Decomposition at the Maximum for Excursions and Bridges of One-dimensional Diffusions*

Jim Pitman and Marc Yor

Department of Statistics, University of California, 367 Evans Hall # 3860, Berkeley, CA 94720-3860, U.S.A.

Laboratoire de Probabilités, Université Pierre et Marie Curie, 4, Place Jussieu - Tour 56, 75252 Paris Cedex 05, France.

This article appeared in *Itô's Stochastic Calculus and Probability Theory*, edited by N. Ikeda, S. Watanabe, M. Fukushima and H. Kunita, pages 293-310, Springer-Verlag (1996). Electronically reprinted from the original tex source, August 3, 2004

1 Introduction

In his fundamental paper [25], Itô showed how to construct a Poisson point process of excursions of a strong Markov process X over time intervals when X is away from a recurrent point a of its statespace. The point process is parameterized by the local time process of X at a. Each point of the excursion process is a path in a suitable space of possible excursions of X, starting at a at time 0, and returning to a for the first time at some strictly positive time ζ, called the *lifetime* of the excursion. The intensity measure of the Poisson process of excursions is a σ-finite measure Λ on the space of excursions, known as Itô's excursion law. Accounts of Itô's theory of excursions can now be found in several textbooks [48, 46, 10]. His theory has also been generalized to excursions of Markov processes away from a set of states [34, 19, 10] and to excursions of stationary, not necessarily Markovian processes [38].

Itô's excursion theory has been applied to the study of the distribution of functionals of the trajectories of one-dimensional Brownian motion and Bessel processes [41, 9, 58, 39], and to the study of random trees [2, 3, 4, 5, 18, 1, 8] and measure valued diffusions [16]. In such studies, the following two descriptions of Itô's law Λ for excursions away from 0 of a reflecting Brownian motion X on $[0,\infty)$ have proved useful. Both involve BES(3), the 3-dimensional Bessel

*Research supported in part by N.S.F. Grants MCS91-07531 and DMS-9404345

process. We recall that for positive integer δ a BES(δ) process can be defined as the radial part of a δ-dimensional Brownian motion, and that this definition can be extended using additivity properties of squares of Bessel processes to define a BES(δ) process for all real $\delta \geq 0$ [49]. The first description of Itô's law Λ is drawn from Itô's definition and observations of Lévy [33], Itô-McKean [26], and Williams [54]. The second description is due to Williams [56] and proved in Rogers [47].

> Description I: Conditioning on the lifetime: *First pick a lifetime t according to the σ-finite density $(2\pi)^{-1/2}t^{-3/2}dt$ on $(0,\infty)$; then given t, run a* BES(3) *bridge from* 0 *to* 0 *over time t.*

> Description II: Conditioning on the maximum: *First pick a maximum value m according to the σ-finite density $m^{-2}dm$ on $(0,\infty)$; then given m, join back to back two independent* BES(3) *processes, each started at* 0 *and run till it first hits m.*

As explained in Biane-Yor [9] and Williams [57], the agreement between these two descriptions of Itô's excursion law, combined with Brownian scaling, implies an identity relating the distribution of the maximum of the standard Brownian excursion (or BES(3) bridge from 0 to 0 over time 1) and the distribution of the sum of two independent copies of the hitting time of 1 by BES(3). These authors show how this identity, expressed in terms of moments, is related to the functional equation for Riemann's zeta function. A central result of this paper is the following generalization of this identity from dimension $\delta = 3$ to arbitrary positive real δ:

Theorem 1 *For each $\delta > 0$, on the space of continuous non-negative paths with a finite lifetime, starting and ending at* 0*, there exists a σ-finite measure Λ_{00}^{δ} that is uniquely determined by either of the following descriptions:*

> Description I: Conditioning on the lifetime: *First pick a lifetime t according to the σ-finite density $2^{-\frac{\delta}{2}}\Gamma(\frac{\delta}{2})^{-1}t^{-\frac{\delta}{2}}dt$ on $(0,\infty)$; then given t, run a* BES(δ) *bridge from* 0 *to* 0 *over time t.*

> Description II: Conditioning on the maximum: *First pick a maximum value m according to the σ-finite density $m^{1-\delta}dm$ on $(0,\infty)$; then given m, join back to back two independent* BES(δ) *processes, each started at* 0 *and run till it first hits m.*

The measures Λ_{00}^{δ} defined by Description II for $\delta > 2$ were considered in [41] and further studied by Biane-Yor [9], who gave Description I in this case. It was shown in [41] that for $2 < \delta < 4$ the measure Λ_{00}^{δ} is Itô's excursion law for excursions of BES($4-\delta$) away from zero. For all $\delta \geq 2$ the measure Λ_{00}^{δ} concentrates on excursion paths starting at 0 and first returning to 0 at their lifetime. But the measure with density $t^{-\frac{\delta}{2}}dt$ on $(0,\infty)$ is a Lévy measure only for $2 < \delta < 4$. So for $\delta \leq 2$ or $\delta \geq 4$ the measure Λ_{00}^{δ} is not the excursion law of any Markov process. Nonetheless, these measures Λ_{00}^{δ} are well defined for all $\delta > 0$, and have some interesting properties and applications. As shown in [41], the measure $4\Lambda_{00}^{4}$ appears, due to the Ray-Knight description of Brownian local times, as the distribution of the square root of the total local time process of a path governed by the Itô's Brownian excursion law Λ_{00}^{3}. Consequently, Λ_{00}^{4} appears also in the Lévy-Khintchine representation of the infinitely divisible family of squares of Bessel processes and Bessel bridges [41, 39]. For $0 < \delta < 2$, the point 0 is a recurrent point for BES (δ), and the measure Λ_{00}^{δ} concentrates on paths which, unlike excursions, return many times to 0 before finally being killed at 0.

Here we establish Theorem 1 for all $\delta > 0$ using a general formulation of Williams' path decomposition at the maximum for one-dimensional diffusion bridges, presented in Section 2. This formulation of Williams' decomposition, due to Fitzsimmons [15], contains an explicit factorization of the joint density of the time and place of the maximum of a one-dimensional diffusion bridge. For Brownian bridge this density factorization appears already in the work of Vincze [50] in 1957, and its extensions to Brownian excursion, Brownian meander and diffusion processes have been derived by several authors [13, 21, 23, 12]. As an application of this decomposition, in Section 3 we describe the law of the standard BES(δ) bridge by its density on path space relative to the law obtained by taking two independent BES(δ) processes started at 0 and run till they first hit 1, joining these processes back to back, and scaling the resultant process with a random lifetime and maximum 1 to have lifetime 1 and a random maximum.

