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Decomposition of complete graphs into stars

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A star is a connected graph in which every vertex but one has valency 1 . This paper concerns the question of when complete graphs can be decomposed into stars, all of the same order, which have pairwise disjoint edge-sets. It is shown that the complete graphs on rm and $rm + 1$ vertices, $r > 1$, can be decomposed into stars with m edges, if and only if r is even or m is odd.

By a graph we shall mean a finite undirected graph without loops or multiple edges. In the complete graph K_p there are p vertices and an edge exists between every pair of vertices. The complete bipartite graph, $K_{p,n}$, has two sets of vertices, V_p and V_n , and two vertices are adjacent if and only if both endpoints do not belong to V_p or to V_n . An m-star is a complete bipartite graph, $K_{1,m}$. We shall write $x - yztu \ldots$ for a star with centre x and terminal vertices $y, z, t, u, \ldots$.

A decomposition or factorization of a graph into stars is a way of expressing the graph as the union of edge-disjoint stars. A uniform decomposition is one in which the stars are the same size. An m-star decomposition decomposes a graph uniformly into m-stars.

The sum of graphs, $G + H$, consists of the union of the vertices and edges in G and H and all possible edges between every pair of vertices

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g and h where g belongs to G and h belongs to H .

Examples of a 4-star and of a 4-star decomposition of K_8 are given in Figure 1.

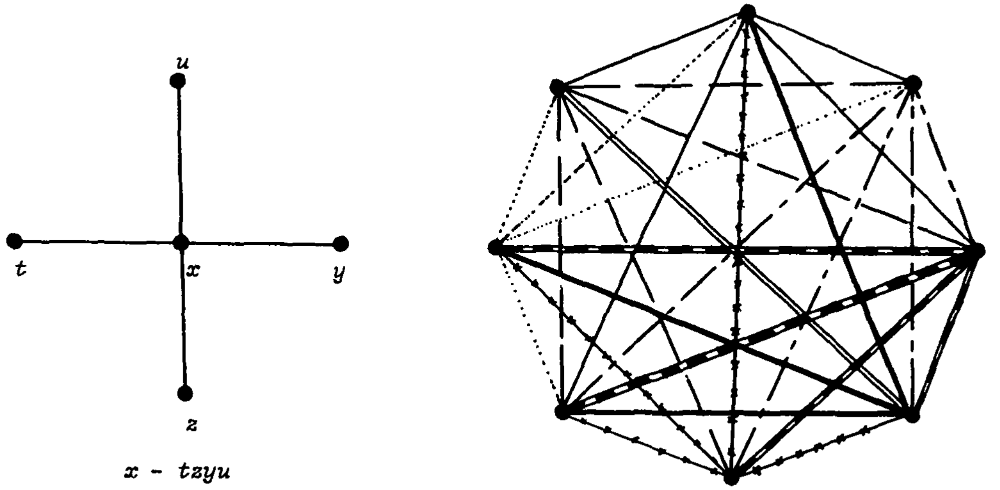


Figure 1

In [1] Ae, Yamamoto, Yoshida have shown that K_{3t} , for t greater than one, is 3-star decomposable.

In this paper we shall show precisely when K_{rm} is m-star decomposable.

LEMMA 1. *If K_p is m-star decomposable then necessarily the number of stars, $p(p-1)/2m$, is integral.*

LEMMA 2. *K_{2m} is m-star decomposable for all m ..*

Proof. K_2 is 1-star decomposable. (All graphs are trivially 1-star decomposable.) K_4 is 2-star decomposable as is shown in Figure 2.

Assume K_{2p} is p-star decomposable for all $p \leq m$. K_{2m+2} may be formed from a $K_2 \cup K_{2m}$ by joining by edges, e_{ij} , each pair of vertices (v_i, w_j) , where v_i , for $i = 1, 2$, belongs to the K_2 and w_j , for $j = 1, \ldots, 2m$, belongs to the K_{2m} . (See Figure 3.)

