Decomposition of a numerical semigroup as an intersection of irreducible numerical semigroups *

J. C. Rosales† M. B. Branco‡

Abstract

Every numerical semigroup S admits a decomposition $S = S_1 \cap \cdots \cap S_n$ with S_i irreducible (that is, S_i is symmetric or pseudo-symmetric) for all i. We give lower and upper bounds for the minimal number of irreducibles in such a decomposition. We also study the problem of determining those numerical semigroups for which all S_i are symmetric, and when all S_i are pseudo-symmetric. We introduce and characterize the concept of atomic numerical semigroup.

1 Introduction

A **numerical semigroup** is a subset S of $\mathbb{N}$ closed under addition, it contains the zero element and generates $\mathbb{Z}$ as a group (here $\mathbb{N}$ and $\mathbb{Z}$ denote the set nonnegative integers and the set of the integers, respectively). From (see [2] or [10]) we know that the set $\mathbb{N} \setminus S$ is finite. We refer to the greatest integer not belonging to S as the **Frobenius number** of S and denote it by $g(S)$.

We say that a numerical semigroup is **irreducible** if it can not be expressed as an intersection of two numerical semigroups containing it properly. In [7] it is show that S is irreducible if and only if S is maximal in the set of all numerical semigroups with Frobenius number $g(S)$. From [2] and [4] we can deduce that the class

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of irreducible numerical semigroups with odd (respectively even) Frobenius number is the same that the class of **symmetric** (respectively **pseudo-symmetric**) numerical semigroups. This kind of numerical semigroups have been widely studied in literature not only from the semigroupist point of view but also by their applications in Ring Theory. In [3] it is show that the semigroup ring associated to an irreducible numerical semigroup is Gorestein or Kunz if the Frobenius number is odd or even, respectively. This work continues the study begun in [7], [8] and [9]. In particular, if S is a numerical semigroup and we denote by $r(S)$ the least positive integer n such that $S = S_1 \cap \cdots \cap S_n$ with S_i an irreducible numerical semigroup, our aim is the study of $r(S)$. The goal in Sections 2 and 3 is to give an upper bound and lower bound for $r(S)$. We will use these results in Section 3 for characterizing the numerical semigroups that are intersection of symmetric numerical semigroups and those that are intersection of pseudo-symmetric numerical semigroups. Also we introduce and study, in this section, a kind of semigroups which we call atomic numerical semigroups. Along this work the concept of pseudo-Frobenius number of a numerical semigroup plays an importante role (this notation was introduced in [9]).

2 An upper bound for $r(S)$

We denote by $\mathcal{L}(g)$ the set of all numerical semigroups with Frobenius number g. In [7] the following result is presented.

Proposition 1. *If S is a numerical semigroup, then the following conditions are equivalent:*
1) *S is irreducible,*
2) *S is maximal in $\mathcal{L}(g(S))$,*
3) *S is maximal in the set of all numerical semigroups that do not contain $g(S)$.*

Let S be a numerical semigroup. We say that an element $x \in \mathbb{Z}$ is a **pseudo-Frobenius number** of S if $x \notin S$ but $x + s \in S$ for all $s \in S \setminus 0$. We denote by $Pg(S)$ the set of pseudo-Frobenius numbers of S. The cardinal of $Pg(S)$ will be called the **type** of S and denoted by type(S).

In [9] we proved the following result showing the connection between the pseudo-Frobenius number in a numerical semigroup and the Frobenius number in a symmetric semigroup.

Lemma 2. *If S is a numerical semigroup and $x \in \mathbb{Z} \setminus S$, then there exists $g' \in Pg(S)$ such that $g' - x \in S$.*

Given $n \in S \setminus \{0\}$, let $0 = w(1) < w(2) < \cdots < w(n)$ be the smallest elements of S in respective congruence classes mod n. We denote by $\mathrm{Ap}(S, n)$ (the **Apéry set** of n in S (see [1])) the set $\{0 = w(1) < w(2) < \cdots < w(n)\}$. We define in S the following partial order:

$$a \leq_S b \text{ if } b - a \in S.$$

By [4, Proposition 7] we deduce the following result.