Our approach to the family of measures $(\Lambda_{00}^{\delta}, \delta > 0)$ leads us to consideration of a σ-finite measure Λ_{xy} associated with a general one-dimensional diffusion process instead of BES(δ), for an arbitrary initial point x and final point y. Some instances of these measures were considered in [40]. Some of the results in this paper were announced in [42].

2 Williams' Decomposition for a One-dimensional diffusion

2.1 Decomposition at the maximum over a finite time interval

Let $X = (X_t, t \geq 0)$ be a regular one-dimensional diffusion on a sub-interval I of the real line. See [26] for background and precise definitions. To keep things simple, assume that I contains $[0, \infty)$, and that X has infinite lifetime. The infinitesimal generator A of X, restricted to smooth functions vanishing in some neighbourhoods of boundary points of I, is of the form

$$A = \frac{d}{dm}\frac{d}{ds}, \tag{1}$$

where $s = s(dx)$ and $m = m(dx)$ are the scale and speed measures of the diffusion. The semigroup of X admits a jointly continuous transition density relative to the speed measure

$$p(t,x,y) = P_x(X_t \in dy)/m(dy), \tag{2}$$

which is symmetric in (x,y). Here $P_x(\cdot) = P(\cdot\,|X_0 = x)$ defines the distribution on a suitable path space of the diffusion process started at $X_0 = x$. Let $P^t_{x,y}$ govern the *diffusion bridge* of length t from x to y:

$$P^t_{x,y}(\cdot) = P_x(\cdot\,|\,X_t = y) \tag{3}$$

Under $P^t_{x,y}$ the process $(X_s, 0 \leq s \leq t)$ is an inhomogeneous Markov process with continuous paths, starting at x at time 0 and ending at y at time t. The one-dimensional and transition probability densities of the diffusion bridge are derived from $p(t,x,y)$ in the obvious way via Bayes rule [14].

Let

$$M_t = \sup_{0\leq s\leq t} X_s; \qquad \rho_t = \inf\{s : X_s = M_t\}. \tag{4}$$

For a diffusion X whose ultimate maximum M_∞ is a.s. finite, Williams [55] gave a path decomposition of X at the time ρ_∞ that X first attains this ultimate maximum value. Since this fundamental work of Williams variations of his idea have been developed and applied in a number of different contexts. See for instance Denisov [13], Millar [35, 36], Jeulin [27], Le Gall [17]. In particular, Fitzsimmons [15] gave the following decomposition at the maximum over a finite time interval, part (i) of which appears also in Csáki et al [12]. The density factorization (7) for Brownian bridge was found already by Vincze [50]. See also Imhof [21, 22] for related results, and Asmussen et al. [6] for an application to discretization errors in the simulation of reflecting Brownian motion. Let

$$f_{xz}(t) = P_x(T_z \in dt)/dt \tag{5}$$

where $T_z = \inf\{t : X_t = z\}$ is the first passage time to z. See [26],p.154, regarding the existence of continuous versions of such first passage densities. This allows rigorous construction of nice versions of the conditioned processes appearing in part (ii) of the following theorem, along the lines of [14].

Theorem 2 [55, 15, 12]
(i) *For $x, y \leq z < \infty, 0 \leq u \leq t$, the P_x joint distribution of M_t, ρ_t and X_t is given by*

$$P_x(M_t \in dz, \rho_t \in du, X_t \in dy) = f_{xz}(u) f_{yz}(t-u) s(dz) du\, m(dy). \tag{6}$$

Consequently the $P^t_{x,y}$ joint distribution of M_t and ρ_t is given by

$$P^t_{x,y}(M_t \in dz, \rho_t \in du) = \frac{f_{xz}(u) f_{yz}(t-u)}{p(t,x,y)} s(dz) du \tag{7}$$

(ii) *Under P_x conditionally given $M_t = z, \rho_t = u$ and $X_t = y$, that is to say under $P^t_{x,y}$ given $M_t = z$ and $\rho_t = u$, the path fragments*

$$(X_s, 0 \leq s \leq u) \text{ and } (X_{t-s}, 0 \leq s \leq t-u)$$

are independent, distributed respectively like

$$(X_s, 0 \leq s \leq T_z) \textit{ under } P_x \textit{ given } T_z = u$$

and

$$(X_s, 0 \leq s \leq T_z) \textit{ under } P_y \textit{ given } T_z = t - u$$

Integrating out u in formula (7) gives an expression of convolution type for the density at z of the maximum M_t of a diffusion bridge from x to y over time t. A second integration then yields

$$P^t_{x,y}(M_t \geq z)p(t,x,y) = \int_0^t du \int_z^\infty s(da) f_{xa}(u) f_{ya}(t-u) \tag{8}$$

$$= \int_0^t f_{xz}(u)p(t-u,z,y)du \tag{9}$$

Here the equality between (9) and the left side of (8) is clear directly by interpreting the latter as

$$P_x(M_t \geq z, X_t \in dy)/m(dy) = P_x(t \geq T_z, X_t \in dy)/m(dy)$$

and conditioning on T_z. Following the method used by Gikhman [20] in the case of Bessel processes, explicit formulae for the bridge probabilities $P^t_{x,y}(M_t \geq z)$ for particular diffusions can be computed using the Laplace transformed version of (9), which is

$$\int_0^\infty e^{-\alpha t} P^t_{x,y}(M_t \geq z)p(t,x,y)dt = \phi^\uparrow(\alpha,x)\phi^\uparrow(\alpha,y)\frac{\phi^\downarrow(\alpha,z)}{\phi^\uparrow(\alpha,z)} \tag{10}$$

where $\phi^\uparrow(\alpha,x)$ and $\phi^\downarrow(\alpha,x)$ are the increasing and decreasing solutions of $Au = \alpha u$, for $\alpha > 0$, normalized so that

$$\int_0^\infty e^{-\alpha t} p(t,x,y)dt = \phi^\uparrow(\alpha, x \wedge y)\phi^\downarrow(\alpha, x \vee y). \tag{11}$$

