Tôhoku Math. Journ.
27 (1975), 413-435.

DECOMPOSABLE OPERATORS IN CONTINUOUS FIELDS OF HILBERT SPACES

HIDEO TAKEMOTO

(Received January 22, 1974)

In the present paper, we shall investigate an extension of Nussbaum's reduction theory [3] for unbounded operators in Hilbert space from the point of view of continuous reduction theory. We can see that Nussbaum's reduction theory, somewhat, depending upon the measure theoretic arguments has been considered in the case that the fibres of field of Hilbert spaces are separable. However, if we have a point of view of continuous field of Hilbert spaces introduced in [6], we can extend Nussbaum's reduction theory for unbounded operators in Hilbert space to unbounded operators in continuous fields of Hilbert spaces of which each fibre is not necessarily separable. Our argument is based on some elementary facts concerning the characteristic matrix of a closed operator which was introduced in [5].

Let Ω be a compact Hausdorff space, then Ω is called a Stonean space if the closure of every open set is open (see [1] and [4]). Let Ω be a Stonean space, $H = C_F(\Omega, H(\omega))$ a continuous field of Hilbert spaces over Ω. Let A be a bounded $C(\Omega)$-module homomorphism of H into H, then there exists an operator field $\{A(\omega)\}$ such that, for each ξ, $\eta \in H$,

$$((A\xi)(\omega) \mid \eta(\omega)) = (A(\omega)\xi(\omega) \mid \eta(\omega))$$

for every $\omega \in \Omega$ where each $A(\omega)$ is a bounded operator on $H(\omega)$ and $C(\Omega)$ is the algebra of all complex-valued continuous functions on Ω [6; Proposition 4.4]. In this paper, we shall investigate to decompose unbounded operators in $H = C_F(\Omega, H(\omega))$ in the above mentioned form for bounded operators. As application of the above representation for unbounded operators in continuous field of Hilbert spaces, we shall show that the square root for densely defined positive operator exists and, further, if A is densely defined decomposable operator, then A can be written uniquely of the form $A = VS$ (the polar decomposition of A), where S is a self-adjoint, positive operator and V is a partially isometric operator.

1. Notation and Preliminaries. In this section, we provide some notations and facts that will be used later. We assume, throughout our discussions, Ω to be a Stonean space, but the reader will notice that the assumption is merely used for convenience and the proofs can be mostly effected without this assumption. Let Ω be a Stonean space, and let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω. For a field $\{H(\omega); \omega \in \Omega\}$ of Hilbert spaces, the elements $\xi = \{\xi(\omega)\}$ of $\prod_{\omega \in \Omega} H(\omega)$ are called vector fields. If $f \in C(\Omega)$ and $\xi = \{\xi(\omega)\}$ is a vector field, $f\xi = \{f(\omega)\xi(\omega)\}$. If $\xi = \{\xi(\omega)\}$ and $\eta = \{\eta(\omega)\}$ are vector fields, the (ξ, η) is the function: $\omega \longrightarrow (\xi(\omega) \mid \eta(\omega))$, and $|\xi|$ is the function: $\omega \longrightarrow \|\xi(\omega)\|$, where $(\xi(\omega) \mid \eta(\omega))$ is the inner product defined in $H(\omega)$ and $\|\xi(\omega)\| = (\xi(\omega) \mid \xi(\omega))^{1/2}$. Let $B(K)$ be the algebra of all bounded operators on a Hilbert space K; then each element of $\prod_{\omega \in \Omega} B(H(\omega))$ is called operator field. If $A = \{A(\omega)\}$ in $\prod_{\omega \in \Omega} B(H(\omega))$ and $\xi = \{\xi(\omega)\}$ in $\prod_{\omega \in \Omega} H(\omega)$, then $A\xi = \{A(\omega)\xi(\omega)\}$.

We give the definition of continuous field of Hilbert spaces over Ω introduced in [6].

DEFINITION 1.1. ([6; Definition 3.1]). Let Ω be a Stonean space, and let $\{H(\omega); \omega \in \Omega\}$ be a field of Hilbert spaces over Ω. A subspace H of $\prod_{\omega \in \Omega} H(\omega)$ is said to be a continuous field of Hilbert spaces over Ω, if there exists a subspace F of $\prod_{\omega \in \Omega} H(\omega)$ such that

(1) for every $\xi \in F$, the function $\omega \longrightarrow \|\xi(\omega)\|$ is continuous on Ω,

(2) for each $\omega \in \Omega$, the subspace $\{\xi(\omega): \xi \in F\}$ is dense in $H(\omega)$,

(3) $H = \{\xi \in \prod_{\omega \in \Omega} H(\omega)$: for each positive number ε and each $\omega_0 \in \Omega$, there exist an element ξ_0 in F and a neighborhood $U(\omega_0)$ of ω_0 such that $\|\xi(\omega) - \xi_0(\omega)\| < \varepsilon$ for every $\omega \in U(\omega_0)\}$,

(4) if $\xi = \{\xi(\omega)\}$ is a vector field such that the function $\omega \longrightarrow \|\xi(\omega)\|$ is bounded, and for each $\eta \in F$, the function $\omega \longrightarrow (\xi(\omega) \mid \eta(\omega))$ is a continuous function on Ω, then $\xi \in H$.

Under Definition 1.1, H is a $C(\Omega)$-module in the ordinary algebraic sense and, for any $\xi = \{\xi(\omega)\}$ and $\eta = \{\eta(\omega)\}$ in H, the function $\omega \longrightarrow (\xi(\omega) \mid \eta(\omega))$ is continuous on Ω, thus (ξ, η) is an element of $C(\Omega)$. Furthermore we can show that, in H, the following properties satisfy:

(1) $(\xi, \eta) = (\eta, \xi)^*$,

(2) $(\xi, \xi) \geqq 0$ and is 0 only for $\xi = 0$,

(3) $(a\xi_1 + b\xi_2, \eta) = a(\xi_1, \eta) + b(\xi_2, \eta)$

for all $\xi, \xi_1, \xi_2, \eta \in H$ and $a, b \in C(\Omega)$. We use the notion

$$|\xi| = (\xi, \xi)^{1/2}, \quad \|\xi\| = \||\xi|\| = \sup\{\|\xi(\omega)\|: \omega \in \Omega\}$$

where on the right we mean the usual positive square root and norm in $C(\Omega)$. Then H is a normed space with respect to the above mentioned norm $\|\cdot\|$. In particular, H is complete with respect to the norm $\|\xi\| = \sup\{\|\xi(\omega)\|: \omega \in \Omega\}$ for $\xi \in H$. Thus we denote $H = C_F(\Omega, H(\omega))$ and we call F as a fundamental subspace for H. Let G be a subspace of $\prod_{\omega\in\Omega} H(\omega)$ such that $H \supset G \supset F$, then G is a fundamental subspace of H, that is, $H = C_G(\Omega, H(\omega))$ is the mean of Definition 1.1. A bounded operator A from a continuous field $H_1 = C_{F_1}(\Omega, H_1(\omega))$ of Hilbert spaces over Ω into a second continuous field $H_2 = C_{F_2}(\Omega, H_1(\omega))$ of Hilbert spaces over Ω is a mapping from H_1 into H_2 that is not only linear and continuous in the usual operator norm, but is also a $C(\Omega)$-module homomorphism. We shall write the element of $C(\Omega)$ (typically $f, g, \cdots, a, b, \cdots$) on the left of the element of H (typically $\xi, \eta, \cdots$). We call A^* the adjoint operator of $A \in B(H)$ if $(A\xi, \eta) = (\xi, A^*\eta)$ for all $\xi, \eta \in H = C_F(\Omega, H(\omega))$. Then A^* is a bounded operator and the algebra $B(H)$ of all bounded operators on H is a C^*-algebra [6; Corollary 3.7]. This permits the formulation of the following definition.

DEFINITION 1.2 ([6; Definition 4.1 and 4.2]). Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω. An element A in $B(H)$ is called a decomposable operator if, for each $\omega \in \Omega$, there exists an element $A(\omega)$ of $B(H(\omega))$ such that for all $\xi, \eta \in H$ and each $\omega \in \Omega$, $((A\xi)(\omega) \mid \eta(\omega)) = (A(\omega)\xi(\omega) \mid \eta(\omega))$.

Let $\{A(\omega)\}$ be an element of $\prod_{\omega\in\Omega} B(H(\omega))$ such that the function $\omega \longrightarrow \|A(\omega)\|$ is bounded. The field $\{A(\omega)\}$ is called continuous if, for every $\xi = \{\xi(\omega)\} \in H = C_F(\Omega, H(\omega))$, the vector field $\{A(\omega)\xi(\omega)\}$ is an element of H.

Definition 1.2 says that every element A of $B(H)$ is represented as an element $\{A(\omega)\}$ of $\prod_{\omega\in\Omega} B(H(\omega))$ satisfying $(A\xi)(\omega) = A(\omega)\xi(\omega)$ for each $\omega \in \Omega$ and $\xi \in H$. Thus we consider, throughout our discussions, every element A of $B(H)$ to be an element of $\prod_{\omega\in\Omega} B(H(\omega))$ in the above mentioned mean. Furthermore, if $A = \{A(\omega)\}$ is an operator field such that the function $\omega \longrightarrow \|A(\omega)\|$ is bounded and the field $\{A(\omega)\}$ is continuous, then A is considered an element of $B(H)$.

We shall contemplate some closed operator which is able to decomposable. Before going to dispute about the above mentioned closed operator, we have some considerations.

We will introduce a notation of continuous submodule. To introduce the notation, we must show that if $\mathscr{M}$ is a closed submodule of H, $\mathscr{M}(\omega)$ is a closed subspace in $H(\omega)$ for every $\omega \in \Omega$. The following proposition says that each $\mathscr{M}(\omega)$ is a closed subspace in $H(\omega)$.