Lemma 3. *If S is a numerical semigroup, $n \in S \setminus \{0\}$ and $\{w_{i1}, \ldots, w_{it}\} =$ $\text{maximals}_{\leq_S} \text{Ap}(S, n)$, then $Pg(S) = \{w_{i1} - n, \ldots, w_{it} - n\}$.*

Let $m = \min\,(S \setminus \{0\})$. Note that if $m \neq 1$, then from the previous lemma with $n = m$ we can deduce that $Pg(S) \subseteq \mathbb{N}$. Note also that if $m = 1$, then $S = \mathbb{N}$. In the sequel we assume that $S \neq \mathbb{N}$ and therefore $Pg(S) \subseteq \mathbb{N}$.

Lemma 4. *If S is a numerical semigroup and $x \in \mathbb{N} \setminus S$, then there exists an irreducible numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$ and $g(\overline{S}) = x$.*

Proof. Let $S' = S \cup \{x+1, x+2, \ldots\}$. It is clear that S' is a numerical semigroup with $g(S') = x$. Let $\overline{S}$ be a maximal element in $\mathcal{L}(x)$ such that $S' \subseteq S$. From Proposition 1 we deduce that $\overline{S}$ is an irreducible numerical semigroup. ∎

Lemma 5. *Let $S_1, \ldots, S_n$ be numerical semigroups containing S. The following conditions are equivalent:*

1) *$S = S_1 \cap \ldots \cap S_n$,*
2) *if $g' \in Pg(S)$, then there exists $i \in \{1, \ldots, n\}$ such that $g' \notin S_i$.*

Proof. 1) ⇒ 2) As $g' \notin S = S_1 \cap \cdots \cap S_n$, then there exist $i \in \{1, \ldots n\}$ such that $g' \notin S_i$.

2) ⇒ 1) It is enough to prove that if $x \in \mathbb{N} \setminus S$, then there exists $i \in \{1, \ldots, n\}$ such that $x \notin S_i$. Suppose that $x \notin S$, from Lemma 2, we obtain that there exists $g' \in Pg(S)$ such that $g' - x \in S$. By hypothesis we can find $i \in \{1, \ldots, n\}$ such that $g' \notin S_i$ and since $g' - x \in S \subseteq S_i$ we obtain that $x \notin S_i$. ∎

From [4] we deduce the following result (see also [6, Proposition 3.1]).

Lemma 6. 1) *If $g(S)$ is an odd positive integer, then S is an irreducible numerical semigroup if and only if for all $h, h' \in \mathbb{Z}$, such that $h + h' = g(S)$, we have that $h \in S$ or $h' \in S$ (that is, S is symmetric).*

2) *If $g(S)$ is an even positive integer, then S is an irreducible numerical semigroup if and only if for all $h, h' \in \mathbb{Z} \setminus \{\frac{g(S)}{2}\}$, such that $h + h' = g(S)$, we have that $h \in S$ or $h' \in S$ (that is, S is pseudo-symmetric).*

As a consequence of [6, Theorem 3.3] we obtain the following result.

Lemma 7. *If S is a numerical semigroup , then there exist $B \subseteq \{x \in \mathbb{N} : \ x > \frac{g(S)}{2}\}$ such that $S \cup B$ is an irreducible numerical semigroup and $g(S \cup B) = g(S)$.*

Let S be a numerical semigroup. Define

$$BPg(S) = \{g' \in Pg(S) : \ g' > \frac{g(S)}{2}\}.$$

Theorem 8. *Let S be a numerical semigroup with $BPg(S) = \{g_1, \ldots, g_r\}$. Then there exist $S_1, \ldots, S_r$ irreducible numerical semigroups such that $S = S_1 \cap \cdots \cap S_r$ and $g(S_i) = g_i$ for all $i \in \{1, \ldots, r\}$.*