Then

$$E_x(e^{-\alpha T_z}) = \int_0^\infty e^{-\alpha t} f_{xz}(t)dt = \frac{\phi^\uparrow(\alpha,x)}{\phi^\uparrow(\alpha,z)} \text{ for } x \leq z \tag{12}$$

and the same holds with $\phi^\downarrow(\alpha,\cdot)$ instead of $\phi^\uparrow(\alpha,\cdot)$ for $x \geq z$. See Itô-McKean [26] for these formulae. In view of (11) and (12), the equality between the right sides of (8) and (9) reduces by Laplace transforms to the classical Wronskian identity

$$\phi^\downarrow(\alpha,x)\frac{d\phi^\uparrow(\alpha,x)}{s(dx)} - \phi^\uparrow(\alpha,x)\frac{d\phi^\downarrow(\alpha,x)}{s(dx)} = 1 \tag{13}$$

To see this, note from (11) and (12) that the Laplace transform of the righthand expression in (8) becomes

$$\phi^\uparrow(\alpha,x)\phi^\uparrow(\alpha,y)\int_z^\infty s(da)\frac{1}{[\phi^\uparrow(\alpha,a)]^2} \tag{14}$$

This equals the Laplace transform of the right-hand side of (9), which, as already remarked, is the expression in (10). Indeed, the Wronskian formula (13) makes

$$\frac{d}{s(dz)}\left(\frac{\phi^{\downarrow}(\alpha,z)}{\phi^{\uparrow}(\alpha,z)}\right)=\frac{-1}{[\phi^{\uparrow}(\alpha,z)]^2}$$

and each of the expressions in (10) and (14) vanishes as $z \uparrow \infty$, because the assumption of infinite lifetime implies $\phi^{\uparrow}(\alpha,z) \uparrow \infty$ as $z \uparrow \infty$. Csáki et al [12] used a variation of this argument to derive (6).

2.2 The agreement formula for a diffusion bridge

With one more simplifying assumption, Theorem 2 can be expressed as in the next corollary. This corollary is a generalization of Theorem 1 suggested by work of Williams [56], Pitman-Yor [41], Biane-Yor [9], Biane [7]. The notation is taken from Section 6 of [9], where more formal definitions can be found. For a distribution Q on path space, and a random time T, let Q^T be the distribution of the path obtained by killing at time T. Let $Q^{\wedge}$ be the image of Q by time reversal. For a second distribution of paths Q', let $Q \circ Q'$, the *concatenation* of Q and Q', be the distribution of the path obtained by first following a path distributed according to Q, then continuing independently according to Q'.

Corollary 3 Agreement Formula for Diffusion Bridges. *Assume that for all $x, y \in I$ with $x < y$, $P_x(T_y < \infty) = 1$. Then for all $x, y \in I$ there is the following identity of measures on path space:*

$$\int_0^{\infty} dt\, p(t,x,y) P^t_{x,y} = \int_{x\vee y}^{\infty} s(dz)(P_x^{T_z}) \circ (P_y^{T_z})^{\wedge} \tag{15}$$

Theorem 1.1 amounts to the following instance of this formula when the basic diffusion is BES(δ) and $x = y = 0$:

$$\Lambda^{\delta}_{00} = \int_0^{\infty} dt \frac{P^t_{0,0}}{(2t)^{\frac{\delta}{2}}\Gamma(\frac{\delta}{2})} = \int_0^{\infty} \frac{dz(P_0^{T_z}) \circ (P_0^{T_z})^{\wedge}}{z^{\delta-1}} \tag{16}$$

Definition 4 *For a one-dimensional diffusion subject to the conditions of Corollary 2.1, let Λ_{xy} denote the measure on path space defined by either side of the agreement formula* (15).

The measure Λ_{xy} is always σ-finite. Its total mass is the 0-potential density

$$\int_0^{\infty} p(t,x,y)dt = s(\infty) - s(x \vee y)$$

which may be either finite or infinite. Informally, the agreement formula states that each of the following two schemes derived from a basic diffusion process X can be used to generate Λ_{xy}:

(LHS) *Pick t according to $p(t,x,y)dt$ and then run an X bridge of length t from x to y*
(RHS) *Pick z according to the speed measure $s(dz)$ restricted to $(x \vee y, \infty)$, then join back to back a copy of X started at x run to T_z and a copy of X started at y run to T_z.*

The (LHS) amounts to conditioning on the lifetime of the path from x to y, while the (RHS) amounts to conditioning on the maximum.

Clearly, Λ_{xy} concentrates on paths starting at x and ending at y, and attaining a maximum value, M say, at a unique intermediate time. Note that $\Lambda_{xy}^{\wedge} = \Lambda_{yx}$. This is obvious from the right side of (15), and can be seen also on the left side, because $p(t,x,y) = p(t,y,x)$, and $(P_{x,y}^t)^{\wedge} = P_{y,x}^t$.

2.3 Relation to last exit times

We now consider the case when X is transient, i.e. $X_t \to \infty$ as $t \to \infty$. We can then choose s such that $s(\infty) = 0$. In this transient case, the measure Λ_{xy} is finite, and in fact is a multiple of the restriction of $P_x^{L_y}$ to $L_y > 0$, where $L_y = \sup\{t > 0 : X_t = y\}$ with the usual convention that $\sup(\emptyset) = 0$. To be precise, by formula (6.e) of [40],

$$P_x(L_y \in dt) = -s(y)^{-1} p(t,x,y)dt \tag{17}$$

where we have dropped a factor of 2 from the formula of [40] due to our definition of the speed measure m here using $A = \frac{d}{dm}\frac{d}{ds}$ rather than $A = \frac{1}{2}\frac{d}{dm}\frac{d}{ds}$ as in [40]. Furthermore, from [40] there is the formula

$$P_x^{L_y}(\ \cdot\ | L_y = t) = P_{x,y}^t \tag{18}$$

so for transient X the agreement formula (15) can be written

$$P_x^{L_y}(\ \cdot\ \cap (L_y > 0)\,) = -\frac{1}{s(y)} \int_{x \vee y}^{\infty} s(dz) (P_x^{T_z}) \circ (P_y^{T_z})^{\wedge} \tag{19}$$

When X is the BES(3) process on $[0,\infty)$, and $x = y = 0$, the σ -finite measure Λ_{00} appearing in (15) is Itô's excursion law. The LHS is the description of Itô's law for Brownian excursions due to Lévy [33] and Itô [25], while the RHS is Williams' [56] description. As noted in Biane-Yor [9] and Williams [57], the agreement between these two descriptions of Itô's law has interesting consequences related to the functional equation for the Riemann zeta function.

