DECOMPOSITION OF CERTAIN KLEINIAN GROUPS

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The purpose of this note is to give an analytic and geometric description of the class of Kleinian groups which are finitely generated and which have an invariant component.

If one starts with a collection of "basic" groups, and forms finite "combinations" of these groups, one gets a class of "constructible" Kleinian groups. In this paper, our combinations occur in the sense of the Combination Theorems appearing in [8] and [9], where the amalgamated subgroups (Combination I) and the conjugated subgroups (Combination II) are trivial or elliptic cyclic or parabolic cyclic.

To describe our basic groups, we recall the following definitions. A point z lies in the *limit set* $\Lambda(G)$ if there is a sequence $\{g_n\}$ of distinct elements of G, and there is a point z_0 with $g_n z_0 \to z$. The *set of discontinuity* $\Omega(G)$ is the complement of $\Lambda(G)$. The connected components of $\Omega(G)$ are called *components* of G. A component Δ_0 of G is *invariant* if $g(\Delta_0) = \Delta_0$ for all $g \in G$.

If $\Lambda(G)$ is a finite set, then G is *elementary*. If G is non-elementary and has a simply-connected invariant component Δ_0, then there is a conformal map φ from Δ_0 onto the unit disc. A parabolic element $g \in G$ is called *accidental* if $\varphi g \varphi^{-1}$ is hyperbolic. By definition, elementary groups do not contain accidental parabolic transformations.

For the purposes of this paper, a *basic group* is a finitely-generated Kleinian group which has a simply-connected invariant component, and which contains no accidental parabolic transformations.

The basic groups are, in a sense, all known, it was shown in [11] (for proof, see Bers [4] and Kra–Maskit [7]) that every basic group is either elementary, degenerate, or quasi-Fuchsian. The *degenerate* groups are such that $\Omega(G)$ is both connected and simply-connected. A *quasi-Fuchsian* group is a quasiconformal deformation of a Fuchsian group.

We define the class $\mathcal{C}_1$ as being the class of Kleinian groups which have an invariant component, and which can be built up in a finite number of steps from the basic groups using the Combination Theorems mentioned above (precise statements of these theorems, and the conventions for their use, appear in section 1).

THEOREM 1. *The class $\mathcal{C}_1$ is the class of finitely-generated Kleinian groups having an invariant component.*

The proof of Theorem 1 appears in section 3.

We remark that the Combination Theorems given in [10] are technically different from those given in [8] and [9]. Starting with the elementary and quasi-Fuchsian groups as basic groups and using the Combinations [10], one obtains the subclass $\mathcal{C}_0 \subset \mathcal{C}_1$ of "nice" Kleinian groups. This subclass will be discussed elsewhere.

Theorem 1 asserts that, given $G \in \mathcal{C}_1$, there is a collection $G_1, \ldots, G_s$ of subgroups of G, so that G is formed from $G_1, \ldots, G_s$ by $s-1$ applications of Combination I, and say t applications of Combination II. Our next main result is that the subgroups $G_1, \ldots, G_s$ and the number t are essentially unique. In order to make them unique, we need some conventions regarding the use of the Combination Theorems. The essence of these conventions is that they guarantee uniqueness, in a simple fashion, for the elementary basic groups. Precise statements of these conventions appear in section 2. With these conventions, the subgroups $G_1, \ldots, G_s$ are then called the *basic subgroups* of G.

In general a subgroup G' of G is called a *factor subgroup* if G' is a maximal subgroup of G with the following properties: the invariant component of G', which contains the invariant component of G, is simply-connected; G' contains no accidental parabolic transformations; if $g \in G$ is parabolic and the fixed point of g lies in $\Lambda(G')$, then $g \in G'$.

The Combination Theorems are geometric versions of simple group-theoretic operations; Combination Theorem I is the free product with amalgamation. The next theorem is a geometric version of the Kurosh Subgroup Theorem.

THEOREM 2. *Every factor subgroup G' of a group $G \in \mathcal{C}_1$ is conjugate in G to a unique basic subgroup of G.*

The proof of Theorem 2 and its corollaries, appears in section 2.

Theorem 2 asserts that the basic subgroups $G_1, \ldots, G_s$ form a complete set of non-conjugate factor subgroups of G, and so the basic subgroups are unique up to order and conjugation.

COROLLARY 1. *Let $G \in \mathcal{C}_1$. Then there are only finitely many conjugacy classes of factor subgroups of G, and each factor subgroup is finitely generated.*

If Δ is a component of G other than the invariant component, then the subgroup G_Δ

of G keeping Δ invariant is called a *component subgroup of* G. It is well known that every component subgroup is a quasi-Fuchsian subgroup.

COROLLARY 2. *For* $G \in \mathcal{C}_1$, *the set of component subgroups equals the set of quasi-Fuchsian factor subgroups.*

COROLLARY 3. *Let* $G \in \mathcal{C}_1$ *and let* $g \in G$ *be parabolic or elliptic. Then* g *is an element of some factor subgroup of* G.

There is a detailed description of the limit set of a general finitely-generated Kleinian group due to Abikoff [1]. The following result is essentially a special case.

COROLLARY 4. *Let* $G \in \mathcal{C}_1$, *and let* z *be a limit point of* G. *Then either there is a factor subgroup* G', *with* z *a limit point of* G', *or the following holds. There is a simple closed curve* γ, *which is invariant under a finite (perhaps trivial) or parabolic cyclic subgroup* H *of* G, *and which lies, except for the fixed point of* H, *in* $\Omega(G)$. *There is a sequence* $\{g_n\}$ *of elements of* G, *where* $g_n(\gamma)$ *nests about* z.

A sequence $\{\gamma_n\}$ of simple closed curves *nests about* z, if the (spherical) diameter of $\gamma_n \to 0$, and for each $n > 1$, γ_n separates z from γ_{n-1}.

COROLLARY 5. *Let* $G \in \mathcal{C}_1$ *and let* G_1' *and* G_2' *be factor subgroups of* G. *Then either* $G_1' \cap G_2' = \phi$, *or* $G_1' \cap G_2' = H$ *is a parabolic or elliptic cyclic group, maximal (as a cyclic subgroup) in* G.

In general, if one has a finitely-generated Kleinian group G with an invariant component Δ_0, then by Ahlfors' Finiteness Theorem [3], $\Delta_0/G = X_0$ is a finite Riemann surface. That is, X_0 is a closed surface of genus g, with finitely many points removed, and with finitely many points where the projection $p:\Delta_0 \to X_0$ is ramified. There are in all, say, n removed points and points of ramification, call them $x_1, \ldots, x_n$. Each x_i has a *branch number* ν_i, $2 \leqslant \nu_i \leqslant \infty$ associated with it, where $\nu_i = \infty$ if x_i has no preimage in Δ_0, otherwise near some preimage of x_i, p is ν_i-to-one. The *signature* of G (or of X_0) is then $(g, n; \nu_1, \ldots, \nu_n)$.

