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DECOMPOSABLE INCOME INEQUALITY MEASURES

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A decomposable inequality measure is defined as a measure such that the total inequality of a population can be broken down into a weighted average of the inequality existing within subgroups of the population and the inequality existing between them. Thus, decomposable measures differ only by the weights given to the inequality within the subgroups of the population. It is proven that the only zero-homogeneous "income-weighted" decomposable measure is Theil's coefficient (T) and that the only zero-homogeneous "population-weighted" decomposable measure is the logarithm of the arithmetic mean over the geometric mean (L). More generally, it is proved that T and L are the only decomposable inequality measures such that the weight of the "within-components" in the total inequality of a partitioned population sum to a constant. More general decomposable measures are also analyzed.

1. INTRODUCTION

MANY EMPIRICAL ANALYSES of income inequality rely heavily on measures of inequality which are decomposable in the sense that, if the population of income-earners is broken down into a certain number of subgroups, the inequality measure for the total population can be expressed as a sum of the inequality measures "within" its subgroups, weighted by coefficients depending on their aggregate characteristics, and of the inequality existing "between" them. The variance coefficient and Theil's coefficient are such decomposable measures of inequality and they have been extensively used in identifying and explaining the main sources of inequality in a given population.[2] By contrast, we shall see that the Gini coefficient is not decomposable in the sense above and has seldom been used in this type of exercise.[3]

If decomposability is a very convenient property, any decomposable measure is not necessarily a satisfactory index of income inequality. The variance, for instance, is not neutral with respect to a scale change of the whole income distribution, which might not seem a very "objective" property for an inequality measure. Likewise, the least normative property that might be expected from an inequality measure is to decrease with any income transfer from rich to less rich people (Pigou-Dalton condition) and it is well-known that some conventional inequality measures do not satisfy that condition.

It seems interesting, under those circumstances, to investigate all inequality measures which are decomposable while satisfying a set of basic requirements. This is what we intend to do in the present paper. Namely, we look for all measures

[1] I thank an anonymous referee and A. Shorrocks for useful comments. I remain responsible for any remaining error.

[2] For a good example of the use of the decomposability of Theil's coefficient, see [**5**]. The use of the decomposability of the variance of incomes or income logarithms, on the other hand, is implicit in the countless regression analyses of income distribution data.

[3] The numerous attempts to disaggregate the Gini coefficient [**4, 8, 9, 10**] shows the relevance and importance of the decomposability property in applied work on income inequality. A proof of the non-decomposability of the Gini coefficient and other inequality measures based on an income-ranking is given at the end of Section 2.

which: (a) are continuous and differentiable in all individual incomes; (b) are symmetric; (c) are income-homogeneous of degree zero; (d) satisfy the symmetry axiom for population; (e) satisfy the Pigou-Dalton condition; and (f) are decomposable.

The continuity requirement seems natural since an infinitesimal change in the value of an individual income may be expected to produce only an infinitesimal change in the inequality measure. The differentiability condition might seem more restrictive since it leads, in particular, to elimination of the wide family of measures in which individual incomes enter with their rank in the whole distribution and which are not differentiable everywhere (Gini coefficient, inter-quantiles mean incomes ratios, etc.). This is not a serious problem, however, because we shall see that such measures are generally not decomposable.[4]

The symmetry requirement corresponds to the idea that the personality of income earners is irrelevant in the measure of inequality (anonymity rule). This seems justified in the context of income inequality but it might not be if a broader definition of economic inequality were considered. The income-zero-homogeneity property, on the other hand, means that the inequality measure is invariant when all incomes are multiplied by the same scalar. Interestingly enough, we shall see that this income-homogeneity property, together, with the undemanding "symmetry axiom for population" (which requires that the inequality of a given distribution be the same as that of the distribution obtained by replicating any number of times each individual income in the initial distribution), implies also a kind of "population-zero-homogeneity" for decomposable inequality measures. It must be pointed out, however, that, if income-zero-homogeneity seems justified for an "objective" inequality measure, it is debatable from a normative point of view.[5]

Finally, we are led to consider two definitions of the decomposability property. First, the inequality of population of individual incomes can be expressed as a function of the inequality within its subgroups and of their aggregate characteristics. This definition corresponds to some kind of "aggregativity" property and permits some decomposition of the total inequality. As it has been presented above and as it is used in empirical works, however, decomposability requires a little more than this general aggregativity property. Namely, it requires some additivity in the decomposition of inequality and we shall focus on this "additive decomposability" property in this paper.

It will appear that additive decomposability is equivalent to expressing the total inequality of a population as the sum of a weighted average of the inequality within subgroups of the population and of the inequality existing between them, although weighting coefficients do not necessarily sum to one. It is clear, then, that additively decomposable inequality measures will differ only by the weight given to the inequalities within the various subgroups of the population. Naturally, the

[4] It must be stressed, however, that we will often require that inequality measures be differentiable to the order two, and sometimes to a higher order, in a neighborhood of some given income distribution. The only justification for this requirement is that it is satisfied by all usual measures, except at some particular points (see the end of Section 2).

[5] On those points see Kolm [**6**, **7**] and Sen [**11**].

most appealing candidates for this weighting system are the population and income shares of the subgroups. This leads to two distinct definitions of the decomposability property: the "population weighted decomposability" and the "income weighted decomposability."

Based on those definitions our calculations yield the interesting result that there is only one inequality measure consistent with each concept of decomposability. The only measure satisfying the "population weighted decomposability" is somewhat original and has seldom been used though, as we will show, it is quite appealing. This measure is simply the average logarithm of the individual incomes expressed as a proportion of the mean income of the population.[6] The only measure consistent with the "income weighted decomposability" is Theil's entropy coefficient.

The preceding measures are only particular cases of the additive decomposability property. In the last section of the paper we investigate some measures consistent with more general weighting systems in the decomposability definition and also the 'income weighted' and 'population weighted' decomposable measures which are homogeneous of any degree. Quite naturally, the resulting measures refer more explicitly to normative judgements than do the average logarithm or Theil's coefficients. On the other hand, the decomposability property permits us to express those judgements in an interesting way.