Corollary 3 allows the identity (8) to be lifted to an identity of measures on path space:

$$\text{the restriction of } \Lambda_{xy} \text{ to } (M > z) \text{ is } P_x^{T_z} \circ \Lambda_{zy} \tag{20}$$

We note also that integration with respect of $m(dy)$ yields the following version of the agreement formula for unconditioned diffusions:

$$\int_0^{\infty} dt\, P_x^t = \int_x^{\infty} s(dz) (P_x^{T_z}) \circ \left(\int_{-\infty}^{z} m(dy) (P_y^{T_z})^{\wedge} \right) \tag{21}$$

Similar representations of the left side of (21) for Brownian motion appear in [9] (see also [46], Ex. (4.18) of Ch. XII). These too can be formulated much as above for a general diffusion.

2.4 Relation to excursion laws

The connection between BES(3) and BM is that BES(3) is BM on $[0, \infty)$ *conditioned to approach* ∞ *before 0*, a concept made precise by Doob's theory of h-transforms. More generally, if 0 is a recurrent point of a regular diffusion Y on an interval I which contains $[0, \infty)$, and X is Y conditioned to approach ∞ before 0, then Λ_{00} derived from X admits a similar interpretation as Itô's law for excursions of Y above 0. See Section 3 of Pitman-Yor [41], where Williams' representation of Λ_{00} is given along with two other representations of the measure in this case, due to Itô and Williams. In view of (20), the second of these two other descriptions also identifies Λ_{x0} derived from X, for $x > 0$, as

$$\Lambda_{x0} = s(x, \infty) Q_x^{T_0}$$

where Q_x is the distribution of Y started at x. The description of Itô's excursion law for a general one-dimensional diffusion Y, via the LHS of (15) for X as above, is less well known. According to this description, the Lévy measure governing the duration of excursions of the recurrent diffusion Y above 0 has density at t identical to $p(t, 0, 0)$ for the diffusion X on $[0, \infty)$ obtained by conditioning Y to approach ∞ before hitting 0. See Knight [31], Kotani-Watanabe [32], concerning the problem of characterizing such Lévy densities.

The two other descriptions of an Itô excursion law, given in Section 3 of [41], do not make sense in the generality of Corollary 3, because they involve conditioning on sets which might have infinite mass. In particular, this is the case if X is recurrent, for example a standard Brownian motion. If y is a recurrent point for X, the measure Λ_{xy}, while σ-finite on path space, has finite dimensional distributions that are not σ-finite. This follows from the LHS of the agreement formula (15) combined with the fact that $\int_v^\infty p(t, x, y) dt = \infty$ for every $v > 0$. The measures Λ_{yy}, as defined by the LHS of the agreement formula for a recurrent point y, were considered in Pitman-Yor [40], and applied in case X is a Bessel process to establish complete monotonicity of some particular ratios of Bessel functions. As noted in [40], if $(\tau_\ell, \ell \geq 0)$ is the inverse of the local time process $(L_t, t \geq 0)$ at a recurrent point y, normalized so that $E_x(dL_t) = p(t, x, y) dt$, then there is the further identity

$$\int_0^\infty dt\, p(t, x, y) P_{x,y}^t = \int_0^\infty d\ell P_x^{\tau_\ell} \tag{22}$$

That is to say , for a recurrent point y a third description of the measure Λ_{xy} in (15) is obtained by first picking ℓ according to Lebesgue measure, then running the diffusion started at x until the time τ_ℓ that its local time at y first equals ℓ.

3 The Agreement Formula for Bessel Processes

3.1 Definition and basic properties of Bessel Processes

Let $R = (R_t, t \geq 0)$ be a BES(δ) process started at $R_0 = 0$. Here δ is a strictly positive real parameter. For $\delta = 1, 2, \ldots$, a BES(δ) diffusion R is obtained as the radial part of BM in $\mathbb{R}^\delta$. See Itô-McKean [26] Section 2.7. For positive integer parameters, this representation displays the *Pythagorean property* of Bessel processes: the sum of squares of independent BES(δ) and BES(δ') processes is the square of a BES($\delta + \delta'$) process. As shown by Shiga-Watanabe [49], the family of BES(δ) processes for real $\delta > 0$ is characterized by extension of this Pythagorean property to all positive real δ and δ'. Typical properties of Bessel processes are consequences of the Brownian representation for integer δ that admit natural extensions to all $\delta > 0$. See [46] for further background and proofs of the basic properties of BES(δ) now recalled.

The BES(δ) process is a diffusion on $[0, \infty)$ whose infinitesimal generator A_δ acts on smooth functions vanishing in a neighbourhood of 0 as

$$A_\delta = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\delta - 1}{2x}\frac{d}{dx} = \frac{d}{dm_\delta}\frac{d}{ds_\delta}. \tag{23}$$

where the scale and speed measures s_δ and m_δ can be chosen to be

$$s_\delta(dx) = x^{1-\delta}dx, \qquad m_\delta(dx) = 2x^{\delta-1}dx. \tag{24}$$

For $0 < \delta < 2$, the definition of the generator is completed by specifying that the boundary point 0 acts as a simple instantaneously reflecting barrier.

The Pythagorean property implies easily that for all $\delta > 0$

$$\text{the law of } R_t^2/2t \text{ is gamma } (\tfrac{\delta}{2}).$$

That is to say

$$P(R_t \in dy) = 2^{1-\frac{1}{2}\delta}\Gamma(\delta/2)^{-1}t^{-\frac{\delta}{2}}y^{\delta-1}e^{-\frac{y^2}{2t}}dy = p_\delta(t, 0, y)m_\delta(dy) \tag{25}$$

where

$$p_\delta(t, 0, y) = (2t)^{-\frac{\delta}{2}}\Gamma(\tfrac{\delta}{2})^{-1}e^{-\frac{y^2}{2t}} \tag{26}$$

is the transition probability density relative to the speed measure. This is the simple special case $x = 0$ of the general formula for the transition probability function $p_\delta(t, x, y)$ of the Bessel diffusion, for which see Itô-McKean [26] Section 2.7, Molchanov and Ostrovski [37].

The BES(δ) process R for each real $\delta > 0$ inherits the familiar *Brownian scaling property* from integer dimensions: for every $c > 0$

$$(c^{-1/2}R_{ct}, t \geq 0) \stackrel{d}{=} (R_t, t \geq 0)$$

A *standard Bessel* (δ) *bridge* is a process

$$(R_u^{\text{br}}, 0 \leq u \leq 1) \stackrel{d}{=} (R_u, 0 \leq u \leq 1 | R_1 = 0).$$

For all $\delta > 0$ a standard BES(δ) bridge R^{br} is conveniently constructed from the unconditioned BES(δ) process R as

$$R_u^{\mathrm{br}} = (1-u)R(u/(1-u)), 0 \le u < 1.$$

In particular, for positive integer δ, the square of the standard BES(δ) bridge is distributed as the sum of squares of δ independent standard one-dimensional Brownian bridges.