PROPOSITION 1.3. *Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω. Let $\mathscr{M}$ be a closed submodule of H. Then each $\mathscr{M}(\omega) = \{\xi(\omega); \xi = \{\xi(\omega)\} \in \mathscr{M}\}$ is a closed subspace of $H(\omega)$.*

PROOF. For an arbitrary fixed element ω_0 of Ω, let λ be the canonical mapping of $\mathscr{M}$ onto $\mathscr{M}(\omega_0)$: $\mathscr{M} \ni \xi \to \xi(\omega_0) \in \mathscr{M}(\omega_0)$, and let V_{ω_0} be the kernel of λ; then λ induces a mapping $\tilde{\lambda}$ of $\mathscr{M}/V_{\omega_0}$ onto $\mathscr{M}(\omega_0)$. Then $\tilde{\lambda}$ is isometric, and so $\mathscr{M}(\omega_0)$ is a closed subspace of $H(\omega_0)$. In fact, since $\|\lambda\| \leqq 1$, $\|\tilde{\lambda}\| \leqq 1$. On the other hand, for each $\xi \in \mathscr{M}$ and an arbitrary positive number ε, the set

$$G = \{\omega : \|\xi(\omega_0)\| + \varepsilon > \|\xi(\omega)\|\}$$

is an open set containing ω_0. Thus let z be the projection in $C(\Omega)$ corresponding to the closure of G, then $\lambda(z\xi) = \lambda(\xi)$. Hence $(1 - z)\xi$ is an element of V_{ω_0}. Furthermore, we have

$$\begin{aligned} \|z\xi\| &= \sup\{z(\omega)\|\xi(\omega)\| : \omega \in \Omega\} \\ &= \sup\{z(\omega)\|\xi(\omega)\| : \omega \in G\} \\ &\leqq \|\xi(\omega_0)\| + \varepsilon . \end{aligned}$$

Hence

$$\inf\{\|\xi + \eta\| : \eta \in V_{\omega_0}\} \leqq \|z\xi\| \leqq \|\xi(\omega_0)\| + \varepsilon .$$

Since ε is an arbitrary positive number,

$$\inf\{\|\xi + \eta\| : \eta \in V_{\omega_0}\} \leqq \|\xi(\omega_0)\| .$$

Therefore, $\tilde{\lambda}$ is isometric and so $\mathscr{M}(\omega_0)$ is a closed subspace of $H(\omega_0)$ for every $\omega_0 \in \Omega$.

For our purpose, we introduce the following definition.

DEFINITION 1.4. Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω. By a continuous submodule $\mathscr{M}$ we mean a subset of H such that $\mathscr{M}$ is a closed submodule and $\mathscr{M} = C_{\mathscr{M}}(\Omega, \mathscr{M}(\omega))$.

The above definition is equivalent to the AW^*-submodule in [2], but our definition can be introduced even if Ω is not a Stonean space.

Next, we shall show that the notation of continuous submodule is equivalent to an another notation.

THEOREM 1.5. *Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω. Let $\mathscr{M}$ be a closed submodule of H. Then the following conditions are equivalent.*

(1) *$\mathscr{M}$ is a continuous submodule,*

(2) *if $\xi = \{\xi(\omega)\}$ is an element of $\prod_{\omega \in \Omega} \mathscr{M}(\omega)$ such that, for every $\eta \in \mathscr{M}$, the function $\omega \to (\xi(\omega) \mid \eta(\omega))$ is continuous on Ω and the function $\omega \to \|\xi(\omega)\|$ is bounded; then ξ is an element of $\mathscr{M}$.*

(3) *let $\{e_\alpha\}$ be orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I$ and $\{\xi_\alpha\}$ a bounded subset of $\mathscr{M}$, then $\sum e_\alpha \xi_\alpha$ is an element of $\mathscr{M}$.*

PROOF. (1) $\Rightarrow$ (2): This assertion was denoted by the remark after Definition 1.1.

(2) $\Rightarrow$ (1): To prove that $\mathcal{M} = C_{\mathcal{M}}(\Omega, \mathcal{M}(\omega))$, we must show that $\mathcal{M}$ satisfies the conditions (1), (2), (3) and (4) in Definition 1.1 with respect to $\mathcal{M}$ itself. It is trivial by the properties of $\mathcal{M}$ that (1), (2) and (4) arise. Thus, we must show the condition (3). If ξ is an element of $\prod_{\omega\in\Omega}\mathcal{M}(\omega)$ such that, for arbitrary positive number ε and each element ω_0 of Ω, there exist ξ' of $\mathcal{M}$ and a neighborhood $U_1(\omega_0)$ of ω_0 satisfying $\|\xi(\omega) - \xi'(\omega)\| < \varepsilon/2$ for every $\omega \in U_1(\omega_0)$. Then, since ξ' is an element of H, there exist an element ξ'' of F and a neighborhood $U_2(\omega_0)$ of ω_0 satisfying $\|\xi'(\omega) - \xi''(\omega)\| < \varepsilon/2$ for every $\omega \in U_2(\omega_0)$. Put $U(\omega_0) = U_1(\omega_0) \cap U_2(\omega_0)$, then $U(\omega_0)$ is a neighborhood of ω_0 and we have the following equation: $\|\xi(\omega)-\xi''(\omega)\| < \varepsilon$ for every $\omega \in U(\omega_0)$. Thus, ξ is an element of H and so, for each $\eta \in \mathcal{M}$, the function $\omega \to (\xi(\omega) \mid \eta(\omega))$ is continuous on Ω. Therefore, by the assumption, ξ is an element of $\mathcal{M}$. Thus, $\mathcal{M} = C_{\mathcal{M}}(\Omega, \mathcal{M}(\omega))$.

(1) $\Leftrightarrow$ (3): We showed this assertion in [6; Theorem 4.5].

Let $B(H)$ be the C^*-algebra of all bounded operators on a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces. We call an element P in $B(H)$ a projection if it is a self-adjoint idempotent: $P^2 = P$ and $P^* = P$. Then we have the following result.

THEOREM 1.6. *Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces. If a subset $\mathcal{M}$ of H is a continuous submodule of H, then there exists a projection P such that $PH = \mathcal{M}$. Conversely let P be a projection on H, then $PH = \mathcal{M}$ is a continuous submodule of H.*

PROOF. Let $\mathcal{M}$ be a continuous submodule of H, then, for every $\omega \in \Omega$, $\mathcal{M}(\omega)$ is a closed subspace of $H(\omega)$ and $\mathcal{M} = C_{\mathcal{M}}(\Omega, \mathcal{M}(\omega))$. Let $P(\omega)$ be the projection of $H(\omega)$ onto $\mathcal{M}(\omega)$, then the operator field $\{P(\omega)\}$ is continuous. In fact, for each $\xi \in H$ and $\eta \in \mathcal{M}$, $\{P(\omega)\xi(\omega)\}$ is an element of $\prod_{\omega\in\Omega}\mathcal{M}(\omega)$ and we have: for each $\omega \in \Omega$,

$$(P(\omega)\xi(\omega) \mid \eta(\omega)) = (\xi(\omega) \mid P(\omega)\eta(\omega)) = (\xi(\omega) \mid \eta(\omega)) .$$

Thus the function $\omega \to (P(\omega)\xi(\omega) \mid \eta(\omega))$ is continuous on Ω, and so the vector field $\{P(\omega)\xi(\omega)\}$ is an element of $\mathcal{M}$ by Theorem 1.5. Hence, for each $\xi, \eta \in H$, the function $\omega \to (P(\omega)\xi(\omega) \mid \eta(\omega))$ is continuous on Ω. Therefore $P = \{P(\omega)\}$ is continuous and it is a projection such that $PH = \mathcal{M}$.

Conversely, let $P = \{P(\omega)\}$ be a projection on $H = C_F(\Omega, H(\omega))$ and put $\mathcal{M} = PH$, then $\mathcal{M}$ is a closed submodule of H because P is an element of $B(H)$. If $\xi = \{\xi(\omega)\}$ is an element of $\prod_{\omega\in\Omega}\mathcal{M}(\omega)$ such that the function $\omega \to \|\xi(\omega)\|$ is bounded and, for each $\eta \in \mathcal{M}$, the function $\omega \to (\xi(\omega) \mid \eta(\omega))$ is continuous on Ω; then since $P\eta \in \mathcal{M}$ for every $\eta \in H$, for each $\eta \in H$, the function

$$\omega \longrightarrow (\xi(\omega) \mid (P\eta)(\omega)) = (\xi(\omega) \mid P(\omega)\eta(\omega))$$

is continuous on Ω. Furthermore, since $\xi(\omega) \in \mathscr{M}(\omega)$ for every $\omega \in \Omega$,

$$(\xi(\omega) \mid P(\omega)\eta(\omega)) = (P(\omega)\xi(\omega) \mid \eta(\omega)) = (\xi(\omega) \mid \eta(\omega)) .$$

This shows that the function $\omega \longrightarrow (\xi(\omega) \mid \eta(\omega))$ is continuous on Ω, and so $\xi \in H$. Hence, since $\xi \in H$ and $P\xi=\xi$, $\xi \in \mathscr{M}$. Therefore $\mathscr{M}$ is a continuous submodule of H.

2. The s-closed operators in continuous fields of Hilbert spaces. Let $H_1 = C_{F_1}(\Omega, H_1(\omega))$ and $H_2 = C_{F_2}(\Omega, H_2(\omega))$ be continuous fields of Hilbert spaces over a Stonean space Ω. Let $H_1 \oplus H_2$ be the direct sum of H_1 and H_2. We define, for each $\{\xi_1, \xi_2\}$ and $\{\eta_1, \eta_2\}$ in $H_1 \oplus H_2$ and $z \in C(\Omega)$,

$$(\{\xi_1, \xi_2\}, \{\eta_1, \eta_2\}) = (\xi_1, \eta_1) + (\xi_2, \eta_2)$$

and

$$z\{\xi_1, \xi_2\} = \{z\xi_1, z\xi_2\} .$$

Then $H_1 \oplus H_2$ is a $C(\Omega)$-moduled Banach space with respect to the norm

$$\|\{\xi_1, \xi_2\}\| = \|\,|\{\xi_1, \xi_2\}|\,\| = \|\,|\xi_1|^2 + |\xi_2|^2\|^{1/2}$$

and has an inner product on $C(\Omega)$ where $|\{\xi_1, \xi_2\}| = \{|\xi_1|^2 + |\xi_2|^2\}^{1/2}$. Furthermore, we can show that $H_1 \oplus H_2$ is a continuous field of Hilbert spaces over Ω with a fundamental subspace $F_1 \oplus F_2$. In fact, $F_1 \oplus F_2$ and $H_1 \oplus H_2$ are subspace of $\prod_{\omega \in \Omega} (H_1(\omega) \oplus H_2(\omega))$.