2. "AGGREGATIVE" AND "DECOMPOSABLE" INEQUALITY MEASURES

In a population of n income earners being partitioned into m groups, let n_i be the number of individuals in group $i (i = 1, 2, \ldots, m)$ and y_{ij} the income of individual j in group i. Let $I^q(x_1, x_2, \ldots, x_q)$ be the inequality measure for a population of q individuals with incomes $x_1, x_2, \ldots, x_q$. It seems natural to require that the inequality associated with an equalitarian distribution be an arbitrary constant. Without loss of generality, let that constant be zero:

$$I^q(x, x, \ldots x) = 0 \qquad \text{for any } q \text{ and any } x.$$

This implies, in particular:

$$I^1(x) = 0 \qquad \text{for any } x.$$

Let us now define the concept of "aggregativity".

DEFINITION 1 (Aggregativity): We shall say that an income inequality measure is *aggregative* if it can be expressed as follows:

$$\text{(A)} \qquad \begin{aligned} &I^n(y_{11}, y_{12}, \ldots, y_{mn_m}) \\ &= F^m\{I^{n_1}(y_{11}, \ldots, y_{1n_1}), \ldots, I^{n_m}(y_{m1}, \ldots, y_{mn_m}); \\ &Y_1, \ldots, Y_m; n_1, \ldots, n_m\} \end{aligned}$$

with $Y_i = \Sigma_{j=1}^{n_i} y_{ij}$, for all partitions $(m; n_1, \ldots, n_m)$ of the population.

[6] Or the mean logarithmic deviation. A reference to that measure is made in [13] but it is not commonly used in applied work.

In other words, aggregativity is a general property according to which there is no need to know the exact distribution of incomes within the subgroups of a population to compute the inequality measure of that population. Only the inequality measures of the subgroups and their aggregate characteristics (Y_i, n_i) are necessary.

Any aggregative inequality measure satisfies an elementary decomposability property. (A) may be rewritten as follows:

$$
\begin{aligned}
(1) \qquad I^n = {} & [F^m(I^{n_1}, \dots, I^{n_m}; Y_1, \dots, Y_m; n_1, \dots n_m) \\
& - F^m(0, \dots, 0; Y_1, \dots, Y_m; n_1, \dots, n_m)] \\
& + F^m(0, \dots, 0; Y_1, \dots, Y_m; n_1, \dots, n_m).
\end{aligned}
$$

The last term on the right-hand side of (1) is clearly the inequality associated with a population of m equalitarian subgroups. Thus, the first term is, by difference, the contribution of the inequality within all subgroups to the total inequality. Let I_B^m and I_W^m be, respectively, those two contributions:

$$I^n = I_W^m + I_B^m.$$

In order to go further into the decomposition of I^n, we need to know the contribution of the inequality within each group i to the total inequality I^n or, equivalently, to the "within-component", I_W^m, in (1). That contribution may be defined simply by the difference between the total actual inequality in the population and the inequality that would be observed if all individuals in group i had the same income. From (A) we have the following definition.

DEFINITION 2: The contribution $I_{W_i}^m$ of the inequality within group i to the total inequality in the case of an aggregative measure satisfying Definition 1 is defined by

$$
\begin{aligned}
I_{W_i}^m = {} & F^m(I^{n_1}, \dots, I^{n_i}, \dots, I^{n_m}; Y_1, \dots, Y_m; n_1, \dots, n_m) \\
& - F^m(I^{n_1}, \dots, 0, \dots, I^{n_m}; Y_1, \dots, Y_m; n_1, \dots, n_m).
\end{aligned}
$$

That definition of the contribution of the inequality within each group i to the total inequality might seem somewhat ambiguous. Those contributions do not necessarily sum to I_W^m, the total contribution of the 'within-group' inequality defined in (1). Under these conditions, there might be some contradiction between the fact that, separately, each group i contributes $I_{W_i}^m$ to the total inequality and the fact that, jointly, they also contribute some additional inequality since $\Sigma_i I_{W_i}^m \neq I_W^m$. The contradiction is only apparent. The point is simply that the preceding argument relies on the implicit assumption that inequalities within groups are additive whereas they are not necessarily so in general. Empirically, however, there is no doubt that it is that additivity property that is expected in a "decomposable" measure. So, we shall adopt the following definition.

DEFINITION 3 (Additive Decomposability): An *additively decomposable* inequality measure is an aggregative measure such that:

$$\sum_{i=1}^{m} I_{W_i}^{m} = I_{W}^{m}$$

for any partition of the population.

The rest of the present paper will mostly focus upon those additively decomposable measures. For the sake of simplicity, however, we will generally omit the word "additively" and simply refer to those measures as "decomposable." The additivity property given in Definition 3 implies that decomposable measures satisfy an aggregative property much more specific than (A).

PROPOSITION 1: *Any differentiable decomposable inequality measure, I^n, can be expressed in the following functional form:*

$$\text{(D)} \qquad I^n(\ ) = \sum_{i=1}^{m} G(Y_i, n_i; Y, n) I^{n_i}(\ ) + I^n(\bar{y}_1, \ldots, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m, \ldots, \bar{y}_m)$$

with $Y = \Sigma_i Y_i$, and $\bar{y}_i = Y_i/n_i$, for any partition $(m; n_1, n_2, \ldots, n_m)$ of the population.

PROOF: Let us prove first that the additivity property in Definition 3 implies that $F^m(\ )$ in (A) is separable with respect to the I^{n_i}'s. That property is obvious for $m = 2$ since the additivity property writes:

$$\begin{aligned} &F^2(I^{n_1}, I^{n_2}; Y_1, Y_2; n_1, n_2) \\ &\quad = [F^2(I^{n_1}, I^{n_2}; Y_1, Y_2; n_1, n_2) - F^2(0, I^{n_2}; Y_1, Y_2; n_1, n_2)] \\ &\qquad + [F^2(I^{n_1}, I^{n_2}; Y_1, Y_2; n_1, n_2) - F^2(I^{n_1}, 0; Y_1, Y_2; n_1, n_2)] \\ &\qquad + F^2(0, 0; Y_1, Y_2; n_1, n_2) \end{aligned}$$

or

$$\begin{aligned} &F^2(I^{n_1}, I^{n_2}; Y_1, Y_2; n_1, n_2) \\ &\quad = F^2(I^{n_1}, 0; Y_1, Y_2; n_1, n_2) - F^2(0, I^{n_2}; Y_1, Y_2; n_1, n_2) \\ &\qquad + F^2(0, 0; Y_1, Y_2; n_1; n_2). \end{aligned}$$