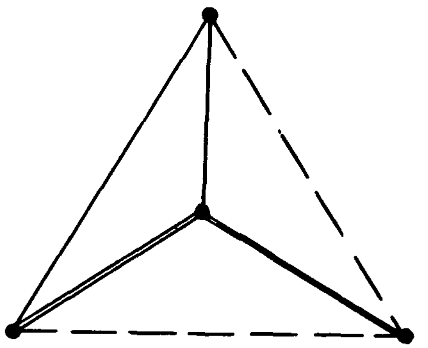
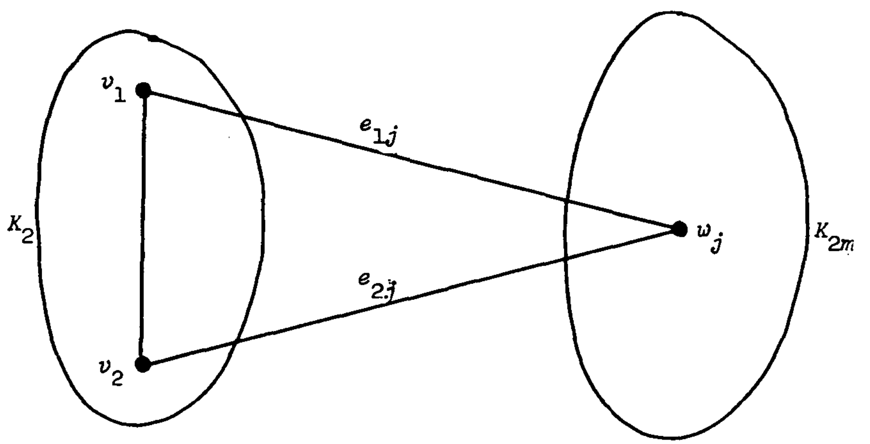


Figure 2



$w_j \in K_{2m}$

$j = 1, 2, \ldots, 2m$

Figure 3

There are $2m - 1$ m-stars in K_{2m}. Since each vertex in K_{2m} has a valency of $2m - 1$, every vertex (of K_{2m}) except one is the centre of an m-star. Label the exception w_{2m}.

For $j = 1, \ldots, m$, attach to the star with centre w_j, the edge e_{1j} to form an $(m+1)$-star. Then form an $(m+1)$-star,

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$v_1 - v_2, w_{m+1}, \ldots, w_{2m-1}, w_{2m}$, exhausting the remaining edges through v_1 .

For $j = m+1, \ldots, 2m-1$, add e_{2j} to the m-star with centre w_j , forming an $(m+1)$-star. The remaining $(m+1)$-star has v_2 as its centre and terminal vertices $w_1, \ldots, w_m, w_{2m}$. We have a total of $2m + 1$ edge-disjoint $(m+1)$-stars.

So the result follows by induction.

THEOREM 1. *If K_x is m-star decomposable then $K_{x+2\alpha m}$ for all positive integral α , is m-star decomposable.*

Proof. $K_{x+2\alpha m}$ is $K_x + K_{2m} + \ldots + K_{2m}$.

Consider $K_x + K_{2m}$.

Let v_i be the vertices of K_x for $i = 1, \ldots, x$ and w_j be the vertices of K_{2m} for $j = 1, \ldots, 2m$. Let e_{ij} be an edge between v_i and w_j .

Both K_x and K_{2m} are m-star decomposable by assumption and Lemma 2 respectively. For $i = 1, \ldots, x$, the graph, E_i , containing the edges e_{ij} where $j = 1, \ldots, 2m$, is decomposable into two m-stars, namely, $v_i - w_1, \ldots, w_m$ and $v_i - w_{m+1}, \ldots, w_{2m}$. So $K_x + K_{2m}$ is decomposable for K_x and every K_{2m} .

The result follows by repeated application.

LEMMA 3. *K_{rm} and K_{rm+1} are not m-star decomposable when r is odd and m is even.*

Proof. Lemma 1 implies that m will divide $rm(rm-1)/2$ and $(rm+1)rm/2$ if K_{rm} and K_{rm+1} are m-star decomposable. This does not happen in the case in question.

LEMMA 4. *K_{3m} is m-star decomposable when m is odd.*

Proof. Write $m = 2n + 1$. If K_{3m} is decomposable the number of

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m-stars will be $3(2n+1)(6n+3-1)/2(2n+1)$ which is an integer, $3(3n+1)$. So, for odd m it is conceivable that K_{3m} may be m-star decomposable.

K_{3m} is $K_m + K_m + K_m$; let us write these K_m with vertices v_i, $i = 1, \ldots, m$, w_j, $j = 1, \ldots, m$ and x_k, $k = 1, \ldots, m$ respectively. (See Figure 4.)