The following result shows the above consideration. The proof of this assertion is shown by considering the properties of continuous field of Hilbert spaces and the direct sum, and so we omit the proof.

PROPOSITION 2.1. *Let $H_1 = C_{F_1}(\Omega, H_1(\omega))$ and $H_2 = C_{F_2}(\Omega, H_2(\omega))$ be continuous fields of Hilbert spaces over Ω, then $H_1 \oplus H_2$ is a continuous field of Hilbert spaces over Ω (defined respect to $F_1 \oplus F_2$).*

Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω and A be an arbitrary operator in H (not necessarily bounded). Throughout in the remainder of this paper, we suppose that the domain $\mathscr{D}(A)$ of A is submodule and A is $C(\Omega)$-module homomorphism on $\mathscr{D}(A)$.

We have a similar notation to operator theory in Hilbert spaces.

DEFINITION 2.2. Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω and A an operator in H with the domain $\mathscr{D}(A)$. The graph of A is the set $G(A)$ of all pairs of vectors $\{\xi, \eta\}$ in the direct sum $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$ such that $\eta = A\xi$ for $\xi \in \mathscr{D}(A)$. A is closed if and only if the relations

$$\xi_n \in \mathscr{D}(A),\ \lim_{n\to\infty} \xi_n = \xi,\ \lim_{n\to\infty} A\xi_n = \eta$$

imply that $\xi \in \mathscr{D}(A)$ and $A\xi = \eta$.

Let $\{\xi_n\}$ and $\{\eta_n\}$ in $\mathscr{D}(A)$ be two sequences such that, if $\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \eta_n$ and both limits of two sequences $\{A\xi_n\}$ and $\{A\eta_n\}$ exist, then $\lim_{n\to\infty} A\xi_n = \lim_{n\to\infty} A\eta_n$. Then A has a closed extension. Among these is the so-called minimal closed extension, which is contained in every closed extension of the operator A. The minimal closed extension is uniquely defined for each operator A. It is denoted by $\bar{A}$ and is called the closure of A. In order to obtain $\bar{A}$, it is sufficient to adjoint to $\mathscr{D}(A)$ all those elements $\xi \in \mathscr{D}(\bar{A})$ which are limit of sequence $\{\xi_n\}$ in $\mathscr{D}(A)$ such that there exists the limit of sequence $\{A\xi_n\}$, and to require that $A\xi = \lim_{n\to\infty} A\xi_n$. Then the closure of $G(A)$ in $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$ is $G(\bar{A})$.

Let A be an operator in H with the domain $\mathscr{D}(A)$. Put

$$\tilde{\mathscr{D}}(A) = \{\xi = \sum e_\alpha \xi_\alpha\colon\ \{\xi_\alpha\} \text{ is a bounded subset of } \mathscr{D}(A) \text{ such that } \{A\xi_\alpha\} \text{ is a bounded subset and } \{e_\alpha\} \text{ are orthogonal projections in } C(\Omega) \text{ with } \sup e_\alpha = I\}\,,$$

and

$$\hat{\mathscr{D}}(A) = \{\xi = \sum e_\alpha \xi_\alpha\colon\ \{\xi_\alpha\} \text{ is a bounded subset of } \mathscr{D}(A) \text{ and } \{e_\alpha\} \text{ are orthogonal projections in } C(\Omega) \text{ with } \sup e_\alpha = I\}\,,$$

then both subsets $\tilde{\mathscr{D}}(A)$ and $\hat{\mathscr{D}}(A)$ are submodules of H by the following result.

LEMMA 2.3. *Let A be an operator with the domain $\mathscr{D}(A)$ in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces over Ω. Then both subsets $\tilde{\mathscr{D}}(A)$ and $\hat{\mathscr{D}}(A)$ are submodules of H.*

PROOF. If we can show that $\tilde{\mathscr{D}}(A)$ is a submodule of H, then we can show by the same way that $\hat{\mathscr{D}}(A)$ is a submodule of H. Thus, we show that $\tilde{\mathscr{D}}(A)$ is a submodule of H. For any $\xi = \sum e_\alpha \xi_\alpha$ and $\eta = \sum f_\beta \eta_\beta$ in $\tilde{\mathscr{D}}(A)$, $\xi = \eta$ if and only if $e_\alpha f_\beta \xi_\alpha = e_\alpha f_\beta \eta_\beta$ for any α, β. Since it is evident that $\tilde{\mathscr{D}}(A)$ is $C(\Omega)$-module, we show that $\tilde{\mathscr{D}}(A)$ is linear. Since the set $\{\xi_\alpha + \eta_\beta\}_{\alpha,\beta}$ is bounded and $\{e_\alpha f_\beta\}_{\alpha,\beta}$ are orthogonal projections with $\sup_{\alpha,\beta} e_\alpha f_\beta = I$, we put $\zeta = \sum e_\alpha f_\beta(\xi_\alpha + \eta_\beta)$ in $\tilde{\mathscr{D}}(A)$. For any α, β,

$$e_\alpha f_\beta \zeta = e_\alpha f_\beta(\xi_\alpha + \eta_\beta) = e_\alpha f_\beta \xi_\alpha + e_\alpha f_\beta \eta_\beta$$
$$= e_\alpha f_\beta \xi + e_\alpha f_\beta \eta = e_\alpha f_\beta(\xi + \eta) .$$

Thus $\zeta = \xi + \eta$ and so $\xi + \eta$ is an element in $\tilde{\mathscr{D}}(A)$.

Under the assumption in Lemma 2.3, put, for every $\xi = \sum e_\alpha \xi_\alpha$ in $\tilde{\mathscr{D}}(A)$,

$$\tilde{A}\xi = \tilde{A}(\sum e_\alpha \xi_\alpha) = \sum e_\alpha A \xi_\alpha ,$$

then $\tilde{A}$ is an operator in $H = C_F(\Omega, H(\omega))$ with the domain $\tilde{\mathscr{D}}(A)$. We must show that $\tilde{A}$ is well-defined. In fact, if $\xi = \sum e_\alpha \xi_\alpha = \sum f_\beta \eta_\beta$, $e_\alpha f_\beta \xi_\alpha = e_\alpha f_\beta \eta_\beta$ for any α, β. Thus, we have

$$e_\alpha f_\beta A \xi_\alpha = A e_\alpha f_\beta \xi_\alpha = A e_\alpha f_\beta \eta_\beta = e_\alpha f_\beta A \eta_\beta$$

and so $\sum e_\alpha A \xi_\alpha = \sum f_\beta A \eta_\beta$.

Furthermore, we have the following fact.

LEMMA 2.4. *Under the assumption in Lemma* 2.3 *and the notation before the lemma, suppose that $\tilde{A}$ is a closed operator; then the graph $G(\tilde{A})$ of $\tilde{A}$ is a continuous submodule of $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$.*

PROOF. By the closedness of $\tilde{A}$ and Lemma 2.3, $G(\tilde{A})$ is a closed submodule. Thus, we must show by Theorem 1.5 that if $\{\{\xi_\alpha, A\xi_\alpha\}\}$ is a bounded subset in $G(A)$ and $\{e_\alpha\}$ are orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I$, then $\sum e_\alpha \{\xi_\alpha, A\xi_\alpha\}$ is an element of $G(\tilde{A})$. Since ξ_α is an element in $\tilde{\mathscr{D}}(A) = \mathscr{D}(\tilde{A})$, we can represent ξ_α by $\xi_\alpha = \sum e^\alpha_{\iota_\alpha} \eta^\alpha_{\iota_\alpha}$ such that $\{\eta^\alpha_{\iota_\alpha}\}$ is a bounded subset of $\mathscr{D}(A)$ with the upperbound $\|\xi_\alpha\|$ and $\{e^\alpha_{\iota_\alpha}\}$ are orthogonal projections in $C(\Omega)$ with $\sup e^\alpha_{\iota_\alpha} = I$. Then $\sum e_\alpha \xi_\alpha = \sum e_\alpha e^\alpha_{\iota_\alpha} \eta^\alpha_{\iota_\alpha}$ and $\{e_\alpha e^\alpha_{\iota_\alpha}\}_{\alpha,\iota_\alpha}$ are orthogonal projections in $C(\Omega)$ with $\sup e_\alpha e^\alpha_{\iota_\alpha} = I$. Thus, $\sum e_\alpha \xi_\alpha$ is in $\tilde{\mathscr{D}}(A)$ and $\tilde{A}(\sum e_\alpha \xi_\alpha) = \sum e_\alpha \tilde{A} \xi_\alpha$. Thus, $G(\tilde{A})$ is a continuous submodule of $H \oplus H$.

The proof of the following lemma is trivial and is left to the readers.

LEMMA 2.5. *Let $\mathscr{M}$ be a submodule of a continuous field $H = C_F(\Omega, H(\omega))$ over Ω such that, for each bounded subset $\{\xi_\alpha\}$ in $\mathscr{M}$ and family $\{e_\alpha\}$ of orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I$, $\sum e_\alpha \xi_\alpha$ is in $\mathscr{M}$; then the norm closure $\bar{\mathscr{M}}$ of $\mathscr{M}$ is a continuous submodule of H.*

From the above lemmas, let A be an operator in a continuous field $H = C_F(\Omega, H(\omega))$ with the domain $\mathscr{D}(A)$, then the closure $\overline{G(\tilde{A})}$ of the graph $G(\tilde{A})$ of $\tilde{A}$ defined before Lemma 2.4 is a continuous submodule of $H \oplus H$ by Lemmas 2.3 and 2.4. If $\tilde{A}$ has a closed extension, then the graph $G(\bar{\tilde{A}})$ of the closure $\bar{\tilde{A}}$ of $\tilde{A}$ is the closure $\overline{G(\tilde{A})}$ of $G(\tilde{A})$. Thus, if $\tilde{A}$ has a closed extension, the graph of $\bar{\tilde{A}}$ is a continuous submodule of $H \oplus H$. Furthermore, if there exists a closed operator B in H such that $B \supset A$ and the graph $G(B)$ of B is a continuous submodule of $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$, then it is evident by the definition of $\tilde{A}$ that $B \supset \tilde{A}$. Thus, $\tilde{A}$ has a closed extension and $B \supset \bar{\tilde{A}}$. Therefore, if $\tilde{A}$ has a closed extension, then the closure $\bar{\tilde{A}}$ is the minimal extension among the operator B such that $B \supset A$ and the graph $G(B)$ of B is a continuous submodule of $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$.