By Brownian scaling, for $t > 0, \delta > 0$, a BES(δ) bridge from 0 to 0 over time t can be represented in terms of the standard BES(δ) bridge R^{br} as

$$\sqrt{t} R_{s/t}^{\mathrm{br}}, 0 \le s \le t$$

We note in passing that an interesting continuum of processes, passing from the Bessel bridges to the free Bessel processes and including the Bessel meanders, is introduced and studied in [45].

3.2 Random Scaling Construction of the Standard Bessel Bridge.

The following theorem is an expression of the agreement formula (15) for Bessel processes. This is a probabilistic expression in terms of standard bridges of Theorem 1.

Theorem 5 *Let R and $\hat{R}$ be two independent BES(δ) processes starting at 0, T and $\hat{T}$ their first hitting times of 1. Define $\tilde{R}$ by connecting the paths of R on $[0,T]$ and $\hat{R}$ on $[0,\hat{T}]$ back to back:*

$$\tilde{R}_t = \begin{cases} R_t \ \text{if}\ t \le T \\ \hat{R}_{T+\hat{T}-t} \ \text{if}\ T \le t \le T+\hat{T}, \end{cases}$$

and let $\tilde{R}^{\mathrm{br}}$ be obtained by Brownian scaling of $\tilde{R}$ onto the time scale $[0,1]$:

$$\tilde{R}_u^{\mathrm{br}} = (T+\hat{T})^{-1/2} \tilde{R}_{u(T+\hat{T})}, \ 0 \le u \le 1.$$

Let R^{br} be a standard BES^{δ} bridge. Then for all positive or bounded measurable functions $F : C[0,1] \to \mathbb{R}$,

$$E[F(R^{\mathrm{br}})] = c_\delta E[F(\tilde{R}^{\mathrm{br}})(\tilde{M}^{\mathrm{br}})^{2-\delta}] \tag{27}$$

where

$$\tilde{M}^{\mathrm{br}} = \sup_{0 \le u \le 1} \tilde{R}_u^{\mathrm{br}} = (T+\hat{T})^{-1/2} \tag{28}$$

$$c_\delta = 2^{\frac{\delta}{2}-1}\Gamma(\tfrac{\delta}{2}). \tag{29}$$

Proof. Fix δ. Let f_z be the density of $T_z = \inf\{t : R_t = z\}$ for the unconditional BES(δ) diffusion R started at $X_0 = 0$. Applied to the standard BES(δ) bridge $R^{\rm br}$, and using (24), (26) and the scaling property

$$f_z(t) = z^{-2} f_1(tz^{-2}),\ t > 0,\ z > 0,$$

formula (7) yields

$$\frac{P(M^{\rm br} \in dz, \rho^{\rm br} \in dt)}{dz\, dt} = 2c_\delta f_1\left(\frac{t}{z^2}\right) f_1\left(\frac{1-t}{z^2}\right) z^{-\delta-3}. \tag{30}$$

On the other hand, $\tilde{R}^{\rm br}$ constructed as above has maximum value

$$\tilde{M}^{\rm br} = (T + \hat{T})^{-1/2} \text{ attained at time } \tilde{\rho}^{\rm br} = \frac{T}{T + \hat{T}} \tag{31}$$

where T and $\hat{T}$ are independent with density f_1. Thus by a change of variables

$$\frac{P(\tilde{M}^{\rm br} \in dz, \tilde{\rho}^{\rm br} \in dt)}{dz\, dt} = 2f_1\left(\frac{t}{z^2}\right) f_1\left(\frac{1-t}{z^2}\right) z^{-5}. \tag{32}$$

Comparison of (30) and (32) shows that (27) holds for F a function of the maximum of the process and the time it is attained. To lift the formula from the above identity of joint laws for the time and level of the maximum, to the identity of laws on the path space $C[0,1]$, it only remains to be seen that the two laws on path space share a common family of conditional laws given the time and level of the maximum: for $y > 0,\ 0 < t < 1$,

$$P(R^{\rm br} \in \cdot\,| M^{\rm br} = y, \rho^{\rm br} = t) = P(\tilde{R}^{\rm br} \in \cdot\,| \tilde{M}^{\rm br} = y, \tilde{\rho}^{\rm br} = t).$$

But this follows immediately from Williams decomposition as stated in part (ii) of Theorem 2, and Brownian scaling.

According to (27), the law of the standard Bessel bridge $R^{\rm br}$ on $C[0,1]$ is mutually absolutely continuous with respect to that of $\tilde{R}^{\rm br}$, with density at $w \in C[0,1]$

$$\frac{P(R^{\rm br} \in dw)}{P(\tilde{R}^{\rm br} \in dw)} = c_\delta \left(\sup_{0 \le u \le 1} w_u\right)^{2-\delta}. \tag{33}$$

Our formulation of Theorems 1 and 5 was suggested by Section 3 of Biane-Yor [9], where some forms of these results are discussed for $\delta > 2$. The present development shows that everything works also for $0 < \delta \le 2$. Recall the well known fact that dimension $\delta = 2$ is the threshold between recurrence and transience of BES(δ) processes:

for $\delta > 2$, there are no recurrent points for the BES(δ) diffusion;

for $\delta = 2$, every $x > 0$ is a recurrent point, but 0 is only neighbourhood-recurrent, not point recurrent;

for $0 < \delta < 2$, every $x \ge 0$ is a recurrent point for BES(δ).