Proof. Suppose that $g_1 = g(S)$ and S_1 is the irreducible numerical semigroup described in Lemma 7. For each $i \in \{2, \ldots, r\}$, let S_i be an irreducible numerical semigroup such that $S \subseteq S_i$ and $g(S_i) = g_i$ (the existence of S_i is guaranteed by Lemma 4). Now for proving that $S = S_1 \cap \cdots \cap S_r$ we use Lemma 5. If $g' \in Pg(S)$ and $g' \leq \frac{g(S)}{2}$, then $g' \notin S_1$. If $g' \in Pg(S)$ and $g' > \frac{g(S)}{2}$, then $g' = g_i$ for some $i \in \{1, \ldots, r\}$ and therefore $g' \notin S_i$. ∎

From [4] we can deduce that if S is an irreducible numerical semigroup then

$$Pg(S) = \begin{cases} \{g(S)\} & \text{if } g(S) \text{ is odd,} \\ \{g(S), \frac{g(S)}{2}\} & \text{if } g(S) \text{ is even.} \end{cases}$$

From this remark and Theorem 8 we obtain the following result.

Corollary 9. *If S is a numerical semigroup, then the following conditions are equivalent:*
1) *S is irreducible.*
2) *$\#BPg(S) = 1$ (where $\#A$ stands for cardinality(A)).*

Let S be a numerical semigroup. Recall that $r(S)$ is the smallest positive integer n such that $S = S_1 \cap \cdots \cap S_n$ with S_i irreducible numerical semigroups for all $i \in \{1, \ldots, n\}$. As a consequence of Theorem 8 we have the following result.

Corollary 10. *If S is a numerical semigroup, then $r(S) \leq \#BPg(S)$.*

The decomposition given in Theorem 8 is not minimal as the following example illustrates.

Example 11. Let $S = \langle 5, 7 \rangle \cap \langle 5, 8 \rangle = \langle 5, 21, 24, 28, 32 \rangle$. Then $\mathrm{Ap}(S, 5) = \{0, 21, 24, 28, 32\}$, using Lemma 3 we get $Pg(S) = \{16, 19, 23, 27\}$ and so $\#BPg(S) = 4$. Note that a numerical semigroup generated by two elements is symmetric (see [5]) and thus $S = \langle 5, 7 \rangle \cap \langle 5, 8 \rangle$ is a decomposition of S as an intersection of irreducibles.

Corollary 12. *If S is a numerical semigroup such that $\#BPg(S) = 2$, then $r(S) = 2$.*

Proof. If $\#BPg(S) = 2$, then by Corollary 9 we have that S is not an irreducible numerical semigroup and thus $r(S) \geq 2$. Besides, applying Corollary 10 we get that $r(S) \leq 2$. Hence we have that $r(S) = 2$. ∎

Note that, from Example 11, we can see that the converse of Corollary 12 is not true. But there are many semigroups verifying the hypothesis of Corollary 12 as we see in the following example.

Example 13. Let m a positive integer greater than or equal to 3 and let $S = (\{x \in \mathbb{N} : x \geq m\} \setminus \{2m-2, 2m-1\}) \cup \{0\}$. The reader can prove that S is a numerical semigroup and $Pg(S) = \{2m-2, 2m-1\}$. Applying Corollary 12 we get that $r(S) = 2$.

3 A lower bound for $r(S)$

Along this section we suppose that S is a numerical semigroup and $BPg(S) = \{g_1, \dots, g_r\}$. For each $i \in \{1, \dots, r\}$, define

$$\xi(g_i) = \{g_i + x : \; x \in \mathbb{N} \text{ and } g_i + x \notin \langle S, x \rangle\}.$$

Theorem 14. *Let $g_i \in \{g_1, \dots, g_r\}$. If $\overline{S}$ is an irreducible numerical semigroup such that $S \subseteq \overline{S}$ and $g_i \notin \overline{S}$, then $g(\overline{S}) \in \xi(g_i)$. Conversely, if $g_i + x \in \xi(g_i)$ then there exists an irreducible numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$, $g_i \notin \overline{S}$ and $g(\overline{S}) = g_i + x$.*

Proof. If $g_i \notin \overline{S}$, then by Lemma 6 we get that $g(\overline{S}) - g_i \in \overline{S}$ (note that $g(\overline{S}) \leq g(S)$ and that $g_i > \frac{g(S)}{2}$ and therefore $g_i \neq \frac{g(\overline{S})}{2}$). Since $g_i + (g(\overline{S}) - g_i) = g(\overline{S}) \notin \overline{S} \supseteq \langle S, g(\overline{S}) - g_i \rangle$ we obtain that $g(\overline{S}) \in \xi(g_i)$.