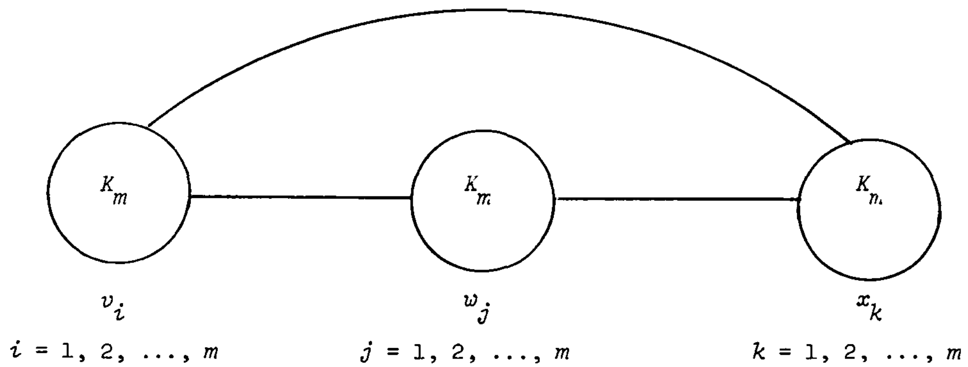

Figure 4

For $1 \le i \le m-1$, form the m-star V_i with centre v_i and terminal vertices $w_1, \ldots, w_i, v_{i+1}, \ldots, w_m$;

for $2 \le i \le m$, form the m-star X_i with centre x_i and terminal vertices $x_1, \ldots, x_{i-1}, w_i, \ldots, w_m$;

for $1 \le i \le m$, form the m-star W_i with centre w_i and terminal vertices $v_1, \ldots, v_{i-1}, v_m, x_{i+1}, \ldots, x_m$;

for $1 \le i \le m$, form the m-star U_i with centre v_i and terminal vertices $x_1, \ldots, x_m$.

We have formed $4m - 2$ m-stars and we require a further $\frac{1}{2}(m+1)$ stars to be formed. We proceed as follows: for $1 \le i \le m$, the star Y_i is centred at w_i and has terminal vertices $w_{i+1}, \ldots, w_m, x_1$. Y_i requires a further $i - 1$ edges to become an m-star.

Y_1 has centre w_1 and terminal vertices $w_2, \ldots, w_m, x_1$. So Y_1

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is an m-star already.

In order to form a further $\frac{1}{2}(m-1)$ m-stars Y_k where $k = 2, \ldots, \frac{1}{2}(m+1)$, we may use the following procedure to modify the V_i, X_i and W_i :

(1) For $2 \leq k \leq \frac{1}{2}(m+1)$:

to Y_k add v_k ;

to V_k delete w_k and add w_{m-k+2} ;

to W_{m-k+2} delete v_k and add x_1 .

Y_2 will now have m terminal vertices, namely, $w_3, \ldots, w_m, x_1, v_2$.

(2) (i) If k is less than $\frac{1}{2}(m+5-j)$, write $\theta = m - k + 5 - j$ and do the following for $3 \leq j \leq \frac{1}{2}(m+1)$ and $j \leq k \leq \frac{1}{2}(m+1)$:

to Y_k add v_θ ;

to V_θ add w_{m-j+3} and delete w_k ;

to W_{m-j+3} delete v_θ and add w_{m-k+2} .

(ii) If k is greater than or equal to $\frac{1}{2}(m+5-j)$, write $j = m + 5 - 2k$ and $\theta = m - k + 5 - j$. Do the following for $0 \leq l \leq k-j$:

to Y_k add $x_{\theta-l}$;

to $X_{\theta-l}$ add x_{m-l} and delete w_k ;

to X_{m-l} add $w_{m-j+3-l}$ and delete $x_{\theta-l}$;

to $W_{m-j+3-l}$ delete x_{m-l} and add w_{m-k+2} .

Using this procedure Y_i gains one edge from (1) and $(i-2)$ edges from (2), thus forming an m-star Y_i for $i = 2, \ldots, \frac{1}{2}(m+1)$. V_i, X_i and W_i are still m-stars since an edge is always added when one is subtracted.

The following situations need to be considered to ensure that the procedure does not break down for large m .

In (1) the procedure may break down if $m-k+2 \le k$, but this implies that $m+2 \le 2k$. Since $2 \le k \le \frac{1}{2}(m+1)$, we have $m+2 \le m+1$, which is false.

In (2) (i) the procedure may break down if $m-k+5-j \le k$, that is $k \ge \frac{1}{2}(m+5-j)$, but this situation is corrected in (2) (ii). Moreover, we are in trouble if $m-j+3 \le m-k+5-j$ $\left(\text{where } j \le k \le \frac{1}{2}(m+1) \text{ and } 3 \le j \le \frac{1}{2}(m+1)\right)$ but this implies $3 \le 5-k \le 5-j \le 5-3 = 2$. Further trouble arises when $m - j + 3 = m - k + 2$, but this implies $k = j-1 \le k-1$.