Now, we may split group 2 into various subgroups and the total inequality will remain separable in I^{n_1}. Since we may choose group 1 as we want, $F^m(\ )$ is certainly separable with respect to all I^{n_i}'s:

$$\text{(2)} \qquad \begin{aligned} I^n = &\sum_{i=1}^{n} A_i^m(I^{n_i}; Y_1, \ldots, Y_m; n_1, \ldots, n_m) \\ &+ B^m(Y_1, \ldots, Y_m; n_1, \ldots, n_m). \end{aligned}$$

We will show now that the functions $A_i^m(\ )$ in (2) are linear with respect to the

I^{n_i}'s. Consider the case $m = 3$:

$$(3)\qquad I^n = \sum_{i=1}^{3} A_i^3(I^{n_i}; Y_1, Y_2, Y_3; n_1, n_2, n_3) + B^3(Y_1, Y_2, Y_3; n_1, n_2, n_3).$$

Aggregate groups 2 and 3 into one group. From (2), I^n rewrites:

$$(4)\qquad I^n = A_1^2(I^{n_1}; Y_1, Y_2+Y_3; n_1, n_2+n_3) + A_2^2(\bar{I}; Y_1, Y_2+Y_3; n_1, n_2+n_3) + B^2(Y_1, Y_2+Y_3; n_1, n_2+n_3)$$

where

$$(5)\qquad \bar{I} = A_1^2(I^{n_2}; Y_2, Y_3; n_2, n_3) + A_2^2(I^{n_3}; Y_2, Y_3; n_2, n_3) + B^2(Y_2, Y_3; n_2, n_3).$$

Equalizing expressions (3) and (4) and taking the partial derivatives with respect to I^{n_2} and I^{n_3} successively yields:

$$\frac{\delta^2 A_2^2}{\delta \bar{I}^2} \cdot \frac{\delta \bar{I}}{\delta I^{n_2}} \cdot \frac{\delta \bar{I}}{\delta I^{n_3}} = 0$$

which implies that A_2^2 is linear in $\bar{I}$, its first argument. It follows that $\bar{I}$ in (5) is linear in I^{n_3} and, using simultaneously (4) and (5), that I^n is linear in I^{n_3}. As group 3 may be chosen as we want, the functions A_i^m in (2) are linear in I^{n_i}. So, (2) becomes:

$$(6)\qquad I^n = \sum_{i=1}^{m} C_i^m(Y_1, \ldots, Y_m; n_1, \ldots, n_m) I^{n_i} + B^m(Y_1, \ldots, Y_n; n_1, \ldots, n_m).$$

It remains to prove that C_i^m depends only on Y_i, n_i, Y, and n. With the partition "group 1–rest of the population," (6) becomes:

$$(7)\qquad I^n = C_1^2(Y_1, Y-Y_1; n_1, n-n_1) I^{n_1} + C_2^2(Y_1, Y-Y_1; n_1, n-n_1)\bar{I} + H^2(Y_1, Y-Y_1; n_1, n-n_1)$$

where $\bar{I}$ is now the inequality measure outside of group 1.

Identifying (6) and (7):

$$C_1^m(Y_1, \ldots, Y_m; n_1, \ldots, n_m) = C_1^2(Y_1, Y-Y_1; n_1, n-n_1).$$

Since the same argument holds for any i, (6) rewrites:

$$I^n = \sum_{i=1}^{m} G(Y_i, n_i; Y, n) I^{n_i} + B^m(Y_1, \ldots, Y_m; n_1, \ldots, n_m).$$

Finally, assume that all subgroups i are egalitarian. Then, according to our convention, $I^{n_i} = 0$, and all individuals in subgroup i receive the income $\bar{y}_i = \bar{Y}_i / n_i$.

From (6), we have the identity:

$$I^n(\bar{y}_1, \ldots, \bar{y}_1; \bar{y}_2, \ldots, \bar{y}_2; \ldots; \bar{y}_m, \ldots, \bar{y}_m)$$
$$= B^m(Y_1, \ldots, Y_n; n_1, \ldots n_m)$$

which achieves the proof of Proposition 1.

We can get a stronger result than Proposition 1 if we assume that the inequality measure is income-homogeneous of degree zero and satisfies the 'symmetry axiom for population'.

PROPOSITION 2: *Any differentiable and decomposable inequality measure which is zero-homogeneous in incomes and satisfies the 'symmetry axiom for population' takes the following functional form:*

$$(\mathrm{D_h}) \qquad I^n = \sum_{i=1}^{m} f(v_i, w_i)I^{n_i} + I^n(\bar{y}_1, \ldots, \bar{y}_1; \ldots; \bar{y}_m, \ldots, \bar{y}_m)$$

for any partition $(m; n_1, \ldots, n_m)$ *of the population, with* $v_i = Y_i/Y$, $w_i = n_i/n$, *and* $f(\ )$ *being a differentiable function homogeneous of degree one in both its arguments.*

PROOF: The income-zero-homogeneity property applied to (D) gives immediately:

$$(8) \qquad G(Y_i, n_i; Y, n) = G(v_i, n_i; 1, n) = g(v_i, n_i, n).$$

Now, the 'symmetry axiom for population' [**3**] postulates that, if a distribution of n individual incomes is replicated r times, the inequality corresponding to the resulting total distribution of rn incomes is the same as that of the initial distribution. Applying that axiom to the decomposability property (D), for the following partitions of the m individuals $(m; rn_1, rn_2, \ldots, rn_i)$ and $(rm; n_1, \ldots, n_1, n_2, \ldots, n_2, \ldots, n_m)$, we get:

$$I^n = \sum_{i=1}^{m} g(v_i, n_i, n)I^{n_i} + I^n(\bar{y}_1, \ldots, \bar{y}_1; \ldots; \bar{y}_m, \ldots, \bar{y}_m)$$
$$= \sum_{i=1}^{m} g(v_i, rn_i, rn) + I^m(\bar{y}_1, \ldots, \bar{y}_1; \ldots; \bar{y}_m, \ldots, \bar{y}_m)$$
$$= \sum_{i=1}^{m} rg\left(\frac{v_i}{r}, n_i, rn\right)I^{n_i} + I^m(\bar{y}_1, \ldots, \bar{y}_1; \ldots; \bar{y}_m, \ldots, \bar{y}_m).$$

Identifying the terms in I^{n_i}, it implies:

$$(9) \qquad g(v_i, n_i, n) = g(v_i, rn_i, rn) = rg\left(\frac{v_i}{r}, n_i, rn\right)$$

for any positive integer r.