Conversely, if $g_i + x \in \xi(g_i)$, then $g_i + x \notin \langle S, x \rangle$. Let $\overline{S}$ be an irreducible numerical semigroup such that $\langle S, x \rangle \subseteq \overline{S}$ and $g_i + x = g(\overline{S})$ (the existence of $\overline{S}$ is guaranteed by Lemma 4). Since $x \in \overline{S}$ and $g_i + x = g(\overline{S}) \notin \overline{S}$, we obtain that $g_i \notin \overline{S}$. ■

Corollary 15. *If $S = S_1 \cap \cdots \cap S_n$ with $S_1, \dots, S_n$ irreducible numerical semigroups, then for each $i \in \{1, \dots, r\}$ there exists $j \in \{1, \dots, n\}$ such that $g(S_j) \in \xi(g_i)$.*

Proof. If $i \in \{1, \dots, r\}$, then $g_i \notin S = S_1 \cap \cdots \cap S_n$ and therefore there exists $j \in \{1 \dots, n\}$ such that $g_i \notin S_j$. Using Theorem 14 we get that $g(S_j) \in \xi(g_i)$. ■

Corollary 16. *Let $x_1, \dots, x_r \in \mathbb{N}$ such that $g_i + x_i \in \xi(g_i)$ for all $i \in \{1, \dots, r\}$. Then there exist irreducible numerical semigroups $S_1, \dots, S_r$ such that $S = S_1 \cap \cdots \cap S_r$ and $\{g(S_1), \dots, g(S_r)\} \subseteq \{g_1 + x_1, \dots, g_r + x_r\}$.*

Proof. Assume that $g_1 = g(S)$. Note that $\xi(g_1) = \{g_1\}$ and thus $x_1 = 0$. Let S_1 be the numerical semigroup $S \cup B$ described in Lemma 7. Now, for each $i \in \{2, \dots, r\}$ let S_i be an irreducible numerical semigroup such that $S \subseteq S_i$, $g_i \notin S_i$ and $g(S_i) = g_i + x_i$ (the existence of S_i is guaranteed by Theorem 14). Applying Lemma 5 we can deduce that $S = S_1 \cap \cdots \cap S_r$. ■

Let A be a subset of $\mathbb{N}$. We say that S is an $A-$semigroup if S can be expressed as an intersection of irreducible numerical semigroups whose Frobenius numbers are in A (that is, $S = S_1 \cap \cdots \cap S_n$ with S_i irreducible numerical semigroups and $g(S_i) \in A$ for all $i \in \{1, \dots, n\}$). Denote by $h(S) = \min\{\#A : \; S \text{ is an } A-\text{semigroup}\}$.

Corollary 17. *If A is a subset of $\mathbb{N}$, then the following conditions are equivalent:*
1) *S is an A-semigroup,*
2) *there exist $(a_1, \dots, a_r) \in \xi(g_1) \times \cdots \times \xi(g_r)$ such that $\{a_1, \dots, a_r\} \subseteq A$.*

Proof. 1) ⇒ 2) This is a consequence of Corollary 15.

2) ⇒ 1) Follows from Corollary 16. ■

Corollary 18. *If S is a numerical semigroup, then $r(S) \geq h(S) = \min\{\#\{a_1, \ldots, a_r\} : (a_1, \ldots, a_r) \in \xi(g_1) \times \cdots \times \xi(g_r)\}$.*

Proof. As a consequence of Corollary 17 we get that

$$h(S) = \min\{\#\{a_1, \ldots, a_r\} : \ (a_1, \ldots, a_r) \in \xi(g_1) \times \cdots \times \xi(g_r)\}.$$

Now we see that $r(S) \geq h(S)$. In fact, if $S_1, \ldots, S_n$ are irreducible numerical semigroups such that $S = S_1 \cap \cdots \cap S_n$, then S is a $\{g(S_1), \ldots, g(S_n)\}$ – semigroup and thus $n \geq \#\{g(S_1), \ldots, g(S_n)\} \geq h(S)$. Hence $r(S) \geq h(S)$. ■

