DECOMPOSITION OF ELASTICITY TENSORS AND TENSORS THAT ARE STRUCTURALLY INVARIANT IN THREE DIMENSIONS

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Summary

The elastic stiffness or compliance is a fourth-order tensor that can be expressed in terms of two second-order symmetric tensors **A** and **B** and a fourth-order completely symmetric and traceless tensor **Z** (or **z**). It is shown that the parts associated with **A**, **B** and **Z** (or **z**) are all *structurally invariant* under a three-dimensional transformation. Thus a linear combination of the three parts gives a general expression for three-dimensional structural invariants. All three-dimensional structural invariants available in the literature are shown to be special cases of this general expression. Invariants that are inherited by each structural invariant are presented.

1. Introduction

In a fixed rectangular coordinate system x_i ($i = 1, 2, 3$), the stress–strain relation for an anisotropic linear elastic material can be written as

$$\sigma_{ij} = C_{ijks}\varepsilon_{ks}, \tag{1.1a}$$

$$C_{ijks} = C_{jiks} = C_{ksij} = C_{ijsk}, \tag{1.1b}$$

where σ_{ij} and ε_{ij} are the stress and strain and C_{ijks} is the elastic stiffness. The C_{ijks} is positive definite and possesses the full symmetry shown in (1.1b). The third equality in (1.1b) is redundant because the first two imply the third (**1**, p. 32). Introducing the contracted notation (**1** to **3**),

$$\begin{aligned} &\sigma_1 = \sigma_{11},\ \ \sigma_2 = \sigma_{22},\ \ \sigma_3 = \sigma_{33},\ \ \sigma_4 = \sigma_{23},\ \ \sigma_5 = \sigma_{31},\ \ \sigma_6 = \sigma_{12},\\ &\varepsilon_1 = \varepsilon_{11},\ \ \varepsilon_2 = \varepsilon_{22},\ \ \varepsilon_3 = \varepsilon_{33},\ \ \varepsilon_4 = 2\varepsilon_{23},\ \ \varepsilon_5 = 2\varepsilon_{31},\ \ \varepsilon_6 = 2\varepsilon_{12}, \end{aligned} \tag{1.2}$$

equations (1.1a,b) can be written as

$$\sigma_\alpha = C_{\alpha\beta}\varepsilon_\beta, \quad C_{\alpha\beta} = C_{\beta\alpha} \tag{1.3}$$

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or, in matrix notation,

$$\boldsymbol{\sigma} = \mathbf{C}\varepsilon, \quad \mathbf{C} = \mathbf{C}^T. \tag{1.4}$$

In the above, the superscript T stands for the transpose and $\mathbf{C}$ is a 6×6 matrix

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{1.5}$$

Only the upper triangle of the matrix is shown because $\mathbf{C}$ is symmetric. It has twenty-one independent elastic constants. The transformation between C_{ijks} and $C_{\alpha\beta}$ is accomplished by replacing the subscripts ij (or ks) by α (or β) using the following rules.

$$\begin{array}{ccc} ij \text{ (or } ks) & \leftrightarrow & \alpha \text{ (or } \beta) \\ 11 & \leftrightarrow & 1 \\ 22 & \leftrightarrow & 2 \\ 33 & \leftrightarrow & 3 \\ 23 \text{ or } 32 & \leftrightarrow & 4 \\ 31 \text{ or } 13 & \leftrightarrow & 5 \\ 12 \text{ or } 21 & \leftrightarrow & 6 \end{array} \tag{1.6}$$

In a new coordinate system x_i^* that is related to x_i by

$$x_i^* = Q_{ij} x_j, \tag{1.7}$$

where Q_{ij} is the rotation tensor satisfying the orthogonality relation

$$Q_{ik} Q_{jk} = \delta_{ij} = Q_{si} Q_{sj}, \tag{1.8}$$

with δ_{ij} being the Kronecker delta, the elastic stiffness C_{ijkt}^* referred to the x_i^* coordinate system is

$$C_{ijks}^* = Q_{ip} Q_{jq} Q_{kr} Q_{st} C_{pqrt}. \tag{1.9}$$

In the contracted notation $C_{\alpha\beta}^*$ is in general different from the $C_{\alpha\beta}$ shown in (1.5) except for isotropic materials.