As r, n_i, and n are restricted to the set of positive integers, it cannot be inferred from the left-hand side equality in (9) that $g(\ )$ depends only on the ratio n_i/n, as with the usual homogeneity property. Nevertheless, it is not difficult to prove that this result still holds. Choose $r = N!/n$ with $n \leq N$ and define the function $h(\ )$:

$$h(v_i, w_i, n) = g(v_i, n_i, n),$$

w_i being the rational number n_i/n. From (9), we have:

$$h(v_i, w_i, n) = g(v_i, w_i N!, N!) \tag{10}$$

for all (n, N) such that $n \leq N$. Thus, holding w_i and v_i constant, the function $h(\ )$ of n is constant for $n \leq N$. If $n > N$, on the other hand, it is always possible to find an integer $N' > n > N$ such that from (9) and (10):

$$g(v_i, w_i N!, N!) = g(v_i, w_i N'!, N'!),$$

$$h(v_i, w_i, n) = g(v_i, w_i N'!, N'!).$$

So (10) holds also for $n > N$ and $h(\ )$ is a constant function of n:

$$h(v_i, w_i, n) = g(v_i, n_i, n) = f(v_i, w_i).$$

Now to prove that $f(\ )$ is homogeneous of degree one, we use the second equality relationship in (9). In terms of $f(\ )$, it becomes:

$$f\left(\frac{v_i}{r}, \frac{w_i}{r}\right) = \frac{1}{r} f(v_i, w_i).$$

Although r is restricted to be an integer, this relationship implies the usual homogeneity property. To see that, call u_i and z_i the rational numbers v_i/r and w_i/r, respectively. We have:

$$rf(u_i, z_i) = f(ru_i, rz_i) \qquad \text{for any} \quad r, u_i, z_i.$$

Combining both relationships for two distinct integers r and r', we get:

$$f\left(\frac{rv_i}{r'}, \frac{rw_i}{r'}\right) = \frac{r}{r'} f(v_i, w_i)$$

and, choosing the rational number r'/r equal to w_i, we get the usual homogeneity property:

$$f(v_i, w_i) = w_i f\left(\frac{v_i}{w_i}, 1\right).$$

This achieves the proof of Proposition 2.

Notice that we would have found the same property assuming I^n homogeneous of *any* degree α in incomes, provided that α would be independent of n. Notice also that the income-homogeneity condition is necessary to get the population-homogeneity property. Without the income-homogeneity property, the symmetry axiom for population would simply write:

$$G(rY_i, rn_i; rY, rn) = G(Y_i, n_i; Y, n)$$

and it would not be possible to get (D_h). It must be pointed out, finally, that the preceding argument applies also to the aggregativity property. When the measure is income-zero-homogeneous, the Y_i's and the n_i's in (A) can be replaced by the v_i's and the w_i's.[7]

We have now proven that (additively) decomposable inequality measures could be expressed as the sum of the inequality existing between subgroups of a population and of a kind of 'weighted average' of the inequality within those groups, although the 'weights' used in this averaging do not necessarily sum up to one. When the measure is assumed to be zero-homogeneous in incomes, on the other hand, the 'weight' of a group depends homogeneously upon its shares of the total income and of the total population. It is clear, then, that decomposable inequality measures will differ only by the weighting systems $f(v_i, w_i)$. From that point of view, the two most obvious candidates are naturally $f(v_i, w_i) = v_i$ and $g(v_i, w_i) = w_i$ or, respectively, 'income-weighted' and 'population-weighted' decomposable measures and we shall devote the next two sections of this paper to them.

Before looking at those particular measures, it might be useful to think a little about the differentiability condition we have imposed at the beginning of this paper to see how restrictive it could actually be. In this respect, it must be stressed that, until now, we have not really used that condition. Proposition 1 calls only for the differentiability of aggregative measures with respect to within-group inequalities and such differentiability does not seem too restrictive, a priori. The rest of the paper, however, will rely heavily upon differentiability with respect to individual incomes.

Among the usual inequality measures (as listed in [**9** or **12**], for instance) the only ones not to be differentiable everywhere with respect to individual incomes are those which are, roughly speaking, based upon the ranking of individual incomes: the Gini coefficient, Elteto-Frigyes indices, and other inter-quantiles mean-income ratios, relative range, average dispersion, etc. All those measures have the property that they can be expressed as a function of individual incomes and of their rank in the distribution. Under these conditions, they are generally not differentiable at some switching points of the income ranking ($y_i = y_j$ or $y_i = \bar{y}$, for instance) because right-hand and left-hand derivatives are not the same. It is possible, however, to prove that such 'ranking-based' measures are generally not aggregative in the sense of (A).

To see that property, consider the case of the Gini coefficient. All individuals are assumed to have distinct incomes and we will consider income transfers small enough to leave invariant overall ranking of individual incomes. If individuals in group i are ranked by decreasing income levels, transfers from the j_ith

[7] It is also interesting to notice that $f(v_i, w_i)$ in (D_h) can also be written

$$w_i \phi\left(\frac{\bar{y}_i}{\bar{y}}\right)$$

where $\bar{y}_i$ is the mean income in group i and $\bar{y}$ the mean income in the whole population. So the "weight" of I^{n_i} in the total inequality is the "population-weight", w_i, corrected by a function of the relative income of group i. For more details, see Section 5.

individual to the k_ith and the l_ith individuals will not change the Gini coefficient within group i if they satisfy:

$$\frac{\Delta y_{k_i}}{\Delta y_{l_i}}=\frac{j_i-l_i}{k_i-j_i}$$

with $l_i<j_i<k_i$. This is because the Gini coefficient may be written as:

$$I^{n_i}=1+\frac{1}{n_i}-(2/n_i^2\bar{y}_i)(y_1+2y_2+\ldots+n_iy_{n_i})$$

with $y_1\geqslant y_2\ldots\geqslant y_{n_i}$. If the Gini coefficient were aggregative in the sense of (A), the total inequality measure would be invariant with such transfers. But, clearly, it is sufficient to assume that the rank of individuals j_i, k_i, and l_i in the whole population are J, K, L such that:

$$\frac{J-K}{L-J}\neq\frac{j_i-k_i}{l_i-j_i}$$

in order for the aggregate Gini coefficient to be affected by those transfers. So, the Gini coefficient is not 'aggregative' in the sense of (A).