In the special case of a factor subgroup G_i of G, we need to enlarge the notion of signature to include some of the interaction of G_i with the rest of G. If G_i has signature $(g_i, n_i; \nu_{i1}, \ldots, \nu_{in_i})$, then there is a correspondence between each of the n_i points and a conjugacy class of elliptic or parabolic cyclic subgroups of G; the order of a cyclic subgroup H_{ij} in the class corresponding to ν_{ij} is ν_{ij}. By Corollary 5, we know that for each other factor subgroup G', either $H_{ij} \subset G'$, or $H_{ij} \cap G' = 1$. The j-th puncture is a *connector* if H_{ij} is contained in some other factor subgroup, or if the normalizer, N_{ij} of H_{ij} in G, contains H_{ij} as a subgroup of infinite index. We let m_i be the number of connectors and let $k_i = n_i - m_i$ be the number of non-connectors. The (*extended*) *signature* is then

$$(g_i, n_i, k_i; \nu_{i1}, \ldots, \nu_{ik_i}, \mu_{i1}, \ldots, \mu_{im_i}).$$

We let $(g, n; \nu_1, \ldots, \nu_n)$ be the signature of G, acting on the invariant component Δ_0.

THEOREM 3.

(a) $$\sum_{i=1}^{s} g_i = g - t$$

(b) $$\sum_{i=1}^{s} k_i = n,$$

and $(\nu_1, \ldots, \nu_n)$ *is a rearrangement of* $(\nu_{11}, \ldots, \nu_{sk_s})$.

We assume that the basic groups have been ordered so that $G_1, \ldots, G_p$ are precisely the quasi-Fuchsian basic subgroups. Then by Corollary 2, we have

$$\Omega(G)/G = \Delta_0/G + \Delta_1/G_1 + \ldots + \Delta_p/G_p = X_0 + X_1 + \ldots + X_p,$$

where we use "+" for disjoint union, and each Δ_i is the component of G_i which does not intersect Δ_0.

The equalities in Theorem 3 can be used to derive inequalities for the component subgroups. Specifically we get inequalities for dim $B_q(X_i)$, $q \geqslant 2$, the dimension of the space of bounded q-forms; $A(X_i)$, the non-Euclidean area; and $\chi(X_i)$ the (negative) Euler characteristic, where all branch points are considered as punctures.

In order to state the inequalities, we need several parameters, most of these are zero unless G has elementary factor subgroups.

The total number of connectors is $r = \sum_{i=1}^{s} m_i$. Some factor subgroups have no connectors; let r_0 be the number of elementary basic subgroups for which $m_i = 0$. The number of basic subgroups which are cyclic but non-trivial is r_1; we write $r_1 = r_1' + r_1''$, where r_1' is the number of these with (extended) signature $(0, 2, 0; \mu, \mu)$, r_1'' is the number with (extended) signature $(0, 2, 2; \nu, \nu)$.

The number of basic subgroups which are elementary with signature $(0, 3; \alpha, \beta, \gamma)$ is r_2.

The number of basic subgroups, necessarily elementary, with signature $(0, 4; 2, 2, 2, 2)$, respectively $(1, 0)$, is denoted by r_3, respectively, r_4.

THEOREM 4. *Let* G *in* $\mathfrak{C}_1$ *have at least two components. Then*

(a) $$g - \sum_{i=1}^{p} g_i \geqslant t + r_4$$

(b) $$\dim B_2(X_0) - \sum_{i=1}^{p} \dim B_2(X_i) \geqslant 3(s+t-1) - r - r_1 + r_3$$

(c) $$\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) \geqslant (2q-1)(s+t-1) - (q-1)(r+r_0) - (s-p),\ q \geqslant 2$$

(d) $$\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) \geqslant (2q-1)(s+t-1) - (q-1)(r+r_0) - r_1',\ q = 2, 4, 6, \ldots$$

(e) $$A(X_0) - \sum_{i=1}^{p} A(X_i) \geqslant 2\pi(2(s+t-1) - r - r_0)$$

(f) $$\chi(X_0) - \sum_{i=1}^{p} \chi(X_i) \geqslant 2(s+t-1) - r + r_2 + 2r_3$$

Inequalities 4(b) *and* 4(f) *are simultaneously sharp; equality occurs if and only if no factor subgroup of G is degenerate.*

Inequalities 4(b), (c) and (d), with RHS zero are not new. Ahlfors [3] discovered 4(b), the others are due to Bers [5]. In the case that Δ_0 is simply connected, inequalities similar to the above appear in [11].

The above inequalities have obscure right hand sides; these are clarified in Theorem 5, where non-negative lower bounds are given.

It should be remarked that one can naively count parameters as one combines groups; this was in fact first done by Klein [6]. One can view equality in 4(b) as asserting that the dimension of the space of bounded quadratic differentials on G is equal to the naive parameter count. One expects that "nice" groups, for which equality in 4(b) holds, are quasiconformally stable in the sense of Bers [4]. This will be pursued elsewhere.

THEOREM 5. *Let G be as in Theorem* 4.

(a) *For* $q \geqslant 2$, $\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) \geqslant p + t - 1$.

(b) *For* $q = 2, 4, 6, \ldots$, $\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) \geqslant s + t - 1$.

(c) *If for some even* $q \geqslant 2$, $\dim B_q(X_0) = \sum_{i=1}^{p} \dim B_q(X_i)$, *then G is quasi-Fuchsian.*

(d) $A(X_0) - \sum_{i=1}^{p} A(X_i) \geqslant 0$; *if equality holds, then Δ_0 is simply-connected and G contains no degenerate factor subgroups.*

(e) $r \leqslant 2(s+t-1) \leqslant 2 \dim B_2(X_0)$.

(f) $p \leqslant \sum_{i=1}^{p} \chi(X_i) \leqslant \chi(X_0)$.

Theorems 3, 4, and 5 are all proven in section 4.

Combining the constructions given in [12] and [11], one can easily show that the second inequality in 4(f) is sharp. That is, if we are given Riemann surfaces X_0, X_1, ..., X_p, satisfying this inequality then, in general, there is a Kleinian group G with $\Omega/G = X_0 + ... + X_p$. However, some of the branch numbers of $X_1 + ... + X_p$ are determined by the branch number of X_0, the others must be chosen in pairs; because of considerations involving elementary groups one must in general exclude branch numbers 2 and 3.

One can view the inequalities given in Theorem 4 as being an analytic description of all groups in $\mathcal{C}_1$. One can also get a precise geometric description. We look at the surface X_0', which is X_0 with the branch points deleted. On X_0' a set of simple disjoint loops $\{w_1, ..., w_q\}$ is called *homotopically independent*, if no w_j bounds either a disc or a punctured disc, and if for $i \neq j$, w_i and $w_j^{\pm 1}$ are not freely homotopic.

THEOREM 6. *Let $G \in \mathcal{C}_1$. Then there is a homotopically independent set of loops $\{w_1, ..., w_q\}$ on X_0', and there is a set of "integers" $\{\alpha_1, ..., \alpha_q\}$, $1 \leq \alpha_i \leq \infty$, as follows. Let Y_1', ..., Y_s' be the connected components of $X_0 - \{w_1 \cup ... \cup w_q\}$. There are $2q$ boundary curves on $Y_1' \cup ... \cup Y_s'$; let Y_1, ..., Y_s be the surfaces obtained by sewing in $2q$ discs along these curves, where we pick one point in each of the two discs bounded by w_i, and give it branch number α_i. The resulting surfaces Y_1, ..., Y_s are topologically equivalent to the surfaces X_1, ..., X_s.*

Essentially all possibilities for homotopically independent sets, branch numbers, and conformal structures on X_0, X_1, ..., X_p, can be realized. These were discussed in [12] and [11] and will not be pursued here.