Dimension 2 plays a special role here, as the unique dimension that makes the density factor (33) identically equal to 1. Thus for $(\tilde{R}_u^{\mathrm{br}}, 0 \leq u \leq 1)$ defined as in Theorem 5 by pasting back to back two independent BES(δ) processes run till they first hit 1 then Brownian scaling the result to have lifetime 1, there is the following immediate consequence of Theorem 5:

Corollary 6 *The process $(\tilde{R}_u^{\mathrm{br}}, 0 \leq u \leq 1)$ is a standard BES(δ) bridge if and only if $\delta = 2$.*

Combined with the skew-product description of planar Brownian motion, (see e.g. [26] or [46]) this yields in turn:

Corollary 7 *Run each of two independent planar Brownian motions Z and $\hat{Z}$ starting at the origin until hitting the unit circle, at times T and $\hat{T}$ respectively. Rotate the entire path of $\hat{Z}$ over the time interval $[0, \hat{T}]$ to make the two paths meet when they first reach the unit circle at times T and $\hat{T}$. Define a path $Z^\dagger$ with lifetime $T + \hat{T}$ by first travelling out to the unit circle over time T via Z, then returning via the reversed and rotated path of $\hat{Z}$ over a following time interval of length $\hat{T}$. Finally, rescale $Z^\dagger$ to have lifetime 1 by Brownian scaling. Then the resultant process is a standard planar Brownian bridge.*

Some applications of this result have been made by Werner [52, 53] to study the shape of the small connected components of the complement of a 2-dimensional Brownian path. We note also the following *asymptotic representation* of the 2-dimensional Brownian bridge as the limit in distribution as $r \to 0$ of

$$\left(\frac{1}{\sqrt{T_r}} Z(uT_r); 0 \leq u \leq 1 \right)$$

where $Z = (Z(t), t \geq 0)$ is a 2-dimensional Brownian motion started at $Z(0) \neq 0$, and T_r is the hitting time of $\{z : |z| = r\}$ by Z.

A construction like that in Corollary 7 can be made starting from δ dimensional Brownian motion for any $\delta = 1, 2, 3, \ldots$. But the result is the standard bridge only for $\delta = 2$. For other dimensions δ the result has distribution absolutely continuous with respect to that of the bridge, with density the function of the maximum of the radial part indicated by (33).

3.3 Applications

A subscript δ will now be used to indicate the dimension of the underlying Bessel process. So

$$\begin{aligned}
T_\delta &= \text{hitting time of 1 for a BES}(\delta) \text{ started at } 0 \\
\hat{T}_\delta &= \text{independent copy of } T_\delta \\
(M_\delta^{\mathrm{br}}, \rho_\delta^{\mathrm{br}}) &= \text{level and time of the maximum for a standard BES}(\delta) \text{ bridge.}
\end{aligned}$$

The distribution of T_δ is determined by its Laplace transform (Kent [29])

$$\varphi_\delta(\lambda) = E\exp(-\lambda T_\delta) = \frac{(2\lambda)^{\mu/2}}{c_\delta I_\mu(\sqrt{2\lambda})} \tag{34}$$

where $\mu = \frac{\delta}{2} - 1$ is the *index* corresponding to dimension δ, and $c_\delta = 2^{\frac{\delta}{2}-1}\Gamma(\frac{\delta}{2}) = 2^\mu\Gamma(\mu+1)$ as in (29). According to Ismail-Kelker ([24], Theorem 4.10) the corresponding density f_δ can be written as a series expansion involving the zeros of J_μ, the usual Bessel function of index μ.

3.3.1 Moment identities

Several consequences of (27), all of which are apparent at the level of the joint laws (30) and (32), were noted for $\delta > 2$, i.e. $\mu > 0$, as formulae (3.k), (3.l), (3.k'), (3.k") of [9]. According to the Theorem 5, these identities in fact hold for all $\delta > 0$: for all positive measurable functions f:

$$E[f(M_\delta^{\mathrm{br}})] = c_\delta E[f((T_\delta + \hat{T}_\delta)^{-1/2})(T_\delta + \hat{T}_\delta)^{\frac{\delta}{2}-1}] \tag{35}$$

$$E[f(\rho_\delta^{\mathrm{br}})] = c_\delta E\left[f\left(\frac{T_\delta}{T_\delta + \hat{T}_\delta}\right)(T_\delta + \hat{T}_\delta)^{\frac{\delta}{2}-1}\right]. \tag{36}$$

In particular,

$$E(M_\delta^{\mathrm{br}})^{\delta-2} = c_\delta \tag{37}$$

$$E(T_\delta + \hat{T}_\delta)^{\frac{\delta}{2}-1} = 1/c_\delta \tag{38}$$

3.3.2 Relation to Kiefer's formula

Let $f_\delta^{(2)}(t) = f_\delta * f_\delta(t)$ denote the density of $T_\delta + \hat{T}_\delta$, with Laplace transform $[\varphi_\delta(\lambda)]^2$. According to (35),

$$P[(M_\delta^{\mathrm{br}})^2 \in da] = c_\delta a^{-\frac{\delta}{2}-1} f_\delta^{(2)}(a^{-1})da. \tag{39}$$

For integer dimensions δ, Kiefer ([30], (3.21)) found a formula for the density of $(M_\delta^{\mathrm{br}})^2$ which also involves the zeros of J_μ. It appears that Kiefer's method and formula are valid also for arbitrary $\delta > 0$. Comparison of Kiefer's formula and (39) using the formula of Ismail-Kelker for f_δ leads to some tricky identities involving the zeros of J_μ. Kiefer [30],p.429 discusses the cases $\delta = 1$ and $\delta = 3$. The second case is of special interest because, as noted by Williams [54], the standard Brownian excursion is a $BES(3)$ bridge. Kiefer's formulae were rediscovered in the context of Brownian excursions by Kennedy [28] and Chung [11].

3.3.3 Moment identities for dimension 2

Differentiation of (38) with respect to δ at $\delta = 2$ yields

$$E[\log(T_2 + \hat{T}_2)] = -\log 2 - \Gamma'(1) \tag{40}$$

From (35) with $\delta = 2$ one also gets

$$2E[\log(M_2^{\mathrm{br}})] = \log 2 + \Gamma'(1) \tag{41}$$

Recently, formula (40) has been useful in checking the following asymptotic result, which is of interest in certain questions related to random environments: For $(B_s, s \geq 0)$ a one-dimensional BM

$$E\left[\log\left(\int_0^t \exp(B_s)ds\right)\right] \; - \; \sqrt{\frac{2t}{\pi}} \; \to \; \log 2 - \Gamma'(1) \text{ as } t \to \infty. \tag{42}$$

This follows from the consequence of Theorem 2 and the Ray-Knight description of Brownian local times that for $S_1 = \sup_{0 \leq s \leq 1} B_s$

$$t \int_0^1 \exp(-\sqrt{t}(S_1 - B_s))ds \; \xrightarrow{d} \; 4(T_2 + \hat{T}_2) \text{ as } t \to \infty \tag{43}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

3.3.4 A check for dimensions less than 2

For $0 < \delta < 2$, corresponding to $-1 < \mu < 0$, we have a check that the evaluation of the constant $c_\delta = 2^\mu \Gamma(\mu + 1)$, in (27), (35) etc. is correct, starting from (34). For a r.v. $X \geq 0$ with Laplace transform $\varphi(\lambda) = Ee^{-\lambda X}$, there is the formula

$$EX^p = \frac{1}{\Gamma(-p)} \int_0^\infty \lambda^{-p-1} \varphi(\lambda) d\lambda, \qquad p < 0 \tag{44}$$

