle$ has connected components $\{E_i: i \in I\}$. Then the *pieces* of $\mathbf{C}$ relative to d are the subgraphs $\{\mathbf{C}_i: i \in I\}$ where $\mathbf{C}_i = \langle E_i \cup d \rangle$, $i \in I$. Finally, let $\mathcal{C}$ and $\{\mathcal{C}_i: i \in I\}$ be the clique hypergraphs corresponding to $\mathbf{C}$ and $\{\mathbf{C}_i: i \in I\}$ respectively.

PROPOSITION 3. $\mathcal{C} = \vee\{\mathcal{C}_i: i \in I\}$ *is a decomposition relative to* d.

PROOF. This is a straightforward checking of definitions and so is omitted.

4. The index

We have seen that any decomposable hypergraph C gives rise to a graph $\mathbf{C}_{\mathcal{C}}$, its 2-section, whose class of cliques is $\mathcal{C}$. Further, we have seen how we may restrict ourselves to those hypergraphs which derive in this way from connected graphs. Thus we may begin afresh by supposing given a connected graph $C = (C, E(\mathbf{C}))$ and denoting its class of cliques by $\mathcal{C} = \mathcal{C}_{\mathbf{C}}$. We also use the notation $\mathbf{C} \setminus s$ for the subgraph $\langle C \setminus s \rangle$ generated by $C \setminus s$ where $s \subseteq C$.

The main purpose of this section is to define an index associated with the complete subsets of C and derive some of its basic properties. Such an index was defined in quite a different way by Haberman (1974), page 174, where it was called the *adjusted replication number*.

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DEFINITION 5. For any complete subset d, let $\beta_0'(\mathbf{C}\backslash d)$ denote the number of pieces of $\mathbf{C}$ relative to d in which d is not a clique. Let

$$\nu(d) = 1 - \beta_0'(\mathbf{C}\backslash d).$$

The notation β_0' is intended to suggest a modification of the number of connected components β_0 (= 0th Betti number) of a graph (= 1-complex), see Harary (1969).

LEMMA 7. *The index ν has the following properties*:

$$\begin{aligned} \nu(d) &= 1 && \textit{if } d \textit{ is a clique},\\ \nu(d) &= 0 && \textit{if } d \textit{ is not an articulation set and not a clique},\\ \nu(d) &< 0 && \textit{implies that } d \textit{ is an articulation set}. \end{aligned}$$

PROOF. If d is a clique, it will be a clique in all the pieces of C relative to d and

$$\nu(d) = 1 - \beta_0'(\mathbf{C}\backslash d) = 1 - 0 = 1.$$

If d is not a clique, nor an articulation set then $\beta_0'(\mathbf{C}\backslash d) = 1$ and thus $\nu(d) = 0$.

If $\nu(d) < 0$, $\beta_0'(\mathbf{C}\backslash d) \geqslant 2$, and d must be an articulation set.

Our major result in this section relates our index across decompositions. More precisely, let the clique hypergraph $\mathcal{C}$ of a connected graph be decomposed into $\mathcal{C}_i$, $i \in I$, relative to $d^* \subseteq C$ and let ν_i denote the indices associated with $\langle C_i \rangle$. Let also $\nu_i(d) = 0$ if $d \not\subseteq C_i$.

LEMMA 8. *For any complete subset $d \subseteq C$ we have, with the notation above*:

$$\nu(d) = \begin{cases} \sum_{i \in I} \nu_i(d) & \textit{if } d \neq d^*,\\ \sum_{i \in I} \nu_i(d^*) - |I| + 1 & \textit{if } d = d^*. \end{cases}$$

PROOF. We readily see that we can restrict ourselves to the case with $|I| = 2$, that is, $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$, $\mathcal{A} \wedge \mathcal{B} = \{d^*\}$. We then consider four cases: i) $d = d^*$, ii) $d \subsetneq d^*$, iii) $d \supset d^*$, iv) $d \not\subset d^*$ and $d \not\supset d^*$.

i) $d = d^*$. If d^* is removed from A we get pieces $\mathbf{A}_1, \ldots, \mathbf{A}_k$, $\mathbf{A}_{k+1}, \ldots, \mathbf{A}_{k+m}$ with $\mathbf{A}_1, \ldots, \mathbf{A}_k$ containing d^* as a clique and $\mathbf{A}_{k+1}, \ldots, \mathbf{A}_{k+m}$ not as a clique. Similarly we get pieces $\mathbf{B}_1, \ldots, \mathbf{B}_p, \mathbf{B}_{p+1}, \ldots, \mathbf{B}_{p+q}$.