More generally, the same argument applies to all measures based upon an income-ranking such that rank-preserving inequality-invariant transfers depend on the rank of transferors and transferees. For those measures, it will always be possible to construct cases where inequality-invariant transfers in one subgroup of the population will change the total inequality measure. The problem comes naturally from the overlapping of the income rankings in two subgroups. This property can easily be checked for all nondifferentiable "income-ranking based" measures indicated above and loosens substantially the restrictiveness of the differentiability assumption under which we shall work now.[8] It must be stressed, moreover, that we do not actually need, in what follows, inequality measures to be differentiable everywhere. Most of the arguments we will be using apply to the "income-ranking based" measures we have just considered in any region where they are differentiable.

[8] Notice, however, that Rawls' criterion leads to an inequality measure which may be considered as a "ranking-based" measure which is nevertheless aggregative. Dalton's measure (see Section 3) associated with Rawls' criterion may be defined as

$$R=1-\min_{i=1,\ldots,n}\left(\frac{y_i}{\bar{y}}\right)$$

which is aggregative since for any partition $(m; n_1, \ldots, n_m)$,

$$R=1-\min\frac{\bar{y}_i}{\bar{y}}(1-R_i) \qquad (i=1,2\ldots m).$$

It is true, on the other hand, that the definition of "ranking-based" inequality measures is not very precise.

3. POPULATION-WEIGHTED DECOMPOSABLE MEASURES

According to what precedes, we want to find the inequality measures which are differentiable, symmetric, and homogeneous of degree zero in all incomes, and which satisfy the Pigou-Dalton and following conditions:

$$I^n(\ )=\sum_{i=1}^{m}\frac{n_i}{n}I^{n_i}(y_{i1},\ldots,y_{in_i})+I^n(\bar{y}_1\ldots\bar{y}_1,\bar{y}_2\ldots,\bar{y}_m\ldots\bar{y}_m) \tag{11}$$

for all partitions $(m; n_1, n_2\ldots, n_m)$ with $Y_i/n_i=\bar{y}_i$.

First let us prove the following proposition:

PROPOSITION 3: *An inequality measure* $I^n(y_1, y_2, \ldots, y_n)$ *is differentiable, symmetric, and satisfies the decomposability property* (11) *if and only if it can be expressed as*

$$I^n(y_1,y_2,\ldots,y_n)=\frac{1}{n}\sum_{i=1}^{m}\left[K(y_i)-K\left(\frac{1}{n}\sum_{i=1}^{n}y_i\right)\right] \tag{12}$$

where K is a differentiable function.

PROOF: The sufficiency part of the proof is obvious. For the necessary part, let us consider $I^{n+1}(y_1, y_2, \ldots, y_{n+1})$. According to (11) and making use of the arbitrary definition, $I^1(y)=0$, introduced at the beginning of the preceding section, it can also be written as

$$I^{n+1}(\ )=\frac{n}{n+1}I^n(y_1,y_2,\ldots,y_n)+I^{n+1}(\bar{y}_n,\ldots,\bar{y}_n,y_{n+1})$$

where $\bar{y}_n=(1/n)\sum_{i=1}^{n}y_i$, or

$$I^{n+1}(y_1,y_2,\ldots,y_{n+1})=A(y_1,y_2,\ldots,y_n)+B(y_1+y_2+\ldots+y_n,y_{n+1}). \tag{13}$$

The symmetry property implies that:

$$\begin{aligned}&A(y_1,y_2,\ldots,y_n)+B(y_1+y_2+\ldots+y_n,y_{n+1})\\&\quad=A(y_2,y_3,\ldots,y_{n+1})+B(y_2+y_3+\ldots y_{n+1},y_1).\end{aligned} \tag{14}$$

Differentiating both sides with respect to y_1 and y_{n+1} yields

$$B_{12}(y_1+y_2+\ldots+y_n,y_{n+1})=B_{12}(y_2+y_3+\ldots+y_{n+1},y_1),$$

which implies that $B_{12}(y_1+y_2+\ldots y_n, y_{n+1})=C(y_1+y_2+\ldots+y_{n+1})$[9] and after integration with respect to the two arguments of $B(\ )$:

$$\begin{aligned}&B(y_1+y_2+\ldots+y_n,y_{n+1})\\&\quad=D(y_1+y_2+\ldots+y_{n+1})+E(y_1+y_2+\ldots+y_n)+F(y_{n+1}).\end{aligned}$$

[9] The proof is as follows. The preceding property for B_{12} is formally equivalent to $F(u+v,w)=F(u+w,v)$. Differentiating along $u+w=cst$ yields $F_1(u+v,w)-F_2(u+w,v)=0$. This implies that the differential of $F(u+v,w)$ along $u+v+w=cst$ is also nil. Thus, $F(u+v,w)$ is a function of $u+v+w$.

Substituting that functional form into functions B in (14) gives:

$$A(y_1,\dots,y_n)-A(y_2,\dots,y_{n+1})$$
$$=E(y_2+\dots+y_{n+1})+F(y_1)-E(y_1+\dots+y_n)-F(y_{n+1}),$$

and identifying with respect to y_1,

$$A(y_1,\dots,y_n)=F(y_1)-E(y_1+\dots+y_n)+H(y_2,\dots,y_n).$$

But $A(\ )$ in (13) stands in fact for $(n/(n+1))I^n(\ )$. Using the symmetry property, we get

$$I^n=\frac{n+1}{n}\sum_{i=1}^{n}[F(y_i)-E(y_1+\dots+y_n)].$$

Since the inequality associated with an equalitarian distribution is arbitrarily zero, $E(\ )$ and $F(\ )$ are related by

$$E(x)=F\left(\frac{x}{n}\right)$$

and we complete the proof of (12) by defining $K(\ )=(n+1)F(\ )$.

Let us introduce now the homogeneity property. We get immediately the final proposition:

PROPOSITION 4: *The only inequality measure which is differentiable, symmetric, homogeneous of degree zero in all incomes, and satisfies the decomposability property* (11), *is given by*

$$L=log\left(\frac{1}{n}\sum_{i=1}^{n}y_i\right)-\frac{1}{n}\sum_{i=1}^{n}log\,y_i. \tag{15}$$

This measure is defined up to a positive multiplicative constant and satisfies the Pigou-Dalton condition.