From the above considerations, we have the following definition.

DEFINITION 2.6. Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over Ω and A an operator in H with the domain $\mathscr{D}(A)$. A is s-closed if and only if the graph $G(A)$ of A is a continuous submodule of $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$. If $\tilde{A}$ has a closed extension, then we say that A has an s-closed extension. The minimal s-closed extension is uniquely defined by $\bar{\tilde{A}}$ and is called the s-closure of A. It is denoted by $\bar{A}^s$.

Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces and A an operator in H with the domain $\mathscr{D}(A)$. If there exist vectors η and η^* in H such that $(A\xi, \eta) = (\xi, \eta^*)$ for every $\xi \in \mathscr{D}(A)$, then we put $A^*\eta = \eta^*$ and call the adjoint operator of A. If $\hat{\mathscr{D}}(A)$ is dense in H, then A has the adjoint operator A^*. If the adjoint operator A^* exists, then A^* is an s-closed operator by the following result.

PROPOSITION 2.7. *Let A be an operator in a continuous field $H = C_F(\Omega, H(\omega))$ such that $\hat{\mathscr{D}}(A)$ is dense in H, then the graph $G(A^*)$ of the adjoint operator A^* of A is a continuous submodule of $H \oplus H$.*

PROOF. Let $\{\{\xi_\alpha, A^*\xi_\alpha\}\}$ be a bounded subset in $G(A^*)$ and $\{e_\alpha\}$ are orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I$. Let G_α be the closed and open set in Ω corresponding to e_α. Put $\xi = \sum e_\alpha \xi_\alpha$. Then for each $\eta \in \mathscr{D}(A)$ and $\omega \in G_{\alpha_0}$, we have

$$\begin{aligned}(A\eta, \xi)(\omega) &= (A\eta, \textstyle\sum e_\alpha \xi_\alpha)(\omega) = (A\eta, e_{\alpha_0}\xi_{\alpha_0})(\omega) \\ &= (A\eta, \xi_{\alpha_0})(\omega) = (\eta, A^*\xi_{\alpha_0})(\omega) \\ &= (\eta, e_{\alpha_0}A^*\xi_{\alpha_0})(\omega) = (\eta, \textstyle\sum e_\alpha A^*\xi_\alpha)(\omega)\,.\end{aligned}$$

Thus $\xi \in \mathscr{D}(A^*)$ and $A^*\xi = A^*(\sum e_\alpha \xi_\alpha) = \sum e_\alpha A^* \xi_\alpha$. Therefore, $\sum e_\alpha\{\xi_\alpha, A^*\xi_\alpha\} = \{\sum e_\alpha \xi_\alpha, \sum e_\alpha A^*\xi_\alpha\}$ is an element of $G(A^*)$. This shows that $G(A^*)$ is a continuous submodule of $H \oplus H$.

By Proposition 2.7, $A^* = \tilde{A}^*$.

Let A be an operator in a continuous field $H = C_F(\Omega, H(\omega))$ with the domain $\mathscr{D}(A)$ such that $\hat{\mathscr{D}}(A)$ is dense in H, that is, the adjoint operator A^* of A exists. We have the following considerations similar to one in operator theory in Hilbert spaces. We now define an operator U on $H \oplus H$ by

$$U\{\xi, \eta\} = \{i\eta, -i\xi\} .$$

Then, the operator U is a unitary operator and $U^2 = I$. Put $G'(\tilde{A}) = UG(\tilde{A})$, then $G(A^*) = (H \oplus H) \ominus G'(\tilde{A})$. Hence, applying the operator U, we get

$$H \oplus H = \overline{G(\tilde{A})} \oplus UG(A^*) = \overline{G(\tilde{A})} \oplus G'(A^*)$$

where $\tilde{A}$ is an operator defined the remark before Lemma 2.4 and $\overline{G(\tilde{A})}$ is the closure of $G(\tilde{A})$. Thus the adjoint operator A^{**} of A^* exists if A has the s-closure.

From the above considerations, we have the following result.

PROPOSITION 2.8. *Let A be an operator in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces such that $\hat{\mathscr{D}}(A)$ is dense in H. Suppose $\tilde{A}$ has the closure (that is, A has the s-closure), then A^{**} exists and $G(\bar{A}^s) = \overline{G(\tilde{A})} = G(A^{**})$.*

COROLLARY 2.9. *Let A be an operator in $H = C_F(\Omega, H(\omega))$ such that $\hat{\mathscr{D}}(A)$ is dense in H and A has the s-closure. Then A is s-closed if and only if $A = \tilde{A} = A^{**}$.*

3. The characteristic matrices of s-closed operators. Let K be a Hilbert space. Each bounded operator S on $K \oplus K$ is uniquely expressible, through the relation

$$S\colon \quad \{\xi_1, \xi_2\} \longrightarrow \{S_{11}\xi_1 + S_{12}\xi_2, S_{21}\xi_1 + S_{22}\xi_2\}$$

is terms of a 2×2 matrix (S_{ij}) of bounded linear operators on K.

Let T be a closed operator in K and P the projection of $K \oplus K$ onto the graph $G(T)$. Then, in [5], Stone called the matrix (P_{ij}) of P the characteristic matrix of T.

In the case of continuous fields of Hilbert spaces, we have the following properties similar to the results mentioned in the preceding sentence. The following result can be shown by the way similar to the way in Hilbert spaces and so we leave its proof to the readers.

PROPOSITION 3.1. *Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over Ω. Let S be a bounded operator on $H \oplus H = C_{F \oplus F}(\Omega, H(\omega) \oplus H(\omega))$, then S can be uniquely expressible by 2×2 matrix (S_{ij}) of bounded operators on $H = C_F(\Omega, H(\omega))$, through the relation*

$$S: \quad \{\xi_1, \xi_2\} \to \{S_{11}\xi_1 + S_{12}\xi_2, S_{21}\xi_1 + S_{22}\xi_2\}$$

for every $\xi_1, \xi_2 \in H = C_F(\Omega, H(\omega))$.

Thus we denote $S = (S_{ij})$. If $\mathscr{M}$ is a continuous submodule of $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$, then the projection P of $H \oplus H$ onto $\mathscr{M}$ has the matrix representation (P_{ij}). Hence we have the following result which extend the characteristic matrix in Hilbert spaces to one of continuous fields of Hilbert spaces.

PROPOSITION 3.2. *Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces. Let $\mathscr{M}$ be a continuous submodule of $H \oplus H$ and P the projection of $H \oplus H$ onto $\mathscr{M}$ with the matrix representation $P = (P_{ij})$. Then $\mathscr{M}$ is the graph of an s-closed operator A if and only if $P_{12}\xi = (I - P_{22})\xi = 0$ implies $\xi = 0$. The operator A is necessarily the uniquely determined s-closed operator, which can be described as the mapping*

$$A: \quad P_{11}\xi_1 + P_{12}\xi_2 \to P_{21}\xi_1 + P_{22}\xi_2 ,$$

where ξ_1 and ξ_2 are arbitrary elements of H. Hence $P_{21} = AP_{11}$ and $P_{22} = AP_{12}$.

PROOF. (Sufficiency): If $\{0, \xi\} \in \mathscr{M}$, then $\{0, \xi\} = P\{0, \xi\} = \{P_{12}\xi, P_{22}\xi\}$. This implies that $P_{12}\xi = (I - P_{22})\xi = 0$. By the assumption, $\xi = 0$. Next, define an operator A in H;

$$A: \quad P_{11}\xi_1 + P_{12}\xi_2 \to P_{21}\xi_1 + P_{22}\xi_2$$

where ξ_1 and ξ_2 are elements in H. Then if $P_{11}\xi_1 + P_{12}\xi_2 = 0$,

$$\begin{aligned} P\{\xi_1, \xi_2\} &= \{P_{11}\xi_1 + P_{12}\xi_2, P_{21}\xi_1 + P_{22}\xi_2\} \\ &= \{0, P_{21}\xi_1 + P_{22}\xi_2\} \in \mathscr{M} . \end{aligned}$$

Hence $P_{21}\xi_1 + P_{22}\xi_2 = 0$, that is, A is well-defined. Since $\mathscr{M}$ is a continuous submodule in $H \oplus H$, A is an s-closed operator. It is evident that $P_{21} = AP_{11}$ and $P_{22} = AP_{12}$.

(Necessity): Since $A(P_{11}\xi_1 + P_{12}\xi_2) = P_{21}\xi_1 + P_{22}\xi_2$ for every $\xi_1, \xi_2 \in H$, if $P_{12}\xi = (I - P_{22})\xi = 0$, then the relation $P\{0, \xi\} = \{0, \xi\} \in G(A)$ implies $\xi = 0$.

COROLLARY 3.3. *A 2×2 matrix $P = (P_{ij})$ of bounded operators on $H \oplus H$ is the projection of $H \oplus H$ onto the graph $G(A)$ of an s-closed operator A in H if and only if it satisfies the relations*

(1) $P_{ij}^* = P_{ji} \quad (i, j = 1, 2)$;

(2) $\sum_{k=1}^{2} P_{ik}P_{kj} = P_{ij} \quad (i, j = 1, 2)$;

(3) $N(I - P_{22}) = \{0\}$

where $N(T)$ denotes the null space of an operator T.

DEFINITION 3.4. The matrix (P_{ij}) of the projection P of $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$ onto the graph $G(A)$ of s-closed operator A is called the characteristic matrix of the operator A.

Then we have the following result. We can show by the way similar to one in Hilbert spaces, and so we omit the proof.

PROPOSITION 3.5. *Let A be an s-closed operator such that $\hat{\mathscr{D}}(A)$ is dense in $H = C_F(\Omega, H(\omega))$. Let (P_{ij}) and (Q_{ij}) be the characteristic matrices of A and its adjoint A^* respectively. Then*

$$Q_{11} = I - P_{22},\ Q_{12} = P_{21},\ Q_{21} = P_{12},\ Q_{22} = I - P_{11}\ .$$

Furthermore, we have the following result. Its proof is evident and is left to the readers.