Applied to $X = T_\delta + \hat{T}_\delta$, $p = \mu$, for $-1 < \mu < 0$, this yields

$$E(T_\delta + \hat{T}_\delta)^\mu = \frac{1}{c_\delta} \int_0^\infty \frac{d}{d\lambda} \, \frac{I_{-\mu}(\sqrt{2\lambda})}{I_\mu(\sqrt{2\lambda})} d\lambda \tag{45}$$

due to the standard formula for the Wronskian of I_μ and $I_{-\mu}$ (Watson [51], p.77) By standard asymptotics of I_μ, this confirms that (38) holds for $-1 < \mu < 0$, with $c_\delta = 2^\mu \Gamma(1 + \mu)$ as in (29).

3.3.5 A check for integer dimensions

In case $\mu = k$ is a positive integer, formula (38) can be checked using

$$E\left[(T_\delta + \hat{T}_\delta)^k\right] = (-1)^k \left.\frac{d^k}{d\lambda^k}\right|_{\lambda=0} \left[\phi_\delta(\lambda)^2\right] \tag{46}$$

Note also the easy equality

$$E\left[(T_\delta + \hat{T}_\delta)\right] = 2E(T_\delta) = \frac{2}{\delta} E\left[(R_\delta(T_\delta))^2\right] = \frac{2}{\delta} \tag{47}$$

which, in case $\delta = 4$, agrees with (38), since $c_4 = 2$.

3.3.6 Chung's identity

To further illustrate formula (35), we now show how it implies the remarkable identity

$$(M_1^{\mathrm{br}})^2 \overset{d}{=} \frac{\pi^2}{4} T_3 \tag{48}$$

which was discovered by Chung [11]. See Biane-Yor [9] and Pitman-Yor [42, 44, 43] for further discussion and related identities.

Take $f(x) = e^{-\frac{1}{2}\lambda^2 x^2}$, so $f(\frac{1}{\sqrt{t}}) = e^{-\lambda^2/2t}$ in (35):

$$E \exp\left(-\frac{\lambda^2}{2}(M_\delta^{\mathrm{br}})^2\right) = c_\delta E\left[(T_\delta + \hat{T}_\delta)^{\frac{\delta}{2}-1} \exp\left(-\frac{\lambda^2}{2(T_\delta + \hat{T}_\delta)}\right)\right].$$

For $\delta = 1$, this expression equals

$$\pi\, E\left[\frac{1}{\sqrt{2\pi}\sqrt{T_1 + \hat{T}_1}} \exp\left(-\frac{\lambda^2}{2(T_1 + \hat{T}_1)}\right)\right] \;=\; \pi P\left(N\sqrt{T_1 + \hat{T}_1} \in d\lambda\right) / \, d\lambda$$

where N is Normal $(0,1)$ independent of $\sqrt{T_1 + \hat{T}_1}$. But

$$E \exp\left(i\lambda N \sqrt{T_1 + \hat{T}_1}\right) = E \exp\left(-\frac{\lambda^2}{2}(T_1 + \hat{T}_1)\right) = \frac{1}{\cosh^2(\lambda)}$$

whence in this case by Fourier inversion

$$\begin{aligned} E \exp\left(-\frac{\lambda^2}{2}(M_1^{\mathrm{br}})^2\right) &= \pi \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta e^{-i\lambda\theta}}{\cosh^2(\theta)} = \int_{-\infty}^{\infty} \frac{dx e^{i\frac{\lambda}{2}x}}{2 \cdot 2\cosh^2(\frac{x}{2})} \\ &= \frac{\pi\frac{\lambda}{2}}{\sinh(\pi\frac{\lambda}{2})} = E \exp\left(-\tfrac{\lambda^2}{2}\,(\tfrac{\pi^2}{4}) T_3\right) \end{aligned}$$

This proves (48).

References

[1] D. Aldous and J. Pitman. Brownian bridge asymptotics for random mappings. *Random Structures and Algorithms*, 5:487–512, 1994.

[2] D.J. Aldous. The continuum random tree I. *Ann. Probab.*, 19:1–28, 1991.

[3] D.J. Aldous. The continuum random tree II: an overview. In M.T. Barlow and N.H. Bingham, editors, *Stochastic Analysis*, pages 23–70. Cambridge University Press, 1991.

[4] D.J. Aldous. The continuum random tree III. *Ann. Probab.*, 21:248–289, 1993.

[5] D.J. Aldous. Recursive self-similarity for random trees, random triangulations and Brownian excursion. *Ann. Probab.*, 22:527–545, 1994.

[6] S. Asmussen, P. Glynn, and J. Pitman. Discretization error in simulation of one-dimensional reflecting Brownian motion. To appear in *Ann. Applied Prob.*, 1996.

[7] Ph. Biane. Decompositions of Brownian trajectories and some applications. In A. Badrikian, P-A Meyer, and J-A Yan, editors, *Probability and Statistics; Rencontres Franco-Chinoises en Probabilités et Statistiques; Proceedings of the Wuhan meeting*, pages 51–76. World Scientific, 1993.

[8] Ph. Biane. Some comments on the paper: "Brownian bridge asymptotics for random mappings" by D. J. Aldous and J. W. Pitman. *Random Structures and Algorithms*, 5:513–516, 1994.

[9] Ph. Biane and M. Yor. Valeurs principales associées aux temps locaux Browniens. *Bull. Sci. Math. (2)*, 111:23–101, 1987.

[10] R. M. Blumenthal. *Excursions of Markov processes.* Birkhäuser, 1992.

[11] K.L. Chung. Excursions in Brownian motion. *Arkiv fur Matematik*, 14:155–177, 1976.

[12] E. Csáki, A. Földes, and P. Salminen. On the joint distribution of the maximum and its location for a linear diffusion. *Annales de l'Institut Henri Poincaré, Section B*, 23:179 – 194, 1987.

[13] I.V. Denisov. A random walk and a Wiener process near a maximum. *Theor. Prob. Appl.*, 28:821–824, 1984.

[14] P. Fitzsimmons, J. Pitman, and M. Yor. Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992*, pages 101–134. Birkhäuser, Boston, 1993.

[15] P.J. Fitzsimmons. Excursions above the minimum for diffusions. Unpublished manuscript, 1985.