But the pieces of $\mathbf{C}$ obtained by removing d must be the same since B_i is always separated from A_j by d^* according to Lemma 6.

Thus we have

$$\nu(d^*) = 1 - k - p, \quad \nu_A(d^*) = 1 - k, \quad \nu_B(d^*) = 1 - p,$$

whereby we see that our formula holds.

ii) $d \subsetneq d^*$. Let the pieces of **A** and **B** relative to d be $\mathbf{A}_1, \ldots, \mathbf{A}_k, \mathbf{A}_{k+1}, \ldots, \mathbf{A}_{k+m}, \mathbf{B}_1, \ldots, \mathbf{B}_p, \mathbf{B}_{p+1}, \ldots, \mathbf{B}_{p+q}$, as before.

Exactly one A-piece and one B-piece contains points of $d^* \setminus d$. Because if there were more, these pieces would be connected via $d^* \setminus d$ when d was removed, thus contradicting the notion of a piece; d^* is not a clique in such a piece since $d \subsetneq d^*$, with d^* complete. So, let those pieces be $\mathbf{A}_{k+1}$ and $\mathbf{B}_{p+1}$. The pieces of **C** relative to d are then

$$\mathbf{A}_{k+1} \cup \mathbf{B}_{p+1}, \mathbf{A}_1, \ldots, \mathbf{A}_k, \mathbf{A}_{k+2}, \ldots, \mathbf{A}_{k+m}, \mathbf{B}_1, \ldots, \mathbf{B}_p, \mathbf{B}_{p+2}, \ldots, \mathbf{B}_{p+q}.$$

Since no two of these can be connected via $d^* \setminus d$ they are therefore only connected via d, again by Lemma 6. Thus

$$\nu_A(d) = 1 - m, \qquad \nu_B(d) = 1 - q,$$
$$\nu(d) = 1 - [(m-1) + (q-1) + 1] = \nu_A(d) + \nu_B(d).$$

iii) $d \supset d^*$. Then we must either have $d \subseteq A$ or $d \subseteq B$. Suppose $d \subseteq A$. Let $\mathbf{A}_1, \ldots, \mathbf{A}_k, \mathbf{A}_{k+1}, \ldots, \mathbf{A}_{k+m}$ be the pieces of **A** relative to d.

Let $\mathbf{B}_1^*, \ldots, \mathbf{B}_p^*, \mathbf{B}_{p+1}^*, \ldots, \mathbf{B}_{p+q}^*$ be the pieces of **B** relative to d^*. Then the pieces of **C** relative to d must be

$$\mathbf{A}_1, \ldots, \mathbf{A}_{k+m}, (\mathbf{B}_1^* \cup d), \ldots, (\mathbf{B}_{p+q}^* \cup d),$$

since $d \cap B = d^*$. But d must be a clique of all the B-pieces, because no vertices in $B \setminus d^*$ are adjacent to those in $d \setminus d^*$ by Lemma 6. Thus $\nu(d) = 1 - m = \nu_A(d)$ and $d \not\subset B$ implies $\nu_B(d) = 0$, that is, that the formula is correct.

iv) $d \not\supset d^*$ and $d^* \not\supset d$. Again, let us assume $d \subset A$, that is, $d \not\subset B$. Let $\mathbf{A}_0$ be the A-piece relative to d containing $d^* \setminus d \neq \varnothing$. Then the pieces of **C** relative to d are

$$\mathbf{A}_0 \cup \mathbf{B}, \mathbf{A}_1, \ldots, \mathbf{A}_k,$$

where $\mathbf{A}_0, \ldots, \mathbf{A}_k$ are the A-pieces relative to d. Note that d is a clique in $\mathbf{A}_0 \cup \mathbf{B}$ if and only if it is in $\mathbf{A}_0$, since no vertices in **B** are adjacent to vertices in $d \setminus d^* \neq \varnothing$. Thus $\nu(d) = \nu_A(d)$ and since $\nu_B(d) = 0$, the proof is complete.