PROPOSITION 3.6. *Let (P_{ij}) be the characteristic matrix of an s-closed operator A in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces. If A^{-1} exists, it is s-closed and its characteristic matrix (Q_{ij}) satisfies the relations*

$$Q_{11} = P_{22},\ Q_{12} = P_{21},\ Q_{21} = P_{12},\ Q_{22} = P_{11}\ .$$

4. Decomposable operators in continuous fields of Hilbert spaces. Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω. For every $\omega \in \Omega$ let $A(\omega)$ be a closed linear operator in $H(\omega)$, then the mapping $\omega \rightarrow A(\omega)$ (or denoted by $\{A(\omega)\}$) will be called a field of closed operators on Ω or simply a field of operators on Ω.

In this section, we shall argue continuous fields of closed operators and decomposable operators in continuous fields of Hilbert spaces.

LEMMA 4.1. *Let $S = \{S(\omega)\}$ be a bounded, self-adjoint operator on $H = C_F(\Omega, H(\omega))$ such that $S(\omega)$ is one-to-one for every $\omega \in \Omega$. If $\xi = \{\xi(\omega)\} \in H = C_F(\Omega, H(\omega))$ is an arbitrary element such that $\xi(\omega) \in \mathscr{D}(S(\omega)^{-1})$ for every $\omega \in \Omega$ and the function $\omega \rightarrow \| S(\omega)^{-1}\xi(\omega) \|$ is bounded, then the field $\omega \rightarrow S(\omega)^{-1}\xi(\omega)$ is a continuous vector field (that is, the field $\{S(\omega)^{-1}\xi(\omega)\}$ is an element of H).*

PROOF. Since $S(\omega)$ is self-adjoint and one-to-one, $\overline{\mathscr{R}(S(\omega))} = H(\omega)$ for every $\omega \in \Omega$. Thus, for each $\xi \in H$, any positive number ε and $\omega \in \Omega$, there exist an element η_ω of $\mathscr{R}(S)$ and a closed and open neighborhood $U(\omega)$ of ω such that

$$\| \xi(\omega') - \eta_\omega(\omega') \| < \varepsilon \qquad \text{for every } \omega' \in U(\omega)\ .$$

Hence there exists a family $\{\eta_\omega, U(\omega)\}_{\omega \in \Omega}$ of pairs of elements in $\mathscr{R}(S)$

and closed and open neighborhood of each ω satisfying

$$\|\xi(\omega') - \eta_\omega(\omega')\| < \varepsilon \qquad \text{for every } \omega' \in U(\omega) .$$

Considering the open covering $\{U(\omega); \omega \in \Omega\}$ of Ω, there exists a finite subcovering $\{U(\omega_i); i = 1, 2, \cdots, n\}$ of $\{U(\omega); \omega \in \Omega\}$. We can suppose that $\{U(\omega_i); i = 1, 2, \cdots, n\}$ are mutually disjoint. Put z_i the projection in $C(\Omega)$ corresponding to $U(\omega_i)$. Then let $\eta = \sum_{i=1}^n z_i \eta_{\omega_i}$, η is an element of $\mathscr{R}(S)$ and satisfies

$$\|\xi(\omega) - \eta(\omega)\| < \varepsilon \qquad \text{for every } \omega \in \Omega .$$

Therefore ξ is an element of $\overline{\mathscr{R}(S)}$. Thus, to prove that the vector field $\{S(\omega)^{-1}\xi(\omega)\}$ is continuous, we show that, for each $\eta \in \mathscr{R}(S)$, the function $\omega \longrightarrow (S(\omega)^{-1}\xi(\omega) \mid \eta(\omega))$ is continuous. For each $\eta = \{\eta(\omega)\} \in \mathscr{R}(S)$, there exists an element $\zeta = \{\zeta(\omega)\}$ in H such that $S\zeta = \eta$, thus we get the relation

$$(S(\omega)^{-1}\xi(\omega) \mid \eta(\omega)) = (S(\omega)^{-1}\xi(\omega) \mid S(\omega)\zeta(\omega)) = (\xi(\omega) \mid \zeta(\omega)) .$$

Therefore, the function $\omega \longrightarrow (S(\omega)^{-1}\xi(\omega) \mid \eta(\omega))$ is continuous.

LEMMA 4.2. *Let $\{A(\omega)\}$ be a field of closed operators over Ω and, for each $\omega \in \Omega$, $(P_{ij}(\omega))$ the characteristic matrix of $A(\omega)$. Suppose that the operator fields $\{P_{ij}(\omega)\}$ $(i, j = 1, 2)$ are continuous. Then if $\xi = \{\xi(\omega)\}$ is an element of $H = C_F(\Omega, H(\omega))$ such that $\xi(\omega) \in \mathscr{D}(A(\omega))$ for all $\omega \in \Omega$ and the function $\omega \longrightarrow \|A(\omega)\xi(\omega)\|$ is bounded, then $\{A(\omega)\xi(\omega)\}$ is an element of H.*

PROOF. Since the operator fields $P_{ij} = \{P_{ij}(\omega)\}$ $(i, j = 1, 2)$ are continuous, every P_{ij} is an element of $B(H)$. By the definition of ξ, we get

$$A(\omega)\xi(\omega) = P_{21}(\omega)\xi(\omega) + P_{22}(\omega)A(\omega)\xi(\omega) .$$

Thus $(I(\omega) - P_{22}(\omega))A(\omega)\xi(\omega) = P_{21}(\omega)\xi(\omega)$ for each $\omega \in \Omega$. Since the vector fields $\{P_{ij}(\omega)\xi(\omega)\}$ $(i, j = 1, 2)$ are continuous and $(I(\omega) - P_{22}(\omega))$ is one-to-one and self-adjoint for all $\omega \in \Omega$ by Proposition 3.2, the vector field $\{A(\omega)\xi(\omega)\} = \{(I(\omega) - P_{22}(\omega))^{-1}P_{21}(\omega)\xi(\omega)\}$ is continuous by Lemma 4.1.

THEOREM 4.3. *Let $H = C_F(\Omega, H(\omega))$ be a continuous field of Hilbert spaces over a Stonean space Ω. Let $\{A(\omega)\} \in \prod_{\omega \in \Omega} B(H(\omega))$ be an operator field such that the function $\omega \longrightarrow \|A(\omega)\|$ is bounded. Let $(P_{ij}(\omega))$ be the characteristic matrix of $A(\omega)$ for each $\omega \in \Omega$. Then the operator field $\{A(\omega)\}$ is continuous if and only if the operator fields $\{P_{ij}(\omega)\}$ $(i, j = 1, 2)$ are continuous.*

PROOF. We showed the sufficiency of theorem in Lemma 4.2. Thus, we show the proof of the necessity of theorem. Put $A = \{A(\omega)\}$, then A is an element of $B(H)$ and two operator fields $\{A^*(\omega)\}$ and $\{I(\omega)+A^*(\omega)A(\omega)\}$ are continuous. Furthermore, $A^*(\omega) = A(\omega)^*$, and two functions $\omega \to \| A(\omega)^* \|$ and $\omega \to \| I(\omega) + A(\omega)^* A(\omega) \|$ are bounded. Hence, since $P_{11}(\omega) = (I(\omega) + A(\omega)^* A(\omega))^{-1}$, by Proposition 3.2 and Corollary 3.3,

$$P_{21}(\omega) = A(\omega)P_{11}(\omega),\ P_{12}(\omega) = P_{21}(\omega)^* \quad \text{and} \quad P_{22}(\omega) = A(\omega)P_{12}(\omega)\ .$$

Therefore the operator fields $\{P_{ij}(\omega)\}$ $(i, j = 1, 2)$ are continuous.

By Theorem 4.3, we give the following definition in which we define the continuity of fields of closed operators.

DEFINITION 4.4. A field $\{A(\omega)\}$ of closed operators is said to be continuous if the operator fields of characteristic matrices of $A(\omega)$ are continuous.

This definition is legitimate because it agrees with the definition of continuous fields $\{A(\omega)\}$ of bounded operators such that the function $\omega \to \| A(\omega) \|$ is bounded by Theorem 4.3.

Under the above mentioned definition, let $\{A(\omega)\}$ be a continuous field of closed operators and $(P_{ij}(\omega))$ the characteristic matrix of $A(\omega)$ for every $\omega \in \Omega$, then the fields $P_{ij} = \{P_{ij}(\omega)\}$ $(i, j = 1, 2)$ are continuous field of bounded operators. Define the set $\mathscr{M} = \{\sum e_\alpha \xi_\alpha; \{\xi_\alpha\}$ is a bounded subset in $H = C_F(\Omega, H(\omega))$ such that $\xi_\alpha(\omega) \in \mathscr{D}(A(\omega))$ for every $\omega \in \Omega$ and α and $\{A(\omega)\xi_\alpha(\omega)\}_{\alpha,\omega}$ is bounded, and $\{e_\alpha\}$ are mutually orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I\}$; then $\mathscr{M}$ is a submodule of H. Let G_α be the closed and open set in Ω corresponding to e_α, then, for each $\xi = \sum e_\alpha \xi_\alpha$ in $\mathscr{M}$ and $\omega \in G_\alpha$, we have the following equation

$$\begin{aligned} A(\omega)\xi(\omega) &= A(\omega)\xi_\alpha(\omega) = P_{21}(\omega)\xi_\alpha(\omega) + P_{22}(\omega)A(\omega)\xi_\alpha(\omega) \\ &= P_{21}(\omega)\xi(\omega) + P_{22}(\omega)A(\omega)\xi(\omega)\ . \end{aligned}$$

Define an operator A' as follows; for each $\xi = \{\xi(\omega)\}$ in H such that $\xi(\omega) \in \mathscr{D}(A(\omega))$ for every $\omega \in \Omega$ and $\{A(\omega)\xi(\omega)\}$ is bounded, $A'\xi = \eta$ where $\eta = \{A(\omega)\xi(\omega)\}$ because, by Lemma 4.2, $\eta = \{A(\omega)\xi(\omega)\}$ is an element of $H = C_F(\Omega, H(\omega))$. Then A' is a $C(\Omega)$-module homomorphism of the submodule $\{\xi \in H; \xi(\omega) \in \mathscr{D}(A(\omega))$ for every $\omega \in \Omega$ and $\{A(\omega)\xi(\omega)\}$ is bounded$\}$. Furthermore define an operator A on $\mathscr{M}$ as follows; for each $\xi = \sum e_\alpha \xi_\alpha \in \mathscr{M}$, $A\xi = \sum e_\alpha A'\xi_\alpha$. Then A is well-defined and A is a $C(\Omega)$-module homomorphism on $\mathscr{M}$. We have the following relations; for each $\xi = \sum e_\alpha \xi_\alpha \in \mathscr{M}$ and $\omega \in G_\alpha$,

$$A(\omega)\xi(\omega) = P_{21}(\omega)\xi(\omega) + P_{22}(\omega)A(\omega)\xi(\omega)$$

$$Ae_\alpha\xi = P_{21}e_\alpha\xi + P_{22}e_\alpha A\xi \quad \text{and} \quad A\xi = P_{21}\xi + P_{22}A\xi\ .$$

From the above considerations, we get the following result.