Consider now the possibility of an anisotropic elastic material for which a uniform pressure produces a uniform contraction. This means that if

$$\sigma_{ij} = -p\delta_{ij}, \quad \varepsilon_{ij} = -\frac{\upsilon}{3}\delta_{ij}, \tag{1.10}$$

where p and υ are the pressure and the volume change, respectively, we must have, from (1.1a) (**4**),

$$C_{ijkk} = 3\kappa\delta_{ij}, \tag{1.11}$$

where $\kappa = p/v$ is the bulk modulus. Equation (1.11) provides the condition on the elastic stiffness of the material for a uniform pressure to produce a uniform contraction. Using the contracted notation, (1.11) can be written in full as

$$C_{\alpha 1} + C_{\alpha 2} + C_{\alpha 3} = 3\kappa \quad \text{for } \alpha = 1, 2, 3, \tag{1.12a}$$

$$C_{\alpha 1} + C_{\alpha 2} + C_{\alpha 3} = 0 \quad \text{for } \alpha = 4, 5, 6. \tag{1.12b}$$

The elastic stiffness matrix **C** that satisfies (1.12) has the structure

$$\mathbf{C} = \begin{bmatrix} 3\kappa - C_{12} - C_{13} & C_{12} & C_{13} & -C_{24} - C_{34} & C_{15} & C_{16} \\ & 3\kappa - C_{12} - C_{23} & C_{23} & C_{24} & -C_{15} - C_{35} & C_{26} \\ & & 3\kappa - C_{13} - C_{23} & C_{34} & C_{35} & -C_{16} - C_{26} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{1.13}$$

There are sixteen independent elastic constants. In a new coordinate system x_i^* we obtain from (1.9), (1.11) and (1.8),

$$C_{ijkk}^* = 3\kappa\delta_{ij}. \tag{1.14}$$

This is identical to (1.11) so that (1.12), and hence (1.13), applies to $C_{\alpha\beta}^*$. Thus even though $C_{\alpha\beta}^*$ is different from $C_{\alpha\beta}$, the structure of $C_{\alpha\beta}^*$ is identical to the $C_{\alpha\beta}$ shown in (1.13). Hence the elastic stiffness **C** shown in (1.13) is *structurally invariant* under a three-dimensional transformation. It is seen from (1.11) and (1.14) that κ is an invariant.

There are several three-dimensional structural invariants available in the literature (**4** to **6**). Like the example shown above, they were all motivated by a physical consideration. In contrast, pure mathematical interests motivated the two-dimensional structural invariants presented by Ting (**7**) and Ahmad (**8**), though the results find some useful physical applications. It is not feasible to extend the mathematical approach employed in (**7**, **8**) for two-dimensional structural invariants to the three-dimensional cases because the algebra would be too complex. The purpose of this paper is to show a different approach that provides a general expression for three-dimensional structural invariants.

2. Decomposition of C_{ijks} and a general three-dimensional structural invariant

Backus (**9**) and Spencer (**10**) independently showed that the elastic stiffness C_{ijks} can be uniquely decomposed as

$$\begin{aligned} C_{ijks} = a\delta_{ij}\delta_{ks} + b(\delta_{ik}\delta_{js} + \delta_{is}\delta_{jk}) + \delta_{ij}\hat{A}_{ks} + \hat{A}_{ij}\delta_{ks} \\ + \delta_{ik}\hat{B}_{js} + \hat{B}_{ik}\delta_{js} + \delta_{is}\hat{B}_{jk} + \hat{B}_{is}\delta_{jk} + Z_{ijks}, \end{aligned} \tag{2.1}$$

where a and b are scalars, $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are second-order symmetric traceless tensors and **Z** is a fourth-order totally symmetric traceless tensor that satisfies

$$Z_{ijks} = Z_{jiks} = Z_{ksij} = Z_{ijsk}, \quad Z_{ijks} = Z_{ikjs}, \quad Z_{ijkk} = 0. \tag{2.2}$$

Setting

$$A_{ij} = \hat{A}_{ij} + \frac{a}{2}\delta_{ij}, \quad B_{ij} = \hat{B}_{ij} + \frac{b}{2}\delta_{ij}, \tag{2.3}$$

(2.1) can be written as

$$C_{ijks} = C_{ijks}(\mathbf{A}) + C_{ijks}(\mathbf{B}) + Z_{ijks}, \tag{2.4}$$

where

$$\begin{aligned} C_{ijks}(\mathbf{A}) &= \delta_{ij}A_{ks} + A_{ij}\delta_{ks}, \\ C_{ijks}(\mathbf{B}) &= \delta_{ik}B_{js} + B_{ik}\delta_{js} + \delta_{is}B_{jk} + B_{is}\delta_{jk}. \end{aligned} \tag{2.5}$$