Problems could occur in (2) (ii), if any of the following situations arise:

(a) $\theta - l < 1$;

(b) $\theta - l > m - l$;

(c) $\theta - l > k$;

(d) $m-j+3-l \ge m-l$;

(e) $m-j+3 \ge m$; or

(f) $m - k + 2 = m - j + 3 - l$.

None of these ever occur.

It is obvious that K_m is not m-star decomposable, not having enough vertices. So the following theorem completely characterizes m-star decomposability of K_{rm} .

THEOREM 2. *If $r \ge 2$ then K_{rm} can be decomposed into m-stars if and only if r is even or m is odd.*

Proof. The result follows from Lemmas 2, 3, and 4 and Theorem 1.

We can use these results to determine when K_{rm+1} is m-star decomposable.

LEMMA 5. *If K_{rm} is m-star decomposable then K_{rm+1} is m-star decomposable.*

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Proof. K_{rm+1} contains K_{rm} and a vertex, say v, with all possible edges between v and K_{rm}. These new edges form r edge-disjoint m-stars. Since K_{rm} is m-star decomposable, K_{rm+1} is m-star decomposable.

COROLLARY. *K_{m+1} is not m-star decomposable. When $r > 1$, K_{rm+1} is m-star decomposable if and only if r is even or m is odd.*

Finally we shall decompose K_{21} into 7-stars, as an example.

V_1	$v_1 - w_1, v_2, v_3, v_4, v_5, v_6, v_7$	W_4	$w_4 - v_1, v_2, v_3, v_7, x_5, x_6, x_7$
V_2	$v_2 - w_1, w_7, v_3, v_4, v_5, v_6, v_7$	W_5	$w_5 - v_1, v_2, v_3, x_1, v_7, x_6, x_7$
V_3	$v_3 - w_1, w_2, w_6, v_4, v_5, v_6, v_7$	W_6	$w_6 - v_1, v_2, x_1, v_4, v_5, v_7, w_5$
V_4	$v_4 - w_1, w_2, w_3, w_5, v_5; v_6, v_7$	W_7	$w_7 - v_1, x_1, v_3, v_4, w_5, w_6, v_7$
V_5	$v_5 - w_1, w_2, w_3, w_7, w_5, v_6, v_7$	U_1	$v_1 - x_1, x_2, x_3, x_4, x_5, x_6, x_7$
V_6	$v_6 - w_1, w_2, w_7, w_4, w_5, w_6, v_7$	U_2	$v_2 - x_1, x_2, x_3, x_4, x_5, x_6, x_7$
X_2	$x_2 - x_1, w_2, w_3, w_4, w_5, w_6, w_7$	U_3	$v_3 - x_1, x_2, x_3, x_4, x_5, x_6, x_7$
X_3	$x_3 - x_1, x_2, w_3, w_4, w_5, w_6, w_7$	U_4	$v_4 - x_1, x_2, x_3, x_4, x_5, x_6, x_7$
X_4	$x_4 - x_1, x_2, x_3, x_7, w_5, w_6, w_7$	U_5	$v_5 - x_1, x_2, x_3, x_4, x_5, x_6, x_7$
X_5	$x_5 - x_1, x_2, x_3, x_4, w_5, w_6, w_7$	U_6	$v_6 - x_1, x_2, x_3, x_4, x_5, x_6, x_7$
X_6	$x_6 - x_1, x_2, x_3, x_4, x_5, w_6, w_7$	U_7	$v_7 - x_1, x_2, x_3, x_4, x_5, x_6, x_7$
X_7	$x_7 - x_1, x_2, x_3, w_6, x_5, x_6, w_7$	Y_1	$w_1 - w_2, w_3, w_4, w_5, w_6, w_7, x_1$
W_1	$w_1 - v_7, x_2, x_3, x_4, x_5, x_6, x_7$	Y_2	$w_2 - w_3, w_4, w_5, w_6, w_7, x_1, v_2$
W_2	$w_2 - v_1, v_7, x_3, x_4, x_5, x_6, x_7$	Y_3	$w_3 - w_4, w_5, w_6, w_7, x_1, v_3, v_6$
W_3	$w_3 - v_1, v_2, v_7, x_4, x_5, x_6, x_7$	Y_4	$w_4 - w_5, w_6, w_7, x_1, v_4, v_5, x_4$

Reference

[1] Tadashi Ae, Seigo Yamamoto, Noriyoshi Yoshida, "Line-disjoint decomposition of complete graph into stars", *J. Combinatorial Theory Ser. B* (to appear).

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