LEMMA 4.5. *Let* $\{A(\omega)\}$ *be a continuous field of closed operators and let* $(P_{ij}(\omega))$ *the characteristic matrix of* $A(\omega)$ *for every* $\omega \in \Omega$; *then there exists an s-closed operator* B *such that* $P = (P_{ij})$ *is the characteristic matrix of* B.

From Lemma 4.5, we have the following theorem.

THEOREM 4.6. *Let* $H = C_F(\Omega, H(\omega))$ *be a continuous field of Hilbert spaces and* $\{A(\omega)\}$ *a continuous field of closed operators; then* $A = B$ *where* A *is the operator defined before Lemma* 4.5 *and* B *is the operator determined in Lemma* 4.5. *Thus,* A *is an s-closed operator.*

PROOF. By the definition of the operator A, the domain $\mathscr{D}(A)$ of A is the submodule $\mathscr{M} = \{\sum e_\alpha \xi_\alpha;\ \{\xi_\alpha\}$ is a bounded subset in H such that $\xi_\alpha(\omega) \in \mathscr{D}(A(\omega))$ for each $\omega \in \Omega$ and α, and $\{A(\omega)\xi_\alpha(\omega)\}_{\alpha,\omega}$ is bounded, and $\{e_\alpha\}$ are mutually orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I\}$. Furthermore, since the operator B is an s-closed operator, the graph $G(B)$ of B is a continuous submodule of $H \oplus H = C_{F\oplus F}(\Omega, H(\omega) \oplus H(\omega))$. Thus, let $\{\xi_\alpha\}$ be a bounded subset of $\mathscr{D}(B)$ such that $\{B\xi_\alpha\}$ is a bounded set in H, and let $\{e_\alpha\}$ a family of mutually orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I$; then $\sum e_\alpha \xi_\alpha$ is an element of $\mathscr{D}(B)$ and $B(\sum e_\alpha \xi_\alpha) = \sum e_\alpha B\xi_\alpha$. For each $\xi \in \mathscr{D}(B)$, there exist two elements ξ_1 and ξ_2 in H such that $\xi = P_{11}\xi_1 + P_{12}\xi_2$. Since $\xi(\omega) = P_{11}(\omega)\xi_1(\omega) + P_{12}(\omega)\xi_2(\omega)$ is an element of $\mathscr{D}(A(\omega))$ and

$$\begin{aligned}(B\xi)(\omega) &= (P_{21}\xi_1)(\omega) + (P_{22}\xi_2)(\omega) = P_{21}(\omega)\xi_1(\omega) + P_{22}(\omega)\xi_2(\omega)\\ &= A(\omega)\xi(\omega)\end{aligned}$$

for every $\omega \in \Omega$. Thus, $B \subset A$.

Conversely, let ξ is an arbitrary element of $\mathscr{D}(A)$, then there exists a bounded set $\{\xi_\alpha\}$ of H such that $\xi_\alpha(\omega) \in \mathscr{D}(A(\omega))$ for every $\omega \in \Omega$ and α, and $\{A(\omega)\xi_\alpha(\omega)\}_{\alpha,\omega}$ is bounded, and the set $\{e_\alpha\}$ of mutually orthogonal projections in $C(\Omega)$ such that $\xi = \sum e_\alpha \xi_\alpha$. Now since $e_\alpha \xi = e_\alpha \xi_\alpha$ for each α and $P(\omega)\{\xi_\alpha(\omega), A(\omega)\xi_\alpha(\omega)\} = \{\xi_\alpha(\omega), A(\omega)\xi_\alpha(\omega)\}$ for every ω and α, $P\{\xi_\alpha, \eta_\alpha\} = \{\xi_\alpha, \eta_\alpha\}$ where $\eta_\alpha = \{A(\omega)\xi_\alpha(\omega)\} \in H = C_F(\Omega, H(\omega))$. Thus, ξ_α is an element of $\mathscr{D}(B)$ and $B\xi_\alpha = \eta_\alpha$ for each α. By the assumption of the boundedness of $\{A(\omega)\xi_\alpha(\omega)\}_{\alpha,\omega}$, the set $\{B\xi_\alpha\}$ is a bounded set. Thus $\xi = \sum e_\alpha \xi_\alpha$ is an element of $\mathscr{D}(B)$. Therefore, $A = B$.

From Theorem 4.6, if a field $\{A(\omega)\}$ of closed operators is continuous, we can considered $\{A(\omega)\}$ as an s-closed operator in H and so we denote $A = \{A(\omega)\}$.

Next, we shall consider whether s-closed operator can be represented as a continuous field of closed operators. We give the following definition which is used for the above mentioned consideration.

DEFINITION 4.7. Let A be an s-closed operator in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces. A is called a decomposable operator if there exists a continuous field $\{A(\omega)\}$ of closed operators such that $A = \{A(\omega)\}$ in the mean of the remark after Theorem 4.6.

We want to any s-closed operator be decomposable, but unfortunately we can give the following example in which we get an s-closed operator that is not decomposable.

EXAMPLE 4.8. Let $H = C_F(\beta N, H(\omega))$ be a continuous field of Hilbert spaces over βN where βN is the Stone-Čech compactification of the set of all positive integers. If we consider a von Neumann algebra of type I with the center $C(\beta N)$, we can show that there exists a continuous field of Hilbert spaces over βN mentioned in the above sentence. Let f be an element of $C(\beta N)$ such that $f(n) = 1/n$ for every $n \in N$ and $f(\omega) = 0$ for every $\omega \in \beta N \backslash N$ and $D = fI$ where I is the identity of $B(H)$, that is, for each $\xi = \{\xi(\omega)\} \in H = C_F(\beta N, H(\omega))$, $(D\xi)(\omega) = f(\omega)\xi(\omega)$ for every $\omega \in \Omega$; then D is a positive operator on H with $0 \leqq D \leqq 1$.

Under the above considerations, define an operator P on $H \oplus H$ as follows:

$$P = \begin{pmatrix} D & (D(I-D))^{1/2} \\ (D(I-D))^{1/2} & I-D \end{pmatrix},$$

further put

$$P_{11} = D\,, \quad P_{12} = P_{21} = (D(I-D))^{1/2}\,, \quad P_{22} = I - D\,.$$

Then we have the following relations;

(1) $P_{ij}^* = P_{ji} \quad (i, j = 1, 2)$,

(2) $\sum_{k=1}^{2} P_{ik}P_{kj} = P_{ij} \quad (i, j = 1, 2)$,

(3) $(I - P_{22})\xi = 0$ implies $\xi = 0$.

The proofs of (1) and (2) are trivial by the definition of the operators P_{ij} $(i, j = 1, 2)$ and so we show the proof of (3).

(3): If $(I - P_{22})\xi = 0$, then $D\xi = 0$. Thus $f(n)\xi(n) = (1/n)\xi(n) = 0$ for every $n \in N$, and so $\xi(n) = 0$ for every $n \in N$. Since the function $\omega \to \|\xi(\omega)\|$ is continuous and N is dense in βN, $\xi(\omega) = 0$ for every $\omega \in \beta N$. Therefore $(I - P_{22})\xi = 0$ implies $\xi = 0$.

By the above consideration and Corollary 3.3, there exists an s-closed operator A of which the characteristic matrix is P. If A is decomposable and we denote $A=\{A(\omega)\}$, then, by Theorem 4.6, the characteristic matrices of $A(\omega)$ must be $(P_{ij}(\omega))$. But, for every $\omega \in \beta N\backslash N$, we have

$$(I - P_{22})(\omega) = I(\omega) - P_{22}(\omega) = D(\omega) = 0 .$$

Thus, by Corollary 3.3, $(P_{ij}(\omega))$ is the characteristic matrix of no s-closed operator. Therefore, A is not decomposable.

In the above Example 4.8, we showed that there exists an s-closed operator A, even if the continuous submodule of $H = C_F(\beta N, H(\omega))$ generated by $\mathscr{D}(A)$ is H, A is not decomposable.

We show that arbitrary densely defined s-closed operator is decomposable. To show it, we have some considerations which can show by considering the properties of operator theory in Hilbert spaces and the characteristic matrix of closed operators, and so we omit the proof.

LEMMA 4.9. *Let $A = \{A(\omega)\}$ be a continuous field of closed operators with the adjoint operator A^* such that $\mathscr{R}(A^*)$ is dense in $H=C_F(\Omega, H(\omega))$. Then, for any $\omega \in \Omega$, $A(\omega)$ is one-to-one, thus A is one-to-one.*

LEMMA 4.10. *Let A be a densely defined s-closed operator in $H = C_F(\Omega, H(\omega))$, let (P_{ij}) be the characteristic matrix of A. Then $\mathscr{R}(I-P_{22}) = \mathscr{D}(I + AA^*) = \mathscr{D}(AA^*)$.*

By Lemmas 4.9 and 4.10, we have the following corollary.

COROLLARY 4.11. *Let A be a densely defined s-closed operator in $H = C_F(\Omega, H(\omega))$. Let (P_{ij}) be the characteristic matrix of A. Then $\mathscr{R}(I-P_{22})$ is dense in H and, for each $\omega \in \Omega$, $I(\omega) - P_{22}(\omega)$ is one-to-one.*

From the above considerations, we have the following theorem which says to any densely defined s-closed operator be decomposable.