COROLLARY 7. *For any connected graph C with the class of cliques* $\mathcal{C}$,

$$\sum_{\substack{\text{all complete} \\ \text{subsets } d}} \nu(d) \geqslant 1.$$

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PROOF. By induction on $|\mathcal{C}|$. If $|\mathcal{C}|= 1$ the result is clearly true. Suppose that C is a connected graph with $|\mathcal{C}|> 1$ and that the assertion is true for all connected graphs with fewer than $|\mathcal{C}|$ cliques. Then either $\nu(d) \geqslant 0$ for all d, in which case the result is true because $\nu(c) = 1$ for all $c \in \mathcal{C}$ by Lemma 7, or there is a d^* with $\nu(d^*) < 0$. But then d^* is an articulation set by Lemma 7 and $\mathcal{C}$ can be decomposed into $\mathcal{C}_i$, $i \in I$, relative to d^*, where $\mathcal{C}_i$ are the clique hypergraphs of the pieces (Proposition 3).

Clearly, $|\mathbf{C}_i|<|\mathbf{C}|$ so the inductive hypothesis and the preceding lemma gives us

$$\begin{aligned}\sum \nu(d) &= \sum_{d \neq d^*} \Big(\sum_{i \in I} \nu_i(d) \Big) + \sum_{i \in I} \nu_i(d^*) - |I| + 1 \\ &= \sum_{i \in I} \Big(\sum_d \nu_i(d) \Big) - |I| + 1 \geqslant |I| - |I| + 1 = 1.\end{aligned}$$

Decomposable graphs

In this section we draw together the notions introduced in the previous two and show that graphs $\mathbf{C}$ whose clique hypergraphs $\mathcal{C}_{\mathbf{C}}$ are decomposable have other interesting properties. We use the notation Z_n for the graph known as the n-cycle Harary (1969), page 13.

THEOREM 2. *The following properties of a connected graph* $\mathbf{C}$ *are equivalent*:
(D) *The clique hypergraph* $\mathcal{C}_C$ *is decomposable.*
(I) $\Sigma_d \, \nu(d) = 1$.
(T) *No subset* $s \subseteq C$ *generates a cyclic subgraph* $\langle s \rangle \simeq Z_n$ *with* $n > 3$.

REMARKS. Vorob'ev (1962) derived condition (T) in his discussion of this topic [Theorem 2,2], see also Kellerer (1964), Satz 3.2, and we note that such graphs are called *triangulated* by Berge (1973). An easy reformulation of (T) is (T′): every polygon in $\mathbf{C}$ of length $k \geqslant 4$ has a chord. Graphs with these properties were apparently first studied by Hajnal and Suranyi (1958).

PROOF. (D) implies (I). This is an easy induction on $|\mathcal{C}|$ using the (Index) Lemma 8. The conclusion is clearly true for $|\mathcal{C}|= 1$, and so we take a decomposable clique hypergraph $\mathcal{C} = \mathcal{C}_C$ with $|\mathcal{C}|> 1$, supposing that the conclusion is true for all decomposable clique hypergraphs with fewer than $|\mathcal{C}|$ edges. Then there must exist a decomposition $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$ relative to a subset $d^* \subseteq C$, of $\mathcal{C}$ into decomposable hypergraphs. By Lemma 5 and Corollary 6, $\mathcal{A}$ and $\mathcal{B}$ are both the clique hypergraphs of connected graphs with fewer elements than $\mathcal{C}$. If (I) is true for $\mathcal{A}$ and for $\mathcal{B}$, as it must be by the inductive hypothesis, it remain true for $\mathcal{C}$ by using Lemma 8, since

$$\sum_d \nu(d) = \sum_{d \neq d^*} (\nu_A(d) + \nu_B(d)) + \nu_A(d^*) + \nu_B(d^*) - 1$$
$$= 1 + 1 - 1 = 1.$$