We remark that Theorem 5(b) gives a characterization of quasi-Fuchsian groups. One can also use these results to characterize other classes of Kleinian groups; for example, using the result in [13], one can characterize the Schottky groups as follows. *A group G in $\mathcal{C}_1$ is a Schottky group if and only if G contains no non-trivial factor subgroups.*

1. Combination Theorems

Let H be a subgroup of the Kleinian group G. A set T is called *precisely invariant* under H if $HT = T$, and $gT \cap T = \phi$, for $g \in G - H$.

For a cyclic subgroup H, a *precisely invariant disc* B is the interior of a closed topological disc, with closure $\bar{B}$, where $\bar{B} - \Lambda(H)$ is precisely invariant under H, and $(\bar{B} - \Lambda(H)) \subset \Omega(G)$.

We need the following forms of the Combination Theorems.

COMBINATION THEOREM I. *For $i=1, 2$, let B_i be a precisely invariant disc under H, a cyclic subgroup of both G_1 and G_2. Assume that B_1 and B_2 have common boundary γ and $B_1 \cap B_2 = \phi$. Let G be the group generated by G_1 and G_2. Then*

(1) *G is Kleinian;*

(2) *G is the free product of G_1 and G_2 with amalgamated subgroup H;*

(3) $\Omega(G)/G = (\Omega(G_1)/G_1 - B_1/H) \cup (\Omega(G_2)/G_2 - B_2/H)$, *where* $\{\Omega(G_1)/G_1 - B_1/H\} \cap \{\Omega(G_2)/G_2 - B_2/H\} = \gamma \cap \Omega(H)/H$.

(4) *If $z \in \Lambda(G)$, and z is not a limit point of a conjugate of either G_1 or G_2, then there is a sequence $\{j_n\}$ of elements of G so that $j_n(\gamma)$ nests about z.*

(5) *If H is its own normalizer in either G_1 or G_2, then every elliptic or parabolic element of G lies in a conjugate of either G_1 or G_2.*

One easily sees that the hypotheses given above are a restatement of the hypotheses in [8], where conclusions (1)–(4) are proven. Conclusion (5) for the case that H is its own normalizer in both G_1 and G_2 is proven in [10]; the more general case is a simple modification of the argument given there.

COMBINATION THEOREM II. *Let G_1 be a Kleinian group. For $i=1, 2$, let B_i be a precisely invariant disc for the cyclic subgroup H_i, and let γ_i be the boundary of B_i. We assume that $g(\bar{B}_1) \cap \bar{B}_2 = \phi$ for all g in G_1. Let G' be cyclic, generated by f, where $f\gamma_1 = \gamma_2$, $f(B_1) \cap B_2 = \phi$, and $f^{-1}H_2 f = H_1$. Let G be the group generated by G_1 and G'. Then*

(1) *G is Kleinian;*

(2) *Every relation in G is a consequence of the relations in G_1 and the relations $f^{-1}H_2 f = H_1$.*

(3) $\Omega(G)/G = \Omega(G_1)/G_1 - (B_1/H_1 \cup B_2/H_2)$, *where $(\gamma_1 \cap \Omega(G))/H_1$ is identified in $\Omega(G)/G$ with $(\gamma_2 \cap \Omega(G))/H_2$.*

(4) *For every point $z \in \Lambda(G)$, where z is not a limit point of a conjugate in G of either G_1 or G', there is a sequence j_n of elements of G, where $j_n(\gamma_1)$ nests about z.*

(5) *If each of H_1 and H_2 is its own normalizer in G_1, then every elliptic or parabolic element of G lies in some conjugate of G_1.*

One easily sees that the hypotheses given above imply those of [9], where conclusions (1)–(4) above are proven. The proof of conclusion (5) appears in [10].

There are two technical assumptions, concerning elementary groups, in the definition of the class $\mathcal{C}_1$. The first assumption is that we want to regard an elementary group with one limit point and signature $(0, 4; 2, 2, 2, 2)$ as a basic group and not as the group formed via Combination Theorem I from two groups each of signature $(0, 3; 2, 2, \infty)$. To this end, we require first, in the use of Combination Theorem I, *that if H is parabolic cyclic, then H must be its own normalizer in either G_1 or G_2*. Our second requirement is that, in using Combination Theorem II, we require that *if H_1 and H_2 are parabolic cyclic, then each is its own normalizer in G_1*.

We remark that the conventions above are precisely those needed for the use of conclusion (5) in the Combination Theorem. Hence, with these conventions, conclusion (5) of Combination Theorem I (II) says that every elliptic or parabolic element of G is a conjugate of some element of G_1 or G_2 (G_1).

Our second assumption is a minimality condition on the number of operations. If we take the free product (Combination Theorem I) of G_1 and a parabolic or elliptic cyclic group G_2, and then adjoin f (Combination Theorem II) which conjugates G_2 and a cyclic subgroup of G_1, we get a group G. This group G could equally well have been obtained from G_1 by adjoining an element f (Combination Theorem II) which conjugates the identity. In this case we do not want the subgroup G_2 to appear as a basic group. We thus require in our use of Combination Theorem II, that the cyclic subgroups H_1 and H_2 must be equal or must each be a proper subgroup of a conjugate of a basic group.

Before going on to the proofs of the theorems, we mention some of the properties of the Combination Theorems that will be used.

We start with Combination Theorem I. It follows from conclusion 2 that for $g \in G$, $g\gamma = \gamma$ if and only if $g \in H$. If $g \notin H$, then $g\gamma \cap \gamma \neq \phi$ if and only if H is parabolic and g belongs to the normalizer of H in G_1 or G_2; i.e. g belongs to either G_1 or G_2 and $g\gamma \cap \gamma$ is the fixed point of H. It follows that the translates of γ under G divide Δ_0 (or the extended complex plane) into regions; each of these regions is invariant under a conjugate of G_1 or G_2; in fact, the subgroup of G keeping one of these regions invariant is precisely a conjugate of G_1 or G_2.

It was proven in [8], that if $g_n(\gamma)$ are all distinct, then their diameters (measured on the sphere) form a null-sequence.

LEMMA 1 (I). *Let G_1' be G_1 or G_2 and let G_2' be gG_1g^{-1} or gG_2g^{-1}, for some $g \in G$. Then either $G_1' = G_2'$, or $G_1' \cap G_2' = 1$, or $G_1' \cap G_2' = \tilde{g}H\tilde{g}^{-1}$, for some $\tilde{g} \in G$.*

Proof. Let R_1 and R_2 be the regions kept invariant by G_1' and G_2', respectively. Let w be a path from an interior point of R_1 to an interior point of R_2, where w crosses each translate of γ at most once. We assume that $R_1 \neq R_2$, so that w crosses at least one translate of γ, say $g_1(\gamma)$. We can assume that $g_1(\gamma)$ lies in the boundary of R_1. Since no element of G_1' which is not also in $g_1Hg_1^{-1}$ can keep R_2 invariant, $G_1' \cap G_2' \subset g_1Hg_1^{-1}$.