Note that if we take again $S = \langle 5, 7\rangle \cap \langle 5, 8\rangle = \langle 5, 21, 24, 28, 32\rangle$ (see Example 11) we know that $r(S) = 2$. Remember that $BPg(S) = \{16, 19, 23, 27\}$ and so $\xi(16) = \{16, 23\}$, $\xi(19) = \{19, 27\}$, $\xi(23) = \{23\}$ and $\xi(27) = \{27\}$. Applying Corollary 18, we obtain that $h(S) = 2$ and therefore $h(S) = r(S)$. Note that there are many examples for which the previous equality does not hold. Observe that if S_1 and S_2 are irreducible numerical semigroups with $g(S_1) = g(S_2)$, then $r(S_1 \cap S_2) = 2$ and $h(S_1 \cap S_2) = 1$.

4 Some remarks

We say that a numerical semigroup is **odd** (respectively **even**) if it can be expressed as an intersection of irreducible numerical semigroups with odd (respectively even) Frobenius numbers. Note that odd (respectively even) numerical semigroups are the numerical semigroups that are intersection of symmetric (respectively pseudo-symmetric) numerical semigroups. If $S, S_1, \ldots, S_n$ are numerical semigroups and $S = S_1 \cap \cdots \cap S_n$, then $g(S) = \max\{g(S_1), \ldots, g(S_n)\}$ and therefore if S is an odd (respectively even) numerical semigroup, then $g(S)$ is odd (respectively even). Note also that every numerical semigroup is odd, even, or an intersection of an odd and an even numerical semigroup.

As a consequence of Corollary 17 we get the following result that is a generalization and an improvement of Theorem 15 of [9].

Corollary 19. *If S is a numerical semigroup and $BPg(S) = \{g_1, \ldots, g_r\}$, then the following conditions are equivalent:*

1) *S is an odd (respectively even) numerical semigroup,*
2) *$\xi(g_i)$ contains at least an odd (respectively even) element for all $i \in \{1, \ldots, r\}$.*

Note that a numerical semigroup is a $\{g\}$ – semigroup if $S = S_1 \cap \cdots \cap S_n$ with S_i an irreducible numerical semigroup and $g(S_i) = g$ for all $i \in \{1, \ldots, n\}$. Observe that S is an $\{g\}$ – semigroup if only if $h(S) = 1$.

As an immediate consequence of Corollary 17 we obtain the following result.

Corollary 20. *If S is a numerical semigroup and $BPg(S) = \{g_1, \ldots, g_r\}$, then the following conditions are equivalent:*

1) *S is an $\{g(S)\}$ – semigroup,*
2) *$g(S) \in \xi(g_i)$ for all $i \in \{1, \ldots, r\}$.*

Let g a positive integer and

$$\widehat{\mathcal{L}(g)} = \{S : \ S \text{ is a numerical semigroup with } g(S) \leq g\}.$$

Note that $(\widehat{\mathcal{L}(g)}, \cap)$ is a semigroup and, as a consequence of Theorem 8, the set of irreducible numerical semigroups of $\widehat{\mathcal{L}(g)}$ is a minimal system of generators for it.

Recall that

$$\mathcal{L}(g) = \{S : \ S \text{ is a numerical semigroup with } g(S) = g\}.$$

Note that $\mathcal{L}(g)$ is a subsemigroup of $(\widehat{\mathcal{L}(g)}, \cap)$. An element in $\mathcal{L}(g)$ is an **atom** if it is not an intersection of two elements of $\mathcal{L}(g)$ containing it properly. Note that an irreducible numerical semigroup of $\mathcal{L}(g)$ is an atom, but in general the converse is not true (see Example 26).