[16] J.-F. Le Gall. Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.*, 19:1399–1439, 1991.

[17] J.F. Le Gall. Une approche élémentaire des théorèmes de décomposition de Williams. In *Séminaire de Probabilités XX*, pages 447–464. Springer, 1986. Lecture Notes in Math. 1204.

[18] J.F. Le Gall. The uniform random tree in a brownian excursion. *Probab. Th. Rel. Fields*, 96:369–383, 1993.

[19] R.K. Getoor and M.J. Sharpe. Excursions of dual processes. *Advances in Mathematics*, 45:259–309, 1982.

[20] I.I. Gikhman. On a nonparametric criterion of homogeneity for k samples. *Theory Probab. Appl.*, 2:369–373, 1957.

[21] J.-P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *Journal of Applied Probability*, 21:500 – 510, 1984.

[22] J.-P. Imhof. On Brownian bridge and excursion. *Studia Sci. Math. Hungar.*, 20:1–10, 1985.

[23] J. P. Imhof. On the range of Brownian motion and its inverse process. *Annals of Probability*, 13:1011 – 1017, 1985.

[24] M. E. Ismail and D.H. Kelker. Special functions, Stieltjes transforms, and infinite divisibility. *SIAM J. Math. Anal.*, 10:884–901, 1979.

[25] K. Itô. Poisson point processes attached to Markov processes. In *Proc. 6th Berk. Symp. Math. Stat. Prob.*, volume 3, pages 225–240, 1971.

[26] K. Itô and H.P. McKean. *Diffusion Processes and their Sample Paths*. Springer, 1965.

[27] Th. Jeulin. Temps local et théorie du grossissement: application de la théorie du grossissement à l'étude des temps locaux browniens. In *Grossissements de filtrations: exemples et applications. Séminaire de Calcul Stochastique, Paris 1982/83*, pages 197–304. Springer-Verlag, 1985. Lecture Notes in Math. 1118.

[28] D.P. Kennedy. The distribution of the maximum Brownian excursion. *J. Appl. Prob.*, 13:371–376, 1976.

[29] J. Kent. Some probabilistic properties of Bessel functions. *Annals of Probability*, 6:760 – 770, 1978.

[30] J. Kiefer. K-sample analogues of the Kolmogorov-Smirnov and Cramér-von Mises tests. *Ann. Math. Stat.*, 30:420–447, 1959.

[31] F.B. Knight. Characterization of the Lévy measure of inverse local times of gap diffusions. In *Seminar on Stochastic Processes, 1981*, pages 53–78. Birkhäuser, Boston, 1981.

[32] S. Kotani and S. Watanabe. Krein's spectral theory of strings and generalized diffusion processes. In *Functional Analysis in Markov Processes*, pages 235–249. Springer, 1982. Lecture Notes in Math. 923.

[33] P. Lévy. *Processus Stochastiques et Mouvement Brownien.* Gauthier-Villars, Paris, 1965. (first ed. 1948).

[34] B. Maisonneuve. Exit systems. *Ann. of Probability*, 3:399 – 411, 1975.

[35] P.W. Millar. Random times and decomposition theorems. In *Proc. of Symp. in Pure Mathematics*, volume 31, pages 91–103, 1977.

[36] P.W. Millar. A path decomposition for Markov processes. *Ann. Probab.*, 6:345–348, 1978.

[37] S. A. Molchanov and E. Ostrovski. Symmetric stable processes as traces of degenerate diffusion processes. *Theor. Prob. Appl.*, 14, No. 1:128–131, 1969.

[38] J. Pitman. Stationary excursions. In *Séminaire de Probabilités XXI*, pages 289–302. Springer, 1986. Lecture Notes in Math. 1247.

[39] J. Pitman. Cyclically stationary Brownian local time proceses. To appear in *Probability Theory and Related Fields*, 1996.

[40] J. Pitman and M. Yor. Bessel processes and infinitely divisible laws. In *Stochastic Integrals*, pages 285–370. Springer, 1981. Lecture Notes in Math. 851.

[41] J. Pitman and M. Yor. A decomposition of Bessel bridges. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 59:425–457, 1982.

[42] J. Pitman and M. Yor. Dilatations d'espace-temps, réarrangements des trajectoires browniennes, et quelques extensions d'une identité de Knight. *C.R. Acad. Sci. Paris*, t. 316, Série I:723–726, 1993.

[43] J. Pitman and M. Yor. Some extensions of Knight's identity for Brownian motion. In preparation, 1995.

[44] J. Pitman and M. Yor. Laws of homogeneous functionals of Brownian motion. In preparation, 1996.

[45] J. Pitman and M. Yor. Quelques identités en loi pour les processus de Bessel. To appear in: *Hommage à P.A. Meyer et J. Neveu*, Astérisque, 1996.

[46] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer, Berlin-Heidelberg, 1994. 2nd edition.

[47] L. C. G. Rogers. Williams' characterization of the Brownian excursion law: proof and applications. In *Séminaire de Probabilités XV*, pages 227–250. Springer-Verlag, 1981. Lecture Notes in Math. 850.

[48] L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales.* Wiley, 1987.

[49] T. Shiga and S. Watanabe. Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 27:37–46, 1973.

[50] I. Vincze. Einige zweidimensionale Verteilungs- und Grenzverteilungssätze in der Theorie der georneten Stichproben. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 2:183–209, 1957.

[51] G.N. Watson. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge University Press, 1944.

[52] W. Werner. *Quelques propriétés du mouvement brownien plan.* Thèse de doctorat, Université Paris VI, March 1993.

[53] W. Werner. Sur la forme des composantes connexes du complémentaire de la courbe brownienne plane. *Prob. Th. and Rel. Fields*, 98:307–337, 1994.

[54] D. Williams. Decomposing the Brownian path. *Bull. Amer. Math. Soc.*, 76:871–873, 1970.

[55] D. Williams. Path decomposition and continuity of local time for one dimensional diffusions I. *Proc. London Math. Soc. (3)*, 28:738–768, 1974.

[56] D. Williams. *Diffusions, Markov Processes, and Martingales, Vol. 1: Foundations.* Wiley, Chichester, New York, 1979.

[57] D. Williams. Brownian motion and the Riemann zeta-function. In *Disorder in Physical Systems*, pages 361–372. Clarendon Press, Oxford, 1990.

[58] M. Yor. *Some Aspects of Brownian Motion.* Lectures in Math., ETH Zürich. Birkhaüser, 1992. Part I: Some Special Functionals.