If $G_1' \cap G_2'$ is non-trivial, then let R_1'' be the region on the other side of $g_1(\gamma)$, and let G_1'' be the subgroup of G keeping R_1'' invariant. Observe that $G_1'' \cap G_1' = g_1 H g_1^{-1} \supset G_1' \cap G_2'$, and we can connect R_1'' to R_2 by a path which passes through one fewer translate of γ.

The same sort of remarks are also true for Combination Theorem II. For every $g \in G$, either $g\gamma_1 = \gamma_1$, in which case $g \in H_1$, or $g\gamma_1 \cap \gamma_1 = \phi$. (We are assuming that our convention dealing with the normalizers of H_1 and H_2 is in force.) The translates of γ_1 under G divide Δ_0 (or the extended complex plane) into regions. Each conjugate of G_1 determines one of these regions; it is the subgroup of G keeping the region invariant. The union of all these regions, together with the translates of γ_1, contains $\Omega(G)$, for any sequence of distinct translates of γ_1 has (spherical) diameter converging to zero [9].

We repeat the proof of Lemma 1 (I), and obtain

LEMMA 1 (II). *If G_1' and G_2' are conjugates of G_1, then $G_1' = G_2'$, or $G_1' \cap G_2' = 1$, or $G_1' \cap G_2' = \tilde{g} H \tilde{g}^{-1}$, for some $\tilde{g} \in G$.*

2. Proof of Theorem 2

In order to prove Theorem 2, it suffices to show that every factor subgroup of G is contained in a conjugate of a basic subgroup, that every basic subgroup is in competition to be a factor subgroup, and that no conjugate of a basic subgroup is contained in another.

We take up the last statement first. Using Lemmas 1 (I) and 1 (II) inductively, we see that the only possibility for a conjugate of a basic group to be contained in another, is if the first group is cyclic and is used as an amalgamating or conjugated subgroup. Our convention specifically forbids this.

We next suppose we are given a factor subgroup J, and we want to show that it is conjugate to a subgroup of a basic group. We have several cases to consider.

Case 1. J is cyclic. J is then necessarily parabolic or elliptic cyclic, or trivial, and the result is immediate from conclusion 5 of the Combination Theorems.

Case 2. J is finite, but not cyclic. The proof is by induction on the Combinations.

Suppose the finite non-cyclic group J is a subgroup of G, the group formed via Combination Theorem I from G_1 and G_2. We pick generators j_1, j_2 for J. By conclusion 5, we can assume that there are conjugates of G_1 or G_2, call them G_1', G_2', so that $j_1 \in G_1'$, $j_2 \in G_2'$. Let R_1, R_2 be the regions bounded by translates of γ, kept invariant under G_1', G_2', respectively. As in the proof of Lemma 1 (I), let w be a path, with minimal crossings of translates of γ, which goes from some point of R_1 to some point of R_2. Each translate γ' of γ crossed by w must be invariant under either j_1 or j_2; for if not, the two topological discs bounded by γ would be precisely invariant under the identity as a subgroup of the cyclic groups generated by j_1 and

j_2. It follows then that either R_1 has a translate of γ on its boundary which is invariant under j_2, or R_2 has a translate of γ on its boundary which is invariant under j_1, or w passes through a region with two transforms of γ on its boundary, one invariant under j_1 and the other invariant under j_2. The subgroup of G keeping that region invariant contains both j_1 and j_2, and hence it contains J.

Suppose next that J is a finite non-cyclic subgroup of G, the group formed from G_1 via Combination Theorem II. We again pick generators j_1, j_2 for J, where we can assume that $j_1 \in G_1$ and that there is a $g \in G$ with $j_2 \in gG_1g^{-1}$. We again pick a path w from R_1, the region kept invariant under G_1, to R_2, the region kept invariant by gG_1g^{-1}. Assume as above that w crosses no translate of γ more than once, and observe that each translate of γ crossed by w must be invariant under either j_1 or j_2. Hence there is some region kept invariant by both j_1 and j_2, and so J is contained in some conjugate of G_1.

Case 3. J has exactly one limit point and is not cyclic. In this case J has a parabolic cyclic subgroup J_1 generated by j_1.

We again use induction on the Combinations; we start with Combination I. Using the result in case 1, we can assume that $j_1 \in G_1$. If j_1 were not conjugate in G_1 to an element of H, then the fixed point of j_1 would not lie on a translate of γ, and so no element of $G-G_1$ would normalize J_1. Of course, J normalizes J_1, hence we can assume that $j_1 \in H$. Now $g(\gamma) \cap \gamma = \phi$, except for g in G_1 or G_2, and so we can assume that the normalizer of J_1 in G_1 is non-trivial. Our convention then requires that $H = J_1$ be its own normalizer in G_2; i.e. $g(\gamma) \cap \gamma = \phi$ for all $g \in G_2$. It now follows that the normalizer of J_1 in G is the normalizer of J_1 in G_1.

For the second Combination, we again can assume that $j_1 \in G_1$, and as above, we observe trivially that the normalizer of J_1 in G is the normalizer of J_1 in G_1 unless $J_1 = H_1$ or $J_1 = H_2$. We can assume that $J_1 = H_1$. Our convention then requires that H_1 and H_2 each be its own normalizer in G_1. Then $g\gamma_1 \cap \gamma_1 = \phi$ for all $g \in G - H_1$; and so $J = J_1$.

Case 4. J has more than one limit point.

In this case, the limit set $\Lambda(J)$ is connected. We again use induction on the Combination Theorems. The proof for Combination Theorem II is essentially the same as the proof for Combination Theorem I, given below.

It suffices to show that $\Lambda(J)$ is contained in one of the regions of the sphere cut out by translates of γ. Since $\Lambda(J)$ is connected, unless H is parabolic it is obvious that $\Lambda(J)$ lies on one side or the other of every translate of γ. If H is parabolic and say γ disconnected $\Lambda(J)$, then the fixed point of γ would lie in $\Lambda(J)$ and so by the definition of factor subgroup, $H \subset J$. Since both topological discs bounded by γ contain limit points of J, H is an accidental parabolic subgroup of J.

In order to complete the proof of Theorem 2, we need to show that every basic subgroup is contained in a factor subgroup. We've chosen our basic subgroups so that they each have a simply-connected invariant component, and so that they have no accidental parabolic transformations.

It remains only to show that if G_i is a basic subgroup and g is a parabolic element of G whose fixed point lies on the limit set of G_i, then $g \in G_i$. By conclusion (5), g must lie in some conjugate of some G_j. The proof of case 4 above can be used here. G_i and a translate of G_j can have a limit point in common only if that limit point lies on a translate of γ; i.e., is a parabolic fixed point. In this case, the full parabolic cyclic subgroup is a common subgroup of both G_i and the conjugate of G_j.

This concludes the proof of Theorem 2. The corollaries require a few words. Corollary 1 is immediate from the definition of the class $\mathcal{C}_1$.

For Corollary 2, it is not immediately obvious that every component subgroup is a factor subgroup. It does, however, follow at once from conclusion 3 of the Combination Theorems, that the component subgroups are precisely the conjugates of basic subgroups having more than one component. Corollary 2 now follows from Theorem 2 and the observation that the quasi-Fuchsian groups are the only basic groups with more than one component.

Corollaries 3 and 4 follow at once from conclusion (5) and (4), respectively of the Combination Theorems. Corollary 5 follows from Lemmas 1 (I) and 1 (II).