Lemma 21. *Let S and $\overline{S}$ be two elements in $\mathcal{L}(g)$ such that $S \subsetneq \overline{S}$ and let $x = \max\,(\overline{S} \setminus S)$. Then $S \cup \{x\} \in \mathcal{L}(g)$.*

Proof. From the definition of x we obtain that $2x \in S$ and $x+s \in S$ for all $s \in S \setminus \{0\}$. Hence $S \cup \{x\}$ is a numerical semigroup. Since $x \in \overline{S}$, then $x \neq g(\overline{S}) = g$ and thus $g(S \cup \{x\}) = g$. ∎

Lemma 22. *If $S \in \mathcal{L}(g)$ and S is not an atom of $\mathcal{L}(g)$, then there exist $x_1, x_2 \in \mathbb{N} \setminus S$ such that $x_1 \neq x_2$ and $S \cup \{x_1\}$ and $S \cup \{x_2\}$ are elements of $\mathcal{L}(g)$.*

Proof. If S is not an atom, then there exist $S_1, S_2 \in \mathcal{L}(g)$ such that $S \subset S_1$ and $S \subset S_2$ and $S = S_1 \cap S_2$. Assume that $x_i = \max(S_i \setminus S)$ for $i = 1, 2$. Applying Lemma 21 we obtain that $S \cup \{x_1\}, S \cup \{x_2\} \in \mathcal{L}(g)$. Note that $x_1 \neq x_2$ because otherwise we would have $x_1 = x_2 \in S_1 \cap S_2 = S$, which contradicts $x_1 \notin S$. ∎

Lemma 23. *Let S be a numerical semigroup and $x \in \mathbb{N} \setminus S$. Then $S \cup \{x\}$ is a numerical semigroup if only if $x \in Pg(S)$ and $2x \notin Pg(S)$.*

Proof. If $S \cup \{x\}$ is a numerical semigroup, then $x + s \in S$ for all $s \in S \setminus \{0\}$ and thus $x \in Pg(S)$. Furthermore $2x \in S$ and whence $2x \notin Pg(S)$.

Conversely, if $x \in Pg(S)$, then $x + s \in S$ for all $s \in S \setminus \{0\}$. If $2x \notin Pg(S)$ then, since $x \in Pg(S)$, we can deduce that $2x \in S$. Hence $S \cup \{x\}$ is a numerical semigroup. ∎

Proposition 24. *If $S \in \mathcal{L}(g)$, then the following conditions are equivalent:*

1) *S is not an atom of $\mathcal{L}(g)$,*
2) *there exist $x_1, x_2 \in Pg(S) \setminus \{g\}$ such that $x_1 \neq x_2$ and $\{2x_1, 2x_2\} \cap Pg(S) = \emptyset$.*

Proof. 1) ⇒ 2) By Lemma 22 we know that there exist $x_1, x_2 \in \mathbb{N}$ such that $x_1 \neq x_2$ and $S \cup \{x_1\}$ and $S \cup \{x_2\}$ are elements of $\mathcal{L}(g)$. Using Lemma 23 and the fact that $g \notin S \cup \{x_1\}$ and $g \notin S \cup \{x_2\}$, we deduce that $x_i \in Pg(S) \setminus \{g\}$ and $2x_i \notin Pg(S)$ for $i = 1, 2$.

2) ⇒ 1) From Lemma 23 we deduce that $S \cup \{x_1\}$, $S \cup \{x_2\} \in \mathcal{L}(g)$. Since $S = (S \cup \{x_1\}) \cap (S \cup \{x_2\})$, we have that S is not an atom of $\mathcal{L}(g)$ ∎

As an immediate consequence of the previous proposition we get the following result.

Corollary 25. *If S is an numerical semigroup and* $\text{type}(S) \in \{1, 2\}$*, then S is an atom of $\mathcal{L}(g(S))$.*

Example 26. Let $S = \langle 4, 5, 11 \rangle$. Then $Pg(S) = \{6, 7\}$ (see Example 13) and therefore $\text{type}(S) = 2$. Applying the previous corollary, we get that S is an atom of $\mathcal{L}(7)$. Note also that S is not irreducible because, using Lemma 23, we have that $S \cup \{6\}$ and $S \cup \{7\}$ are numerical semigroups and $S = (S \cup \{6\}) \cap (S \cup \{7\})$.

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Departamento de Álgebra, Universidad de Granada,
E-18071 Granada, Spain
email : jrosales@ugr.es

Departamento de Matemática, Universidade de Évora,
7000 Évora, Portugal
email : mbb@dmat.uevora.pt