The second-order tensors **A** and **B** are symmetric but no longer traceless. Substitution of (2.1) into (1.9) yields

$$C^*_{ijks} = C_{ijks}(\mathbf{A}^*) + C_{ijks}(\mathbf{B}^*) + Z^*_{ijks}, \tag{2.6}$$

where

$$\mathbf{A}^* = \mathbf{QAQ}^T, \quad \mathbf{B}^* = \mathbf{QBQ}^T, \tag{2.7}$$

$$Z^*_{ijks} = Q_{ip}Q_{jq}Q_{kr}Q_{st}Z_{pqrt}. \tag{2.8}$$

It is clear that $C_{ijks}(\mathbf{A})$ and $C_{ijks}(\mathbf{B})$ are structurally invariant. In fact Z_{ijks} is also structurally invariant because the decomposition (2.1) applies to any coordinate system so that Z^*_{ijks} also enjoys the totally symmetric traceless property (2.2). In conclusion, the C_{ijks} shown in (2.4) is structurally invariant in terms of **A**, **B** and **Z**.

In the contracted notation we write (2.4) as

$$C_{\alpha\beta} = C_{\alpha\beta}(\mathbf{A}) + C_{\alpha\beta}(\mathbf{B}) + Z_{\alpha\beta}, \tag{2.9}$$

where

$$\mathbf{C}(\mathbf{A}) = \begin{bmatrix} 2A_{11} & A_{11}+A_{22} & A_{11}+A_{33} & A_{23} & A_{13} & A_{12} \\ & 2A_{22} & A_{22}+A_{33} & A_{23} & A_{13} & A_{12} \\ & & 2A_{33} & A_{23} & A_{13} & A_{12} \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}, \tag{2.10}$$

$$\mathbf{C}(\mathbf{B}) = \begin{bmatrix} 4B_{11} & 0 & 0 & 0 & 2B_{13} & 2B_{12} \\ & 4B_{22} & 0 & 2B_{23} & 0 & 2B_{12} \\ & & 4B_{33} & 2B_{23} & 2B_{13} & 0 \\ & & & B_{22}+B_{33} & B_{12} & B_{13} \\ & & & & B_{33}+B_{11} & B_{23} \\ & & & & & B_{11}+B_{22} \end{bmatrix}, \tag{2.11}$$

$$\mathbf{Z} = \begin{bmatrix} -z_2 - z_3 & z_3 & z_2 & 2z_4 & -z_8 - z_5 & z_9 - z_6 \\ & -z_3 - z_1 & z_1 & z_7 - z_4 & 2z_5 & -z_9 - z_6 \\ & & -z_1 - z_2 & -z_7 - z_4 & z_8 - z_5 & 2z_6 \\ & & & z_1 & 2z_6 & 2z_5 \\ & & & & z_2 & 2z_4 \\ & & & & & z_3 \end{bmatrix}. \qquad (2.12)$$

The $\mathbf{C}(\mathbf{A})$ and $\mathbf{C}(\mathbf{B})$ are obtained from (2.5) by applying the contracted notation. As to $Z_{\alpha\beta}$, the first set of three equalities in (2.2) suggests that $Z_{\alpha\beta}$ is symmetric. The second equation in (2.2) implies that

$$Z_{44} = Z_{23}, \quad Z_{55} = Z_{13}, \quad Z_{66} = Z_{12}, \quad Z_{45} = Z_{36}, \quad Z_{46} = Z_{25}, \quad Z_{56} = Z_{14}. \qquad (2.13)$$

The last equation in (2.2) means that

$$Z_{\alpha 1} + Z_{\alpha 2} + Z_{\alpha 3} = 0 \quad \text{for } \alpha = 1, 2, \ldots, 6. \qquad (2.14)$$

It is then readily shown that (2.13) and (2.14) lead to the $Z_{\alpha\beta}$ shown in (2.12). Equation (2.12), including the z-notation, was first obtained by Cowin (**11**).

The tensors $\mathbf{A}$ and $\mathbf{B}$ in $C_{\alpha\beta}(\mathbf{A})$ and $C_{\alpha\beta}(\mathbf{B})$ have six independent constants each. There are nine independent constants $z_k (k = 1, 2, \ldots, 9)$ for $Z_{\alpha\beta}$. Thus the total number of independent constants in (2.9) is twenty-one, as with the $C_{\alpha\beta}$ in (1.5). Since $\mathbf{C}(\mathbf{A})$, $\mathbf{C}(\mathbf{B})$ and $\mathbf{Z}$ are structurally invariant, (2.9) to (2.12) apply to $\mathbf{C}^*$, $\mathbf{C}(\mathbf{A}^*)$, $\mathbf{C}(\mathbf{B}^*)$ and $\mathbf{Z}^*$.