3. The basic decomposition

We start now with a finitely-generated group G having an invariant component Δ_0. By Ahlfors' Finiteness Theorem [3], $X_0 = \Delta_0/G$ is a finite Riemann surface. We remark that, rather than appealing to Ahlfors' Theorem, we could have started with the assumption that X_0 be finite.

We remark that if G is non-elementary, and Δ_0 is simply-connected, then the proof of Theorem 1 is in [11] (where it appears as the proof of Theorem 5); for completeness, we will include an outline of the proof here.

We start by assuming that Δ_0 is not simply-connected. Let Δ' be Δ_0 with the elliptic fixed points of G removed, and set $X' = \Delta'/G$. Then $p: \Delta' \to X'$ is a regular covering of X', and so by the planarity theorem [14], there is a simple loop w on X', and there is a minimal positive integer α, so that w^α lifts to a loop γ in X'. Since we have assumed that Δ_0 is not simply-connected, we can assume that γ is not null-homotopic in Δ_0 (this requires starting with a loop W which lies in Δ' and is homotopically non-trivial in Δ_0, and then using the planarity theorem in a neighborhood of $p(W)$).

We first assume that w divides X' into two subsurfaces Y_1' and Y_2'. We choose a simply connected open set U on X', which intersects w along an arc; we choose a lifting $\tilde{U}$ of U which intersects γ; we choose base points $o_i \in Y_i' \cap U$, and $\tilde{o}_i \in \tilde{U}$, with $p\tilde{o}_i = o_i$, $i = 1, 2$.

Having chosen these base points, there is a natural identification of $\pi_1(X', o_1)$ with $\pi_1(X', o_2)$, and with this identification, there is a natural homomorphism $\tau: \pi_1(X', o) \to G$, where o is either o_1 or o_2. For $i = 1, 2$, let π^i be that subgroup of $\pi_1(X', o_i)$ generated by loops at o_i, which do not cross w. Set $G_i = \tau(\pi^i)$.

One easily sees that G_i can equivalently be defined as follows. Let $\tilde{Y}_i$ be the connected component of $p^{-1}(Y_i')$ which contains $\tilde{o}_i$. Then G_i is the subgroup of G which keeps $\tilde{Y}_i$ invariant.

We next observe that $G_1 \cap G_2$ contains a subgroup H, of order α, which keeps γ invari ant. For $i = 1, 2$, let B_i be the open topological disc bounded by γ which does not contain $\tilde{o}_i$.

LEMMA 2 (I). *For $i = 1, 2$, B_i is a precisely invariant disc for H as a subgroup of G_i.*

Proof. Since w is a simple loop, every translate of B_i is either equal to B_i, disjoint from B_i, or is a relatively compact subset of B_i. Since for $g \in G_i$, $g \in H$ if and only if $gB_i = B_i$, it suffices to show that the last possibility cannot occur. If $g(B_i)$ were a relatively compact subset of B_i, then every path from $\tilde{o}_i$ to $g(\tilde{o}_i)$ would cross γ, and so g could not be in G_i.

Using Lemma 2 (I), we see that G_1 and G_2 satisfy the hypotheses of Combination Theorem I. Since $\pi_1(X', o)$ is generated by π^1 and π^2, $G = \tau(\pi_1(X', o))$ is generated by G_1 and G_2; hence G is formed from G_1 and G_2 via Combination Theorem I.

The group G_i has an invariant component $\Delta^i \supset \Delta$. In order to describe Δ^i/G_i, we fill in the branch points on Y_i', i.e., let Y_i^* be the component of $(p(\Delta_0) - w)$ which contains Y_i'.

LEMMA 3 (I). *Δ^i/G_i is the surface Y_i^* with a disc T sewn in along w. T contains exactly one branch point of G_i of order α (if $\alpha = 1$, then $p^{-1}(T)$ contains no elliptic fixed points).*

Proof. We already know that $p^{-1}(Y_i^*) \subset \Delta^i$, and, since B_i is precisely invariant under the finite cyclic subgroup H, that $B_i \subset \Delta^i$. We let $\tilde{\Delta}_i = p^{-1}(Y_i^*) \cup \bigcup_{g \in G_i} g(B_i)$. We have to show that $\tilde{\Delta}_i = \Delta^i$. For this, we have to show that if z is a point on the boundary of $\tilde{\Delta}_i$, then $z \in \Lambda(G_i)$. Every point on the boundary of $\tilde{\Delta}_i$ is a point of $\Lambda(G)$; hence there is a sequence $\{g_n\}$ of elements of G with $g_n(\gamma) \to z$. Now either $g_n \in G_i$, or there is a $g_n' \in G_i$ with $g_n(\gamma) \subset g_n'(B_i)$. Then $g_n'(\gamma) \to z$, and so $z \in \Lambda(G_i)$.

Before going on to the next case, we need to make one more observation, which is an immediate corollary of conclusion (3) of Combination Theorem I.

LEMMA 4 (I). $\Omega(G)/G = \Delta_0/G + \Omega(G_1)/G_1 - \Delta^1/G_1 + \Omega(G_2)/G_2 - \Delta^2/G_2$.

We next take up the case that w is non-dividing. Then there is a loop v on X' which crosses w exactly once. There is a lifting $\tilde{v}$ of v, which starts at some point of $\gamma=\gamma_1$ and ends at some $\gamma_2=f(\gamma_1)$. We choose base points o on v, but not on w, and $\tilde{o}$ on $\tilde{v}$, lying over o. Let π^1 be the subgroup of $\pi_1(X', o)$ generated by loops which do not cross w. Let $G_1=\tau(\pi^1)$, where $\tau: \pi_1(X, o)\to G$ is the natural homomorphism.

As in the preceding case, we observe that G can equivalently be defined as follows. Let $\tilde{Y}_1$ be the connected component of $p^{-1}(X'-w)$ which contains $\tilde{o}$. Then G_1 is the subgroup of G which keeps $\tilde{Y}_1$ invariant.

For $i=1, 2$, we let H_i be the subgroup of G_1 which keeps γ_1 invariant, and we let B_i be that topological disc bounded by γ_i which does not contain $\tilde{o}$. We already know that $f(\gamma_1)=\gamma_2$; we easily see that $f^{-1}\circ H_2\circ f=H_1$; following orientation along $\tilde{v}$, we see that $f(B_1)\cap B_2=\phi$.

LEMMA 2 (II). *For $i=1, 2$, B_i is a precisely invariant disc under H_i, as a subgroup of G_1; for every $g\in G_1$, $g(\bar{B}_1)\cap \bar{B}_2=\phi$.*

Proof. Since $p(\gamma_1)=p(\gamma_2)=w$ is a simply loop, for every $g\in G$, there are four possibilities: $g(B_1)=B_1$; $g(B_1)=B_2$; $g(\bar{B}_1)$ does not intersect either $\bar{B}_1$ or $\bar{B}_2$; $g(B_1)$ is a relatively compact subset of either B_1 or B_2.

For $g\in G_1$, the last possibility cannot occur, since we must be able to connect $\tilde{o}$ to $g(\tilde{o})$ without crossing either γ_1 or γ_2. We also cannot have $g(B_1)=B_2$, for then $f^{-1}\circ g$ would be fixed-point free on γ_1 while mapping B_1 onto its complementary component.