PROOF: We look at the functions satisfying (12) which are homogeneous of degree zero. Euler's theorem for these functions is written:

$$\sum_{i=1}^{n}y_iK'(y_i)-K'\left(\frac{1}{n}\sum_{i=1}^{n}y_i\right)\sum_{i=1}^{n}y_i=0.$$

Differentiating under the condition that the aggregate income $\Sigma_{i=1}^{n}y_i$ be constant yields

$$y_iK''(y_i)+K'(y_i)=y_jK''(y_j)+K'(y_j)$$

for any pair (y_i, y_j) or

$$y_iK''(y_i)+K'(y_i)=K, \tag{16}$$

where K is a constant. Integrating (16), we get

$$K(y_i)=M\log_i+Ky_i+N$$

where M and N are constant. The last two terms of $K(y_i)$ cancel in (12) which proves (15), with the multiplicative constant M.

The Pigou-Dalton condition is written:

$$(17)\qquad \left(\frac{\delta L}{\delta y_i}-\frac{\delta L}{\delta y_j}\right)(y_i-y_j)>0.$$

It is a simple matter to prove that (15), with the multiplicative constant M, implies that

$$(18)\qquad \frac{\delta L}{\delta y_i}-\frac{\delta L}{\delta y_j}=M\frac{y_i-y_j}{ny_iy_j}.$$

Thus L satisfies the Pigou-Dalton condition if M is positive, which achieves the proof of Proposition 4.

That the inequality measure L has seldom been used in applied works on income distribution is somewhat surprising because it has very much to commend it. Besides the fact that it is decomposable (as already noticed by Theil [**13**]) and satisfies the basic properties of an inequality measure, L lends itself to a very simple interpretation in terms of social welfare. In the utilitarian framework, the social welfare function is the sum of identical concave individual utility functions. If we choose the logarithm form for those utility functions, L is simply the difference between the maximum social welfare for a given total income, which corresponds to the equalitarian distribution, and the actual social welfare. It is the 'distance' between the actual and the 'optimal' situation. From that point of view, L is very close to the measure proposed by Dalton [**2**; **11**, p. 37]. Dalton's measure (D_L) is the ratio of actual utilitarian welfare to maximum utilitarian welfare, whereas L is the difference between these two values.

$$D_L=\frac{\sum_{i=1}^{n}\log y_i}{n\log\bar{y}},\qquad L=\log\bar{y}-\frac{1}{n}\sum_{i=1}^{n}\log y_i.$$

As a matter of fact, D_L and L are so close that one may wonder whether D_L would not be decomposable. It is easy to see that this is the case. Define $D'_L=1-D_L$ to have a positive measure of inequality taking the value zero for an equalitarian distribution. Some transformations on D'_L permit us to express that measure in the following form:

$$D'_L=\sum_{i=1}^{m}\frac{n_i}{n}\frac{\log\bar{y}_i}{\log\bar{y}}D'^{i}_L+1-\frac{\sum_{i=1}^{m}n_i\log y_i}{n\log\bar{y}}.$$

In other words, D'_L is a decomposable measure of inequality. However, it is not income-homogeneous of degree zero and it satisfies the decomposability definition (D) (with $G(Y_i, n_i, y, n)=n_i\log(Y_i/n_i)/n\log(Y/n)$) rather than definition (D_h). More generally, it is obvious that any Dalton measure

$$D'_u = 1 - \sum_{i=1}^{n} U(y_i)/nU(\bar{y})$$

or any measure of the type

$$L_u = U(\bar{y}) - \frac{1}{n}\sum_{i=1}^{n} U(y_i)$$

is decomposable in the sense of (D) but, for the latter, only a logarithmic utility function leads to a measure which is income-homogeneous of degree zero.[10]

4. INCOME-WEIGHTED DECOMPOSABLE MEASURES

In accordance with Section 2, the income-weighted decomposability corresponds to the following property of the inequality measure:

$$I^n(\ ) = \sum_{i=1}^{m} \frac{Y_i}{Y} I^{n_i}(y_{i1}, y_{i2}, \ldots, y_{in_i}) + I^n(\bar{y}_1 \ldots \bar{y}_1; \ldots; \bar{y}_m \ldots \bar{y}_m) \tag{19}$$

for all partitions $(m; n_1, n_2 \ldots n_m)$, $Y = \Sigma_{i=1}^m Y_i = \Sigma_{i=1}^m \Sigma_{j=1}^{n_i} y_{ij}$, and $\bar{y}_i = Y_i/n_i$.

Finding the symmetric and homogeneous inequality measures satisfying (19) is easy. First, let us define

$$J^n(y_1, y_2 \ldots y_n) = (y_1 + y_2 + \ldots + y_n) I^n(y_1, y_2, \ldots, y_n). \tag{20}$$

(19) is then equivalent to

$$J^n(y_{11}, y_{12}, \ldots, y_{mn_m}) = \sum_{i=1}^{m} J^{n_i}(y_{i1}, y_{i2}, \ldots y_{in_i}) + J^n(\bar{y}_1 \ldots \bar{y}_1; \ldots; \bar{y}_m \ldots \bar{y}_m)$$

which itself is almost identical to the population-weighted decomposability definition (11). It is easy to prove that Proposition 3 applies to J^n (the coefficient (n_i/n) in (11) was unimportant in the proof of Proposition 3). Thus

$$J^n(y_1, y_2, \ldots, y_n) = \frac{1}{n}\sum_{i=1}^{n} K(y_i) - K\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) \tag{21}$$

and we look for functions of type (21) which are homogeneous of degree one. (In

[10] Though (11) corresponds to the decomposition of the variance, no variance-like inequality measure has been derived in the present section. This is natural. On one hand, the variance is not zero income-homogeneous. On the other hand, the variance of logarithms would be decomposable if we were considering income logarithms rather than incomes. As a matter of fact, it can be proven that the variance of logarithms is not even aggregative in terms of individual incomes. Aggregativity (A) requires that inequality-preserving transfers in one group i be neutral on total inequality. This implies in particular that

$$I_j^n - I_k^n / I_l^n - I_j^n \qquad (I_j^n = \text{derivative of } I^n \text{ with respect to } y_j)$$

depend only on y_j, y_k, and y_l. This is the property we used to prove the non-aggregativity of the Gini coefficient. It can be checked that the variance of logarithms does not satisfy that condition.