THEOREM 4.12. *Let A be an s-closed operator in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces such that $\mathscr{D}(A)$ is dense in H, then A is decomposable.*

PROOF. Let (P_{ij}) be the characteristic matrix of A and $P_{ij} = (P_{ij}(\omega))$ $(i, j = 1, 2)$. Then $P_{ij}(\omega)$ $(i, j = 1, 2)$ are bounded operators on $H(\omega)$ for every $\omega \in \Omega$. By the properties of characteristic matrix, we have the following relations

(1) $P_{ij}(\omega)^* = P_{ij}^*(\omega) = P_{ji}(\omega) \quad (i, j = 1, 2)$,

(2) $\sum_{k=1}^{2} P_{ik}(\omega)P_{kj}(\omega) = P_{ij}(\omega) \quad (i, j = 1, 2)$.

Furthermore, by Corollary 4.11, $N(I(\omega) - P_{22}(\omega)) = \{0\}$. Thus, by Corollary 3.3, there exists a closed operator $A(\omega)$ for every $\omega \in \Omega$ such that the characteristic matrix of $A(\omega)$ is $(P_{ij}(\omega))$ for every $\omega \in \Omega$. Since for $i, j =$

1, 2, the fields $\{P_{ij}(\omega)\}$ are continuous, the field $\{A(\omega)\}$ is a continuous field of closed operators. Furthermore, we have the following relation;

$$A(\omega):\quad P_{11}(\omega)\xi_1(\omega) + P_{12}(\omega)\xi_2(\omega) \to P_{21}(\omega)\xi_1(\omega) + P_{22}(\omega)\xi_2(\omega)$$

for any $\xi_1, \xi_2 \in H$ and $\omega \in \Omega$. Therefore, the relation

$$A:\quad P_{11}\xi_1 + P_{12}\xi_2 \to P_{21}\xi_1 + P_{22}\xi_2\ ,$$

implies $A = \{A(\omega)\}$ and $\mathscr{D}(A(\omega)) = \{\xi(\omega); \xi = \{\xi(\omega)\} \in \mathscr{D}(A)\}$.

In Theorem 4.12, we supposed that an operator A is a densely defined s-closed operator. Then if we write as $A=\{A(\omega)\}$, $A(\omega)$ are densely defined closed operators. The following proposition shows that if every $A(\omega)$ is densely defined and the field $A = \{A(\omega)\}$ is continuous, then both operators A and A^* are densely defined.

PROPOSITION 4.13. *Let $A = \{A(\omega)\}$ be a continuous field of closed operators such that, for each $\omega \in \Omega$, $\mathscr{D}(A(\omega))$ is dense in $H(\omega)$, then $\mathscr{D}(A)$ and $\mathscr{D}(A^*)$ are dense in H.*

The above result is shown by considering the properties of the characteristic matrices and the continuous fields of Hilbert spaces and we omit the proof.

Considering the properties of the characteristic matrices of the s-closed operators, Theorem 4.12 and Proposition 4.13, we have the following corollaries.

COROLLARY 4.14. *Let $A = \{A(\omega)\}$ be a decomposable operator in $H = C_F(\Omega, H(\omega))$ with the dense domain $\mathscr{D}(A)$, then $A^*(\omega) = A(\omega)^*$ for every $\omega \in \Omega$.*

COROLLARY 4.15. *Let $A = \{A(\omega)\}$ and $B = \{B(\omega)\}$ be two decomposable operators. Then $A \subset B$ if and only if $A(\omega) \subset B(\omega)$ for every $\omega \in \Omega$.*

COROLLARY 4.16. *Let $A = \{A(\omega)\}$ be a densely defined decomposable operator, then A is symmetric if and only if $A(\omega)$ are symmetric.*

5. Applications of decomposable operators. In this section, we shall show that the square root for a densely defined self-adjoint, positive operator exists and if A is a densely defined s-closed operator then it can be written uniquely of the form $A = VS$ (the polar decomposition of A), where S is a self-adjoint, positive operator and V is a partially isometric operator and V is a partially isometric operator. We recall that a bounded operator B and an arbitrary operator A are said to be permutable if $BA \subset AB$. If $\mathfrak{A}$ is a $C(\Omega)$-moduled C^*-subalgebra of $B(H)$ with $\mathfrak{A} = \mathfrak{A}''$, then we define that an s-closed A is affiliated to $\mathfrak{A}$ (we denote $A_\eta\mathfrak{A}$) if

$BA \subset AB$ for every $B \in \mathfrak{A}$. Then, by [6; Theorem 5.7], we have a remark of a relation of the field $\{A(\omega)\}$ and the field $\{\mathfrak{A}(\omega)\}$.

Before the proof of the polar decomposition, we have some considerations. From the remark before Proposition 2.8, we have the following lemma.

LEMMA 5.1. *Let $A = \{A(\omega)\}$ be a densely defined s-closed operator in a continuous field $H = C_F(\Omega, H(\omega))$, then A^*A is self-adjoint and positive.*

PROOF. For each $\xi, \eta \in \mathscr{D}(A^*A)$, $(A^*A\xi, \eta) = (A\xi, A\eta) = (\xi, A^*A\eta)$ and $(A^*A\xi, \xi) = (A\xi, A\xi) \geqq 0$. Thus A^*A is a positive operator.

By Lemma 4.10, A^*A and $I + A^*A$ are symmetric. Now, since $\mathscr{R}(I + A^*A) = H$, the operator $I + A^*A$ is a self-adjoint operator. In fact, if B is a densely defined symmetric operator with $\mathscr{R}(B) = H$, then, for each $\eta \in \mathscr{D}(B^*)$, there exists an element $\xi \in \mathscr{D}(B)$ with $B\xi = B^*\eta$. Thus, for each $\zeta \in \mathscr{D}(B)$,

$$(B\zeta, \eta) = (\zeta, B^*\eta) = (B\zeta, \xi) .$$

Therefore since $\mathscr{R}(B) = H$, $\eta = \xi \in \mathscr{D}(A)$. Thus, $I + A^*A$ is a self-adjoint operator. Therefore A^*A is a self-adjoint operator.

Before the definition of square root of a self-adjoint, positive operator, we consider the following corollary by Lemma 5.1 and Corollary 4.14.

COROLLARY 5.2. *Let $A = \{A(\omega)\}$ be an s-closed operator such that $\overline{\mathscr{D}(A)} = H$, and let $A^* = \{A^*(\omega)\}$ the adjoint operator of A, then the field $\{A^*(\omega)A(\omega)\}$ is a continuous field of self-adjoint, positive operators and $A^*A = \{A^*(\omega)A(\omega)\}$.*

THEOREM 5.3. *Let $A = \{A(\omega)\}$ be a self-adjoint, positive operator in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces, then there exists uniquely the self-adjoint, positive operator $B = \{B(\omega)\}$ in $H = C_F(\Omega, H(\omega))$ such that $B^2 = A$.*

PROOF. Since $A(\omega)$ are self-adjoint, positive operator, for each $\omega \in \Omega$, there exists the square root $A(\omega)^{1/2}$ of $A(\omega)$ which is a self-adjoint, positive operator. We shall show that the field $\{A(\omega)^{1/2}\}$ is a continuous field of closed operators. Put, for each $\omega \in \Omega$,

$$Q_{11}(\omega) = (I(\omega) + A(\omega))^{-1} , \quad Q_{21}(\omega) = A(\omega)^{1/2}(I(\omega) + A(\omega))^{-1} ,$$
$$Q_{12}(\omega) = A(\omega)^{1/2}(I(\omega) + A(\omega))^{-1} \quad \text{and} \quad Q_{22}(\omega) = A(\omega)(I(\omega) + A(\omega))^{-1} .$$

Then, since the fields $\{A(\omega)\}$ and $\{I(\omega) + A(\omega)\}$ are continuous, the field $\{(I(\omega) + A(\omega))^{-1}\}$ is a continuous field of closed operators. Furthermore, since $\|(I(\omega) + A(\omega))^{-1}\| \leqq 1$, the field $\{Q_{12}(\omega)\}$ is a continuous field of bounded operators. By the same way, the field $\{Q_{22}(\omega)\}$ is continuous. Furthermore, by the relation $Q_{12}(\omega)^2 = A(\omega)(I(\omega) + A(\omega))^{-2}$, the field $\{Q_{12}(\omega)^2\}$ is continuous. Thus, there exists the positive operator S of $B(H)$ with $S = \{Q_{12}(\omega)^2\}$. Then $S^{1/2}(\omega) = Q_{12}(\omega)$. In fact, let $\{p_n\}_{n=1}^{\infty}$ be a sequence of polynomials such that $p_n(S) \to S^{1/2}$ as $n \to \infty$ (in the norm topology). Then $p_n(S)(\omega) = p_n(S(\omega)) = p_n(Q_{12}(\omega)^2)$, $p_n(S)(\omega) \to S^{1/2}(\omega)$ as $n \to \infty$ and $p_n(Q_{12}(\omega)^2) \to Q_{12}(\omega)$ as $n \to \infty$ for every $\omega \in \Omega$. Thus, $S^{1/2}(\omega) = Q_{12}(\omega)$ for every $\omega \in \Omega$. Therefore, the field $\{Q_{12}(\omega)\}$ is continuous and so the field $\{Q_{21}(\omega)\}$ is continuous and so the field $\{Q_{21}(\omega)\}$ is continuous. Next, we shall show that the field $\{A(\omega)^{1/2}\}$ is a continuous field of closed operators. From Proposition 3.2 and Definition 4.4, we say that the matrix $(Q_{ij}(\omega))$ is the characteristic matrix of $A(\omega)^{1/2}$. If $\xi(\omega)$ is an element such that $Q_{12}(\omega)\xi(\omega) = (I(\omega) - Q_{22}(\omega))\xi(\omega) = 0$, then

$$A(\omega)^{1/2}(I(\omega) + A(\omega))^{-1}\xi(\omega) = 0 \quad \text{and} \quad A(\omega)(I(\omega) + A(\omega))^{-1}\xi(\omega) = \xi(\omega) .$$

Then, $\xi(\omega) = A(\omega)(I(\omega) + A(\omega))^{-1}\xi(\omega) = A(\omega)^{1/2}A(\omega)^{1/2}(I(\omega) + A(\omega))^{-1}\xi(\omega) = 0$. Furthermore, for each ξ_1 and $\xi_2 \in H$, we have

$$\begin{aligned} &A(\omega)^{1/2}\{Q_{11}(\omega)\xi_1(\omega) + Q_{12}(\omega)\xi_2(\omega)\} \\ &\quad = A(\omega)^{1/2}\{(I(\omega) + A(\omega))^{-1}\xi_1(\omega) + A(\omega)^{1/2}(I(\omega) + A(\omega))^{-1}\xi_2(\omega)\} \\ &\quad = A(\omega)^{1/2}(I(\omega) + A(\omega))^{-1}\xi_1(\omega) + A(\omega)(I(\omega) + A(\omega))^{-1}\xi_2(\omega) \\ &\quad = Q_{21}(\omega)\xi_1(\omega) + Q_{22}(\omega)\xi_2(\omega) . \end{aligned}$$

Thus, $(Q_{ij}(\omega))$ is the characteristic matrix of $A(\omega)^{1/2}$ for every $\omega \in \Omega$ by Proposition 3.2. Put $Q_{ij} = \{Q_{ij}(\omega)\}$ $(i, j = 1, 2)$ and $B = \{A(\omega)^{1/2}\}$, then B is a self-adjoint, positive operator such that the characteristic matrix of B is (Q_{ij}). Furthermore, by Theorem 4.6, $B^2(\omega) = B(\omega)^2 = (A(\omega)^{1/2})^2 = A(\omega)$, and so $B^2 = A$. The unicity of existence is obvious.