The proof of Lemma 2 (II) is completed by observing that the same remarks apply to translates of B_2.

Since $\pi_1(X', o)$ is generated by π^1 and v, G is generated by G_1 together with f; hence G is formed from G_1 via Combination Theorem II.

We let Δ^1 be that invariant component of G_1 which contains Δ_0, and we let Y_1^* be Y_1' with the branch points filled in. Then Y_1^* has two boundary components corresponding to w; call them w_1 and w_2.

LEMMA 3 (II). *Δ^1/G_1 is Y_1^* with two discs attached; one each along w_1 and w_2. Each of these discs contains exactly one branch point of order α.*

The proof of the above is essentially the same as the proof of Lemma 3 (I).

We again make explicit the meaning of conclusion (3) of the Combination Theorems.

LEMMA 4 (II). $\Omega(G)/G-\Delta_0/G=\Omega(G_1)/G_1-\Delta^1/G_1$.

We remark at this point that we have not as yet used the fact that X' is a finite Riemann surface.

Suppose that X' has signature $\{g, n, \nu_1, ..., \nu_n\}$. In the first case, where w divides X' into Y_1' and Y_2', since w^α lifts to a homotopically non-trivial loop in Δ_0, both Y_1' and Y_2' must either have positive genus, or have more than one puncture. Then as a corollary of Lemma 4 (I), we get that since Δ^i/G_i is a finite Riemann surface, $i=1, 2$, G_1 and G_2 are both finitely generated. We see further that the signatures $(g_i, n_i, \nu_{i1}, ..., \nu_{in_i})$ of $Y_i=\Delta^i/G_i$ satisfy

LEMMA 5 (I).

(a) $g_1+g_2=g$.

(b) *If* $\alpha>1$, *then* $n_1+n_2=n+2$, *where up to order*

$$\{\nu_1, ..., \nu_n, \alpha, \alpha\} = \{\nu_{11}, ..., \nu_{1n}, \nu_{21}, ..., \nu_{2n_2}\}.$$

If $\alpha=1$, *then* $n_1+n_2=n$, *where up to order*

$$\{\nu_1, ..., \nu_n\}=\{\nu_{11}, ..., \nu_{2n_2}\}.$$

(c) $3(g_i-1)+n_i<3(g-1)+n$, $i=1, 2$.

Proof. Only the last inequality needs proving. The minimum possible for $3(g_i-1)+n_i$ is -1, which can occur only if G_i is cyclic, and then $\alpha=1$. Inequality (c) now follows from (a) and (b).

Similar considerations show that if w is non-dividing, then we get again that G_1 is finitely generated.

LEMMA 5 (II).

(a) $g_1=g-1$.

(b) *If* $\alpha>1$, *then* $n_1=n+2$, *where up to order*

$$\{\nu_1, ..., \nu_n, \alpha, \alpha\}=\{\nu_{11}, ..., \nu_{1n_1}\}.$$

If $\alpha=1$, *then* $n=n_1$ *and up to order*

$$\{\nu_1, ..., \nu_n\}=\{\nu_{11}, ..., \nu_{1n_1}\}.$$

(c) $3(g_1-1)+n_1<3(g-1)+n$.

Since (-3) is an absolute minimum for the quantity $3(g-1)+n$, we can use the last inequality in Lemmas 5 (I) and 5 (II) for inductive purposes. Thus we have shown that *every finitely-generated Kleinian group with an invariant component can be built up, using Combination Theorems I and II, from finitely-generated Kleinian groups which have a simply connected invariant component.*

We remark that in this inductive process, our old loop w appears on our new surface, say X_1', as bounding a disc or punctured disc. Hence *we can choose our next simple loop* w_1 *on* X_1' *to be disjoint from our old loop* w.

In order to complete the proof of Theorem 1, we have to decompose a group G which has a simply-connected invariant component. It suffices to consider the case that G is a B-group; i.e. non-elementary. Then there is a conformal map $\varphi: \Delta_0 \to U$, the unit disc. One easily sees [11] that if $h \in G$ is an accidental parabolic transformation, then the axis of $\varphi h \varphi^{-1}$ does not intersect any of its translates under $\varphi G \varphi^{-1}$. The projection of this axis to X_0 is a simple loop, unless the axis has elliptic fixed points, necessarily of order 2, on it. In the latter case, a simple modification yields a simple loop on X_0.

We thus have a simple loop w on X_0, in fact on X', and a connected component γ' of $p^{-1}(w)$ which is invariant under the (accidental) parabolic cyclic subgroup H. We adjoin the fixed point of H to γ'; the resulting simple closed curve γ is precisely invariant under H.

If w divides X', into Y_1' and Y_2', then, exactly as in the preceding case when H is finite, we pick base points near w, and define G_1 and G_2 by loops which do not cross w. We observe that γ bounds two topological discs, B_1 and B_2, where B_i is a precisely invariant disc under H_i, and we observe that Lemmas 3 (I), 4 (I) and 5 (I) all hold in this case where $\alpha = \infty$.

We also remark that since our original group G is non-elementary, H has index at most 2 in its normalizer N. Since $G_1 \cap G_2 = H$, H must be its own normalizer in at least one of G_1, G_2.

If w doesn't divide X', then proceeding as before, we again choose a loop v which crosses w at exactly one point; we let f be the element of G corresponding to the lifting of v, starting at $\gamma = \gamma_1$; we let $\gamma_2 = f(\gamma_1)$; for $i = 1, 2$, we let H_i be the (parabolic cyclic) subgroup of G keeping γ_i invariant; and we let G_1 be the subgroup of G defined by loops on X' which do not cross w. Exactly as in the preceding case, we see that each γ_i bounds a precisely invariant disc B_i.

In order to see that Combination Theorem II is again applicable, we need to know that $H_1(H_2)$ is its own normalizer in G_1, and that no translate of γ_1 under G_1 intersects γ_2. Since Δ_0 is simply-connected, the first possibility follows from the known fact about Fuchsian groups that if a hyperbolic cyclic subgroup is not its own normalizer, then a simple deformation of its axis projects to a dividing loop (one side is a disc containing exactly two, branch points, each of order 2). For the second possibility, since f is definitely not in G_1, we could have $g_1(\gamma_1) \cap \gamma_2 \neq \phi$, $g_1(\gamma_1) \neq \gamma_2$, $g_1 \in G_1$, only if the corresponding Fuchsian group contained a rank 2 free abelian subgroup, which it doesn't.

Hence Combination Theorem II is applicable, and Lemmas 3 (II), 4 (II) and 5 (II) again hold, where $\alpha = \infty$.

Theorem 1 now follows by induction on the quantity $3(g-1)+n$.

Lemmas 3 (I) and 3 (II) show that our induction process yields new surfaces, which except for a finite number of discs, are disjointly embedded in our old surface. The loop w, which we use for our construction, appears as the boundary of these discs in the new surfaces. Hence, as we proceed with the induction, we can choose the new loops to be disjoint from all the old loops.

Combining the above remark with Lemmas 3 (I), 3 (II) and Theorem 2, we obtain a proof of Theorem 6.

4. Equalities and inequalities

In this section, we apply the results of the preceding sections to obtain proofs of Theorems 3, 4 and 5.

Statement (a) in Theorem 3 is an immediate consequence of statement (a) in Lemmas 5 (I) and 5 (II).