(20), I^n is homogeneous of degree zero and Σy_i is homogeneous of degree one.) This leads to the following proposition:

PROPOSITION 5: *The only inequality measure which is differentiable, symmetric, homogeneous of degree zero in all incomes, and which satisfies the income-weighted decomposability property* (19) *is Theil's coefficient,*

$$T=\sum_{i=1}^{n} x_i \log x_i, \qquad x_i = y_i\Big/\sum_{i=1}^{n} y_i.$$

It is defined up to a positive multiplicative constant and satisfies the Pigou-Dalton condition.

PROOF: Euler's theorem applied to (21) yields the identity:

$$\sum_{i=1}^{n} y_i\left[\frac{1}{n}K'(y_i)-\frac{1}{n}K'\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)\right]=\frac{1}{n}\sum_{i=1}^{n} K(y_i)-K\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right).$$

Differentiating with respect to y_i and y_j under the condition that the aggregate income is constant yields

$$yK''(y)=H,$$

where H is a constant. After a double integration, K can be expressed as

$$K(y)=Hy\log y+My+N,$$

where M and N are constant. Thus,

$$J^n(y_1, y_2, \ldots, y_n)=\frac{H}{n}\sum_{i=1}^{n} y_i \log y_i-\frac{H}{n}\sum_{i=1}^{n} y_i \log\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)$$

and, after dividing by $\Sigma_{i=1}^{n} y_i$, Proposition 5 follows.

It is interesting to notice that the inequality measure L derived in the preceding section and Theil's coefficient are in some sense dual measures. If the population of income earners is broken down into groups $i (i=1, 2 \ldots m)$ with weight w_i in the total population and share v_i of the total income, we have

$$L=\sum_{i=1}^{m} w_i L_i+\sum_{i=1}^{m} w_i \log\frac{w_i}{v_i}, \qquad T=\sum_{i=1}^{m} v_i T_i+\sum_{i=1}^{m} v_i \log\frac{v_i}{w_i}.$$

In other words, the inequality measure L is the same as Theil's coefficient except that the roles of the w_i's and v_i's are inverted.[11]

5. SOME OTHER DECOMPOSABLE MEASURES

With respect to the decomposability property (D_h), the income-weighted and population-weighted measures have the property that the weights of the within-

[11] On that property, see [9].

group inequalities in the total inequality sum to one. In that sense, it can be said that the total 'within' inequality is truly a weighted average of the inequality within the subgroups of the total population. One may wonder, under these conditions, whether that intuitively appealing property is satisfied by measures other than L and T. The answer is negative.

PROPOSITION 6: *The only differentiable decomposable and zero-homogeneous inequality measures such that the sum of the weights of the 'within-group' inequalities in the decomposability definition* (D_h) *is constant are L and T.*

PROOF: According to (D_h),

$$I^n(\ ) = \sum_{i=1}^{m} f(v_i, w_i) I^{n_i}(\ ) + I^n(\bar{y}_1, \ldots, \bar{y}_1; \ldots; \bar{y}_m, \ldots, \bar{y}_m)$$

with $w_i = n_i/n$, $v_i = Y_i/Y$.

Now, we want in addition that:

$$\sum_{i=1}^{m} f(v_i, w_i) = k \tag{22}$$

where k is some constant.

But, this condition, together with $\Sigma_i v_i = \Sigma_i w_i = 1$ and the homogeneity of degree one of $f(v_i, w_i)$ imply that $f(\ )$ has the form $\beta w_i + \gamma v_i$.[12] This leads to decomposable measures satisfying

$$I^n(\ ) = \sum_{i=1}^{m} \left(\beta \frac{n_i}{n} + \gamma \frac{Y_i}{Y}\right) I^{n_i}(\ ) + I^n(\bar{y}_1, \ldots, \bar{y}_1; \ldots; \bar{y}_m, \ldots, \bar{y}_m).$$

It is possible to prove that no homogeneous and symmetric function satisfies this functional equation if both γ and β are strictly positive. The proof is tedious and a sketch of it is given in the Appendix.

If the weighting functions $f(v_i, w_i)$ are not restricted by (22), the choice is unlimited. However, that choice generally implies strong explicit normative judgements; $f(v_i, w_i)$ being homogeneous of degree one, it can be written:

$$f(v_i, w_i) = \frac{n_i}{n} \phi\left(\frac{\bar{y}_i}{\bar{y}}\right). \tag{23}$$

In other words, any decomposable measure relies on the population-weighting system (n_i/n) corrected by a factor which takes into account the relative income of each group. So, the choice of a function $f(v_i, w_i)$ amounts to deciding whether

[12] To prove that point, it is sufficient to differentiate the relationship

$$f\left(1 - \sum_{i=2}^{m} v_i, 1 - \sum_{i=2}^{m} w_i\right) = k - \sum_{i=2}^{m} f(v_i, w_i)$$

twice with respect to v_i and v_j or with respect to w_i and w_j.

inequality in rich groups matter more than in poor groups and by how much. From that point of view, the measure L does not make any distinction between groups, whereas T assigns to each group a weight which is proportional to its relative income. One might very well make the opposite normative judgement, however, and require that the weight of a group in the total inequality be a decreasing function of its relative income. A simple general inequality measure satisfying that condition, for instance, could be based upon the weighting function

$$f(v_i, w_i) = w_i^{1+b} v_i^{-b} = \frac{n_i}{n}\left(\frac{\bar{y}_i}{\bar{y}}\right)^{-b} \qquad \text{with } b > 0.$$

Using the same argument as in the preceding section the decomposable measure consistent with that weighting system is the Dalton measure associated with the utility function $U = -ky^{-b}$:

$$P_b = 1 - \frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_i}{\bar{y}}\right)^{-b}.$$

But, obviously, we would have found a different measure by postulating a distinct decreasing relationship between the weight of a group in the total inequality and its relative income.

In the same way as decomposable measures require some value judgements in the choice of the weighting system $f(v_i, w_i)$, the decomposability of some usual inequality measures may reveal normative properties which are not immediately apparent. The preceding example shows, for instance, that the square of the coefficient of variation,[13] and, more generally, Dalton measures based upon the utility function $U = ky^{\alpha}(0 < \alpha)$ give more weight to the rich groups in the decomposition of inequality.