By Lemma 5.1 and Theorem 5.3, we introduce the following definition.

DEFINITION 5.4. Let $A = \{A(\omega)\}$ be a densely defined s-closed operator. Then we denote the square root $(A^*A)^{1/2}$ of A^*A as $|A|$ and call the absolute value of A.

Then $|A|(\omega)^2 = (A^*A)(\omega) = A^*(\omega)A(\omega)$ and $|A|(\omega) = |A(\omega)|$ for every $\omega \in \Omega$. Furthermore, we can show by the elementary examination that $\mathscr{D}(|A|) = \mathscr{D}(A)$.

Next, we shall consider the polar decomposition of s-closed operators.

Let $A = \{A(\omega)\}$ be a densely defined s-closed operator in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces. Define two subsets H_1 and H_2 of H as follows;

$H_1 =$ the closure of $\{\sum e_\alpha |A| \xi_\alpha$: $\{|A|\xi_\alpha\}$ is bounded, $\{\xi_\alpha\} \subset \mathscr{D}(|A|)$ and $\{e_\alpha\}$ are orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I\}$,

$H_2 =$ the closure of $\{\sum e_\alpha A\xi_\alpha$: $\{A\xi_\alpha\}$ is bounded, $\{\xi_\alpha\} \subset \mathscr{D}(A)$ and $\{e_\alpha\}$ are orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I\}$.

Then both H_1 and H_2 are continuous submodules of H. In fact, let $\{\xi_\iota\}$ be a bounded set in $\{\sum e_\alpha |A| \xi_\alpha$: $\{|A|\xi_\alpha\}$ is bounded and $\{e_\alpha\}$ are orthogonal projections in $C(\Omega)$ with $\sup e_\alpha = I\}$ and $\{e_\iota\}$ are orthogonal projections in $C(\Omega)$ with $\sup e_\iota = I$, then each ξ_ι is represented as follows; $\xi_\iota = \sum_{\iota_\alpha} e'_{\iota_\alpha} |A| \xi'_{\iota_\alpha}$ where $\{|A|\xi'_{\iota_\alpha}\}_{\iota_\alpha}$ is bounded with upper bound $||\xi_\iota||$ and $\{e'_{\iota_\alpha}\}_{\iota_\alpha}$ are orthogonal projections in $C(\Omega)$ with $\sup_{\iota_\alpha} e'_{\iota_\alpha} = I$. Then $\sum_\iota e_\iota \xi_\iota = \sum_\iota \sum_{\iota_\alpha} e_\iota e'_{\iota_\alpha} |A| \xi'_{\iota_\alpha}$, $\{|A|\xi'_{\iota_\alpha}\}_{\iota,\iota_\alpha}$ is a bounded subset of H and $\{e_\iota e'_{\iota_\alpha}\}_{\iota,\iota_\alpha}$ are orthogonal projections in $C(\Omega)$ and $\sup e_\iota e'_{\iota_\alpha} = I$. Thus $\sum e_\iota \xi_\iota \in H_1$. Therefore, by Theorem 1.5, H_1 is a continuous submodule of H, similarily H_2 is also a continuous submodule of H.

Define an operator V' of H_1 to H_2 as follows; for each $\xi = \sum e_\alpha |A| \xi_\alpha$ in H_1,

$$V'(\sum e_\alpha |A| \xi_\alpha) = \sum e_\alpha A\xi_\alpha .$$

Then, we have

$$\begin{aligned} ||\sum e_\alpha |A| \xi_\alpha||^2 &= \sup_\alpha ||e_\alpha |A| \xi_\alpha||^2 \\ &= \sup_\alpha ||e_\alpha A\xi_\alpha||^2 = ||\sum e_\alpha A\xi_\alpha||^2 . \end{aligned}$$

Thus, the above defined V' is well-defined and V' is an isometrical operator. Hence we have the extension V'' of V' to H_1 that is an isometrical opepator of H_1 onto H_2. Let E_i $(i = 1, 2)$ be the projections of H onto H_i $(i = 1, 2)$ respectively, and let $V = E_2 V'' E_1$, then V is a partially isometrical operator on H with the initial domain H_1 and the final domain H_2. Furthermore, we can show that $A = V|A|$ and if H_1 and H_2 are fixed, V is determined uniquely. From the above consideration, we have the following theorem.

THEOREM 5.5. *Let $A = \{A(\omega)\}$ be a densely defined s-closed operator in a continuous field $H = C_F(\Omega, H(\omega))$ of Hilbert spaces over a Stonean space Ω, then A can be written uniquely of the form $A = V|A|$ where V is a partially isometrical operator with the initial domain $H_1 =$ the closure of $\{\sum e_\alpha |A| \xi_\alpha$: $\{|A|\xi_\alpha\}$ is bounded, $\{\xi_\alpha\} \subset \mathscr{D}(|A|)$ and $\{e_\alpha\}$ are* *orthogonal projections in* $C(\Omega)$ *with* $\sup e_\alpha = I\}$ *and the final domain* $H_2 =$ *the closure of* $\{\sum e_\alpha A\xi_\alpha$: $\{A\xi_\alpha\}$ *is bounded,* $\{\xi_\alpha\} \subset \mathscr{D}(A)$ *and* $\{e_\alpha\}$ *are orthogonal projections in* $C(\Omega)$ *with* $\sup e_\alpha = I\}$.

The representation of A in the above theorem is called the polar decomposition of A.

In Theorem 5.5, we gave the polar decomposition of a densely defined s-closed operator. Let A be a densely defined s-closed operator and $V|A|$ the polar decomposition of A, then we have a relation: $A(\omega) = V(\omega)\,|A|\,(\omega) = V(\omega)\,|A(\omega)|$ for every $\omega \in \Omega$. But, we cannot assert in general that $V(\omega)$ is the partially isometrical operator with the initial domain $\overline{\mathscr{R}(|A(\omega)|)}$ and the final domain $\overline{\mathscr{R}(A(\omega))}$. For example, let Ω be the spectrum space of $L^\infty(0, 1)$ and $H = C_F(\Omega, H(\omega))$ a continuous field of Hilbert spaces over Ω, then there exists a continuous function f on Ω such that $f(\omega_0) = 0$ and $f(\omega) \neq 0$ if $\omega \neq \omega_0$. Put $A = fI$ in $B(H)$, then $H_i = H$ $(i = 1, 2)$, but $|A(\omega_0)| = 0$ and so $A(\omega_0) = 0$.

If $\mathfrak{A}$ is a $C(\Omega)$-moduled C^*-subalgebra of $B(H)$ with $\mathfrak{A} = \mathfrak{A}''$ where $H = C_F(\Omega, H(\omega))$ is a continuous field of Hilbert spaces over Ω, then, by [6; Theorem 5.7], $\mathfrak{A} = C(\Omega, \widetilde{\mathfrak{A}(\omega)})$ where $\widetilde{\mathfrak{A}(\omega)}$ is the weak closure of $\mathfrak{A}(\omega) = \{A(\omega)$: $A \in \mathfrak{A}$ with $A = \{A(\omega)\}\}$. Then, we can show the following result which can be proved an elementary examination, and so the proof is left to the readers.

PROPOSITION 5.6. *Let* $\mathfrak{A}$ *be a* $C(\Omega)$*-moduled* C^**-subalgebra of* $B(H)$ *with* $\mathfrak{A} = \mathfrak{A}''$ *where* $H = C_F(\Omega, H(\omega))$ *is a continuous field of Hilbert spaces, and let* $A = \{A(\omega)\}$ *be an arbitrary densely defined s-closed operator in* H; *then* $A\eta\mathfrak{A}$ *if and only if* $A(\omega)\eta\widetilde{\mathfrak{A}(\omega)}$ *for every* $\omega \in \Omega$.

NOTE. When this paper was firstly typewritten and sent to many authors, the title of this paper was "Continuous reduction theory of unbounded operators in continuous fields of Hilbert spaces".

REFERENCES

[1] J. DIXMIER, Sur certains espaces consideres par M. H. Stone, Summa Brasil Math., 2 (1951), 1–32.
[2] I. KAPLANSKY, Modules over operator algebras, Amer. J. Math., 75 (1953), 839–853.
[3] A. NUSSBAUM, Reduction theory for unbounded closed operators in Hilbert space, Duke Math. J., 31 (1964), 33–44.
[4] S. SAKAI, C^*-algebras and W^*-algebras, Springer-Verlag, 1971.
[5] M. STONE, On unbounded operators in Hilbert space, J. Indian Math. Soc., 15 (1951), 155–192.
[6] H. TAKEMOTO, On a characterization of AW^*-modules and a representation of Gelfand

type of non-commutative operator algebras, Michigan Math. J., 20 (1973), 115-127.

DEPARTMENT OF MATHEMATICS
THE COLLEGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY
SENDAI, JAPAN