(I) implies (T). Again the proof is by induction on $|\mathcal{C}|$. If $|\mathcal{C}| = 1$, the corresponding graph is complete and (T) always holds. Suppose now $|\mathcal{C}| > 1$ and that the assertion is true for all connected graphs with fewer than $|\mathcal{C}|$ cliques. If (I) holds for **C** and $|\mathcal{C}| > 1$ there must be a d^* with $\nu(d^*) < 0$ since $\nu(c) = 1$ for all $c \in \mathcal{C}$ by Lemma 7. As in Corollary 7 we deduce that there is a decomposition of $\mathcal{C}$ into $\mathcal{C}_i$, $i \in I$, relative to d^* and with $|\mathcal{C}_i| < |\mathcal{C}|$. Using the inductive hypothesis, Lemma 8 and Corollary 7, we deduce that $\mathbf{C}_i$ satisfy (T). That **C** satisfies (T) now follows from Corollaries 1 and 6.

(T) implies (D). The final implication is also proved by induction on $|\mathcal{C}|$. As before it is easy to see that the conclusion desired is true when $|\mathcal{C}| = 1$ and so we make the now familiar inductive hypothesis. Then with $|\mathcal{C}| > 1$ there is either (i) a decomposition $\mathcal{C} = \mathcal{A} \vee \mathcal{B}$ of $\mathcal{C}$ relative to some $d \subseteq C$, or (ii) there is no such decomposition. Since property (T) is preserved upon passing to generated subgraphs, we note that in case (i) **A** and **B** must satisfy (T). But then the inductive hypothesis implies that $\mathcal{A}$ and $\mathcal{B}$ are both decomposable, and so we conclude that $\mathcal{C}$ is decomposable.

Our proof will be complete when we show that case (ii) cannot arise. To prove this, let $\mathcal{D}$ be the set of intersections of *distinct* cliques and let d be an element in $\mathcal{D}$ which is maximal in $\mathcal{D}$ under set inclusion. We shall show that d is an articulation set and hence by Proposition 2 defines a decomposition. Suppose $\mathbf{C} \setminus d$ is connected. Since $d \in \mathcal{D}$ there are $a, b \in \mathcal{C}$ such that $ab = d$, $a \setminus d \neq \varnothing$, $b \setminus d \neq \varnothing$ and $a \setminus d$ is connected to $b \setminus d$ outside d. Amongst the pairs $\alpha \in a \setminus d$ and $\beta \in b \setminus d$, select a pair, α^*, β^*, say, for which the shortest connecting path is of shortest length. Then

$$\alpha^* = \gamma_0, \gamma_1, \ldots, \gamma_m = \beta^*$$

is of length $m \geqslant 2$, and because it is a shortest path, $\gamma_i \notin a$ for $i > 0$.

Let us note that γ_2 cannot be adjacent in **C** to every $\delta \in d = ab$. For if this was the case, $\{\gamma_1, d^*\} \cup d$ would be a complete subset of **C** and so contained in a clique $c \in \mathcal{C} \setminus \{a, b\}$ whence we would have $\mathcal{D} \ni ac \supseteq d \cup \{\alpha^*\}$, contradicting the maximality of d. Thus there exists $\delta^* \in d = ab$ with $\{\gamma_1, \delta^*\} \notin E(\mathbf{C})$.

Now let $k = \min\{j\colon\ \{\gamma_j, \delta^*\} \in E(\mathbf{C})\}$. By the foregoing, $j \geqslant 2$ and since $\beta^* = \gamma_m$ is adjacent to δ^*, $j \leqslant m$. Then $\alpha^* = \gamma_0, \gamma_1, \ldots, \gamma_{j-1}, \gamma_j, \delta^*, \alpha^*$ is a cycle of length $j + 2 \geqslant 4$ in **C**. It has no chords, since the path from α^* to β^* has shortest length. But this contradicts (T) and so $\mathbf{C} \setminus d$ must be connected. The proof is now complete.

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