We might also look at measures which are not zero-homogeneous, or in other words, at measures of inequality which are not invariant to a change in the scale of the distribution. Such measures would correspond to the idea that inequality is more burdensome for a poor than a rich population or inversely. If we restrict ourselves to measures which are homogeneous of some degree α (with $\alpha \neq 0$), the argument of Sections 3 and 4 may be generalized to get the corresponding "population-weighted" and "income-weighted" decomposable measures. They are close to P_b above and may be written, respectively,

$$L_\alpha = \bar{y}^\alpha \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_i}{\bar{y}}\right)^{\alpha} - 1\right],$$

$$T_\alpha = \bar{y}^\alpha \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_i}{\bar{y}}\right)^{\alpha+1} - 1\right].$$

To conclude this analysis we might go back to our definition of "aggregativity" to notice the interesting property that any transformation of a decomposable measure by a monotonic function gives a measure of inequality which is aggre-

[13] This corresponds to $-P_b$ with $b = -2$.

gative.[14] This explains why no reference has been made until now to Atkinson's measure, which is often considered as decomposable. It can be seen that Atkinson's measure is a transformation of P_b above and, therefore, that it is aggregative. But it is not (additively) decomposable which explains the ambiguities found by the authors who have studied the decomposition of that measure.[15]

6. CONCLUSION

The main results obtained in the above analysis can be summarized as follows (see Table I):

(a) Among the usual inequality measures, three are additively decomposable and zero-homogeneous in incomes: the average logarithm of relative incomes (L), Theil's entropy coefficient (T), and the square of the coefficient of variation. The last one, however, offers the inconvenience of referring implicitly to a utilitarian welfare function with convex individual utilities.

(b) Dalton's measures are additively decomposable but only those relying on individual utilities of the power type are income-zero-homogeneous.

(c) Three usual measures are aggregative without being decomposable: the coefficient of variation, Atkinson's measure, and Rawl's criterion. In addition, any transformation of a decomposable measure by a monotonic function gives an aggregative measure (as is the case for the first two preceding measures).

(d) L and T are the only zero-homogeneous decomposable measures such that the weights of the 'within-group-inequalities' in the total inequality sum to a constant.

Although these results are interesting, they rely on the restricted definition we have given to the concept of decomposability. "Aggregativity" would appear, a priori, as a much more powerful and practical concept. As we have seen in the case of more general weighting systems for decomposable measures, however, a careful normative analysis of the aggregativity property would be necessary before trying to generalize the results obtained in the present paper to that concept.

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APPENDIX

NONEXISTENCE OF A ZERO-HOMOGENEOUS DECOMPOSABLE MEASURE WITH $f(v_i, w_i) = \beta v_i + \gamma w_i$

The decomposability condition is written:

$$(1) \qquad I^n = \sum_{i=1}^{n} \left(\beta \frac{y_i}{y} + \gamma \frac{n_i}{n}\right) I^{n_i} + I^n(\bar{y}_1, \ldots, \bar{y}_1; \ldots; \bar{y}_m, \ldots \bar{y}_m).$$

[14] Naturally, this is interesting only to the extent that an inequality measure is considered as a cardinal measure.

[15] See [1] and [12].

TABLE I

AGGREGATIVITY, DECOMPOSABILITY AND SOME OTHER PROPERTIES OF CURRENT INEQUALITY MEASURES

Inequality measure	Aggregativity	Additive decomposability	Income-zero -homogeneity	Pigou-Dalton condition
Relative maximum range	No	No	Yes	(No)[b]
Relative mean deviation	No	No	Yes	(No)[b]
Variance	Yes	Yes	No	Yes
Coefficient of variation	Yes	No	Yes	Yes
Square of the coefficient of variation	Yes	Yes	Yes	Yes
Variance of logarithms	No	No	Yes	No
Gini coefficient	No	No	Yes	Yes
Elteto-Frigyes indices	No	No	Yes	(No)[b]
Quantiles mean incomes ratios	No	No	Yes	(No)[b]
Rawls' criterion	Yes	No	Yes	(No)[b]
Theil's entropy coefficient (T)	Yes	Yes	Yes	Yes
Mean logarithmic deviation (L)	Yes	Yes	Yes	Yes
Atkinson's measure	Yes	No	Yes	Yes
Dalton's measures	Yes	(No)[a]	(No)[a]	Yes

[a] Except with power functions for individual utilities.
[b] In the strong sense of condition (17).

After multiplication by nY and using the symmetry property in the case $n=3$, (1) yields:

$$(2)\qquad [2\gamma(y_1+y_2+y_3)+3\beta(y_2+y_3)]I^2(y_2,y_3)+3(y_1+y_2+y_3)I^3\left(y_1,\frac{y_2+y_3}{2},\frac{y_2+y_3}{2}\right)$$
$$=[2\gamma(y_1+y_2+y_3)+3\beta(y_1+y_3)]I^2(y_1,y_3)+3(y_1+y_2+y_3)I^3\left(y_2,\frac{y_1+y_3}{2},\frac{y_1+y_3}{2}\right).$$

Unlike the cases $L(\beta=0)$ and $T(\gamma=0)$, this relationship does not lend itself to any nice simplification. Differentiating twice with respect to y_1 and y_2 and defining $J(x,z)=3(x+z)I^3(x,z/2,z/2)$ leads to

$$(3)\qquad \frac{\delta^4 J(y_1,y_2+y_3)}{\delta y_1^2\,\delta y_2^2}=\frac{\delta^4 J(y_2,y_1+y_3)}{\delta y_2^2\,\delta y_1^2}=F(y_1+y_2+y_3).$$

But $J(\ )$ is homogeneous of degree one, which implies

$$F(x+z)=A(x+z)^{-3},$$

A being a constant. Integrating,

$$\frac{\delta^4 J(x,z)}{\delta x^2\,\delta z^2}=A(x+z)^{-3},$$

and taking into account the homogeneity of $J(\ )$ leads to

$$J(x,z)=P(x+z)\log(x+z)-(P+N)z\log z-(P+N')x\log x+M\log x+M'x\log z+Nz+N'x$$

where capital letters stand for constants. Quite naturally, we find here a mix of the expression of L and T.

Now, when $y_1=y_3$, (2) gives

$$I^2(y_2,y_3)=\frac{J(y_3,y_2+y_3)-J(y_2,2y_3)}{2\gamma(y_2+2y_3)+3\beta(y_2+y_3)}.$$

Identifying this expression of $I^2(y_2, y_3)$ with its symmetric permits us to determine all the constants which appear in (5). This tedious operation shows that all these constants are necessarily nil if β and γ are simultaneously strictly positive. This argument can be extended to any $n > 3$.

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