Statement (b) of Lemmas 5 (I) and 5 (II) asserts the following. There is a branch-number preserving correspondence between the n distinguished points of Δ_0/G, and a subset of the $\sum_{i=1}^{s} n_i$ distinguished points of $\bigcup_{i=1}^{s} \Delta^i/G_i$. Each of these distinguished points corresponds to a conjugacy class of maximal elliptic or parabolic cyclic subgroups of G (or G_i). A distinguished point of some Δ^i/G_i actually corresponds to a distinguished point of Δ_0/G if and only if no cyclic subgroup of the corresponding conjugacy class is used as a subgroup H or H_i in one of the Combinations.

Our proof of Theorem 3 is thus reduced to showing that the conjugates of cyclic subgroups which are used in the Combinations are precisely those cyclic subgroups which represent punctures in their factor subgroups and which are either contained in two factor subgroups or which are of infinite index in their normalizer in G.

We already know, via Lemmas 1 (I) and 1 (II) that the intersection of two conjugates of basic groups is either trivial or is an elliptic or parabolic cyclic subgroup used in one of the Combinations. A parabolic cyclic subgroup H of infinite index in its normalizer N, does not represent a puncture in $\Omega(N)/N$, and so does not represent a puncture in the factor group containing H. An elliptic cyclic subgroup H of infinite index in its normalizer, represents at least one puncture in the factor subgroup containing H, but represents no puncture in $\Omega(N)/N$; hence by the remark above, H is a conjugate of a subgroup used in one of the combinations.

To go the other way, it is obvious in the use of Combination Theorem I, that the amalgamated subgroup H is a subgroup of both G_1 and G_2. In the use of Combination Theorem II, H_1 is a subgroup of both G_1 and $f^{-1}G_1 g$. If, however, $G_1 = f^{-1}G_1 f$, then there will be infinitely many regions bounded by translates of γ which are invariant under G_1; i.e., if R is a region kept invariant by G_1, then $f^n(R)$, $n = \pm 1, \pm 2, \ldots$, is also invariant under G_1. By Accola's Theorem [2], G_1 is cyclic. Then $G_1 = H_1 = H_2$, and f commutes with H_1.

This concludes the proof of Theorem 3.

We come now to the proof of Theorem 4. To this end, we reorder the basic subgroups $G_1, \ldots, G_s$, so that $G_1, \ldots, G_p$ are the quasi-Fuchsian basic subgroups and $G_{p+1}, \ldots, G_s$ are the elementary and degenerate basic subgroups.

The inequalities in Theorems 4 and 5 follow from Theorem 3, via some complicated counting arguments by observing that $\Omega(G)/G - \Delta_0/G$ is anti-conformally equivalent to

$$X_1 = \Delta^1/G_1 + \ldots + X_p = \Delta^p/G_p.$$

Theorem 3(a) asserts that

$$g = \sum_{i=1}^{s} g_i + t = \sum_{i=1}^{p} g_i + \sum_{i=p+1}^{s} g_i + t. \tag{1}$$

If G_i is elementary, $g_i \neq 0$ if and only if G_i has signature $(1, 0)$. Hence

$$g - \sum_{i=1}^{p} g_i = t + \sum_{i=p+1}^{s} g_i \geqslant t + r_4. \tag{4(a)}$$

We recall that if G_i is quasi-Fuchsian, then $\dim B_2(X_i) = 3(g_i - 1) + n_i$. For convenience in writing, we use this formula to define $B_2(X_i)$ if G_i is elementary or degenerate. Without further mention, we will similarly define $B_q(X_i)$, $A(X_i)$ and $\chi(X_i)$.

Since G has at least two components, it is surely not elementary. Using Theorem 3, we observe that

$$\begin{aligned} \dim B_2(X_0) = 3(g-1) + n &= \sum_{i=1}^{s} \{3(g_i - 1) + k_i\} + 3(s + t - 1) \\ &= \sum_{i=1}^{s} \dim B_2(X_i) + 3(s + t - 1) - r \\ &= \sum_{i=1}^{p} \dim B_2(X_i) + \sum_{i=p+1}^{s} \dim B_2(X_i) + 3(s + t - 1) - r. \end{aligned} \tag{2}$$

Hence $$\dim B_2(X_0) - \sum_{i=1}^{p} \dim B_2(X_i) = 3(s + t - 1) - r + \sum_{i=p+1}^{s} \dim B_2(X_i). \tag{3}$$

Elementary computations show that for G_i elementary, $\dim B_2(X_i) \neq 0$ only if G_i has signature $(0, 2; \alpha, \alpha)$, in which case $\dim B_2(X_i) = -1$; or G_i has signature $(0, 4; 2, 2, 2, 2)$, in which case $\dim B_2(X_i) = +1$.

Combining the above remarks with (3), we get inequality 4(b), together with the remark as to when equality holds.

We go on to inequality 4(c) and recall that in general $\dim B_q(X_i) = (2q-1)(g_i - 1) + \Sigma_j [q - q/\nu_{ij}] + \Sigma_j [q - q/\mu_{ij}]$ where $[x]$ is the integral part of x, and $[q - q/\infty] = q - 1$. We compute, using Theorem 3,

$$\dim B_q(X_0) = (2q-1)(g-1) + \sum_j [q - q/\nu_j]$$
$$= \sum_{i=1}^{s} \{(2q-1)(g_i - 1) + \sum_j [q - q/\nu_{ij}]\} + (2q-1)(s+t-1) \tag{4}$$
$$= \sum_{i=1}^{s} \dim B_q(X_i) - \sum_{i,j} [q - q/\mu_{ij}] + (2q-1)(s+t-1).$$

Hence

$$\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) = \sum_{i=p+1}^{s} \dim B_q(X_i) - \sum_{i,j} [q - q/\mu_{ij}] + (2q-1)(s+t-1). \tag{5}$$

We estimate the RHS of (5) as follows. First, there are r_1' terms in the first sum with extended) signature $(0, 2, 0; \mu, \mu)$. For these

$$\dim B_q(X_i) - \Sigma_j [q - q/\mu_{ij}] = -(2q-1). \tag{6}$$

Next there are r_0 terms in the first sum for which G_i is elementary and $m_i = 0$. For these, one easily sees that

$$\dim B_q(X_i) \geqslant -(2q-1) + 2\left(\frac{q-1}{2}\right) = -q. \tag{7}$$

Then there are at most $(s - p - r_0 - r_1')$ terms in the first sum for which $m_i > 0$. For each of these, we choose a specific μ_{ij}, call it μ_i', and observe that since X_i does not have signature $(0, 2; \alpha, \alpha)$,

$$\dim B_q(X_i) - [q - q/\mu_i'] \geqslant -q. \tag{8}$$

All the terms in the first sum, not yet considered, correspond to degenerate groups, so $\dim B_q(X_i)$ is the dimension of some space, hence non-negative.

The number of terms left in the second sum is $r - 2r_1' - (s - p - r_0 - r_1')$. For each of these terms

$$[q - q/\mu_{ij}] \leqslant q - 1. \tag{9}$$

Using the results of (6)–(9) in (5) we obtain

$$\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) \geqslant -(2q-1)r_1' - qr_0$$
$$- q(s - p - r_0 - r_1') - (q-1)[r - 2r_1' - (s - p - r_0 - r_1')] + (2q-1)(s+t-1). \tag{10}$$

Simplifying the RHS, we get 4(c).

In the case that q is even, we can estimate $[q - q/\mu] \geqslant \frac{1}{2}q$, rather than $[q - q/\mu] \geqslant \frac{1}{2}(q-1)$, as used above. Then we can replace the RHS of (7) and (8) by $(-q+1)$. With these new inequalities, instead of (10), we get

$$\begin{aligned}\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) \geqslant & -(2q-1)\, r_1' - (q-1)\, r_0 \\ & -(q-1)\,(s-p-r_0-r_1') - (q-1)\,[r-2r_1'-(s-p-r_0-r_1')] \\ & +(2q-1)\,(s+t-1),\ q=2,4,\ldots.\end{aligned} \tag{11}$$

Simplifying the RHS of (11), we get 4(d).

Following Bers [5], we get 4(e) from 4(c) by multiplying by $2\pi q^{-1}$ and taking the limit as $q \to \infty$.

Inequality 4(f) is simpler. We again write

$$\chi(X_0) = 2(g-1) + n = \sum_{i=1}^{s} 2(g_i - 1) + \sum_i k_i + 2(s+t-1) = \sum_{i=1}^{s} \chi(X_i) - \sum_i m_i + 2(s+t-1). \tag{12}$$

Hence
$$\chi(X_0) - \sum_{i=1}^{p} \chi(X_i) = \sum_{i=p+1}^{s} \chi(X_i) - r + 2(s+t-1). \tag{13}$$

We have to observe that $\chi(X_i)$ is positive if G_i is degenerate, and is different from zero only for certain elementary basic groups. If G_i is elementary, and $\chi(X_i) \neq 0$, then G_i must have signature $(0, 3; \alpha, \beta, \gamma)$, in which case $\chi(X_i) = 1$, or G_i has signature $(0, 4; 2, 2, 2, 2)$, in which case $\chi(X_i) = 2$.

This concludes the proof of Theorem 4.

Theorem 5 is a fairly simple corollary of Theorem 4. We need to recall that the total number of operations in the construction of G is $(s+t-1) \geqslant 0$. Each cyclic basic group uses up at least one of these operations where the cyclic subgroup is trivial. The same can be said for each of the r_0 basic groups for which $m_i = 0$. Hence r, the total number of connectors, satisfies

$$r \leqslant 2(s+t-1-r_0-r_1'). \tag{14}$$

Substituting (14) into the RHS of 4(c) yields

$$\dim B_q(X_0) - \sum_{i=1}^{p} \dim B_q(X_i) \geqslant p + t - 1 + (q-1)\,(r_0 + 2r_1'), \tag{15}$$

and 5(a) follows by dropping the (non-negative) last term.

Similarly substituting (14) into 4(d) yields 5(b). Remark 5(c) follows from the fact that $(s+t-1) = 0$ if and only if G is itself a basic group.

To get 5(d), we first substitute (14) into 4(e) to obtain

$$A(X_0) - \sum_{i=1}^{p} A(X_i) \geqslant 2\pi(r_0 + 2r_1') \geqslant 0 \tag{16}$$

In order to get equality, we need $r_0 = r_1' = 0$, and equality in (14); i.e.,

$$r = 2(s+t-1). \tag{17}$$

Now, with these facts, we need to rederive 4(e). We start with

$$(2\pi)^{-1}A(X_0) = 2(g-1) + \sum (1-1/\nu_j) = \sum_{i=1}^{s} 2(g_i - 1) + \sum_{i,j} (1 - 1/\nu_{ij}) + 2(s+t-1)$$
$$= \sum_{i=1}^{s} (2\pi)^{-1} A(X_i) - \sum_{i,j} (1 - 1/\mu_{ij}) + 2(s+t-1). \tag{18}$$

Then

$$(2\pi)^{-1}\{A(X_0) - \sum_{i=1}^{p} A(X_i)\} = \sum_{i=p+1}^{s} (2\pi)^{-1} A(X_i) - \sum_{i,j} (1 - 1/\mu_{ij}) + 2(s+t-1). \tag{19}$$

Since $r = 2(s+t-1)$, and $p \geq 1$, we can choose a μ_{ij} call it μ_i', for every $i = p+1, \ldots, s$. We observe further that the μ_{ij} are paired (i.e., every connector is a common subgroup of *two* factor subgroups), and that we can choose at most one μ_i' from each pair. We now rewrite (19) as

$$(2\pi)^{-1}\{A(X_0) - \sum_{i=1}^{p} A(X_i)\}$$
$$= \sum_{i=p+1}^{s} \{(2\pi)^{-1} A(X_i) - (1 - 1/\mu_i')\} - \sum_{i,j}{}' (1 - 1/\mu_{ij}) + 2(s+t-1), \tag{20}$$

where the second summation extends over all $\mu_{ij} \neq \mu_i'$.

One consequence of (17) is that $r_1 = 0$, and so one can estimate

$$(2\pi)^{-1} A(X_i) - (1 - 1/\mu_i') \geq -1, \tag{21}$$

where equality occurs only if $\mu_i' = \infty$ and G_i has signature $(0, 3; 2, 2, \infty)$.

The number of terms in the second sum of the RHS of (20) is $r - (s-p)$, hence

$$\sum_{i,j}{}' (1 - 1/\mu_{ij}) \leq r - (s-p) - \tfrac{1}{2} \sum_{i,j} (1/\mu_{ij}). \tag{22}$$

Combining (20), (21) and (22), we obtain

$$(2\pi)^{-1}\{A(X_0) - \sum_{i=1}^{p} A(X_i)\} \geq \tfrac{1}{2} \sum_{i,j} 1/\mu_{ij}. \tag{23}$$

We conclude that each $\mu_{ij} = \infty$, and that each $G_i, p+1 \leq i \leq s$, has signature $(0, 3; 2, 2, \infty)$.

To complete 5(d), we have to prove that Δ_0 is simply-connected. Rather than go back through the proof of Theorem 1, we observe trivially that if G_1 and G_2 both have connected limit sets; and if $\Lambda(G_1) \cap \Lambda(G_2) \neq \phi$, then the group generated by G_1 and G_2 has a connected limit set. Likewise, if $\Lambda(G_1)$ is connected, and if $f(\Lambda(G_1))$ and $f^{-1}(\Lambda(G_1))$ both intersect $\Lambda(G_1)$, then the group generated by G_1 and f has connected limit set.

Putting the above remarks together with the fact that G is constructed, from groups with non-trivial limit sets, using Combinations with parabolic cyclic subgroups, we conclude that $\Lambda(G)$ is connected. Since Λ_0 is invariant, $\Lambda(G)$ is the boundary of Δ_0, and so Δ_0 is simply connected.

Only the second half of 5(e) needs explanation. Recall that $s+t-1$ is the total number of combinations, and so (Theorem 6) $(s+t-1)$ is at most the number of simple loops in a homotopically independent set. It was shown in [11] that the maximum number of elements in a homotopically independent set is $3(g-1)+n$.

Finally, 4(f) together with 5(e) shows that $\sum_{i=1}^{p}\chi(X_i)\leqslant\chi(X_0)$, and of course $\chi(X_i)\geqslant 1$.

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