To obtain $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Z}$ in terms of $C_{\alpha\beta}$, we introduce the tensors

$$U_{ij} = C_{ijkk}, \quad V_{ik} = C_{ijkj}. \qquad (2.15)$$

It is readily shown from (2.4) that

$$\mathbf{U} = (\text{tr}\,\mathbf{A})\mathbf{I} + 3\mathbf{A} + 4\mathbf{B}, \quad \mathbf{V} = (\text{tr}\,\mathbf{B})\mathbf{I} + 2\mathbf{A} + 5\mathbf{B}. \qquad (2.16)$$

Taking the trace on both sides of the equations in (2.16) yields

$$\text{tr}\,\mathbf{A} = (2\text{tr}\,\mathbf{U} - \text{tr}\,\mathbf{V})/10, \quad \text{tr}\,\mathbf{B} = (3\text{tr}\,\mathbf{V} - \text{tr}\,\mathbf{U})/20. \qquad (2.17)$$

Equation (2.16) now provides the solution

$$\begin{aligned} \mathbf{A} &= \tfrac{1}{7}(5\mathbf{U} - 4\mathbf{V}) + \tfrac{1}{70}(11\,\text{tr}\,\mathbf{V} - 12\,\text{tr}\,\mathbf{U})\mathbf{I}, \\ \mathbf{B} &= \tfrac{1}{7}(3\mathbf{V} - 2\mathbf{U}) + \tfrac{1}{140}(11\,\text{tr}\,\mathbf{U} - 13\,\text{tr}\,\mathbf{V})\mathbf{I}. \end{aligned} \qquad (2.18)$$

The tensor Z_{ijks} is computed from (2.4). It can be shown that this is equivalent to the solution first obtained by Cowin (**11**); see also (**12**).

There are two linear invariants for the elastic stiffness C_{ijks} (**13** to **15**). They are

$$C_{iikk} = C_{11} + C_{22} + C_{33} + 2(C_{12} + C_{23} + C_{31}) = 6(\text{tr}\,\mathbf{A}) + 4(\text{tr}\,\mathbf{B}), \qquad (2.19\text{a})$$

$$C_{ijij} = C_{11} + C_{22} + C_{33} + 2(C_{44} + C_{55} + C_{66}) = 2(\text{tr}\,\mathbf{A}) + 8(\text{tr}\,\mathbf{B}). \qquad (2.19\text{b})$$

The second equalities are obtained by inserting $C_{\alpha\beta}$ from (2.9). Since (tr $\mathbf{A}$) and (tr $\mathbf{B}$) are invariants, (2.19) provides an alternative proof that C_{iikk} and C_{ijij} are invariants. It should be noted that $Z_{\alpha\beta}$ does not appear in (2.19).

3. Elastic stiffnesses $C_{\alpha\beta}$ that are structurally invariant

There are anisotropic elastic materials that behave like isotropic elastic materials for a certain physical property. The uniform contraction under a uniform pressure discussed in section 1 is an example. Constant Young's modulus or constant shear modulus is another example. All these isotropic-like behaviours seem to demand that the elastic stiffness or compliance be structurally invariant. In this section we specialize (2.9) to (2.12) to obtain the elastic stiffness that has isotropic-like behaviour for a certain physical property. It should be emphasized that (2.9) to (2.12) apply not only to the x_i-coordinate system, but to any rotated coordinate system x_i^*.

Case C1. *Constant* C_{11}

In an isotropic elastic material, a longitudinal wave can propagate in any direction with the wave speed $\sqrt{C_{11}/\rho}$, where ρ is the mass density. Can a longitudinal wave propagate in any direction in an anisotropic elastic material? If it can, the elastic stiffness C_{11}^* referred to any coordinate system must be identical to C_{11}. This means that C_{11}^* must be an invariant. This problem was investigated by Ting (**6**) who showed that a longitudinal wave with wave speed $\sqrt{C_{11}/\rho}$ can propagate in any direction if the elastic stiffness has the structure

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{11}-2C_{66} & C_{11}-2C_{55} & -2C_{56} & 0 & 0 \\ & C_{11} & C_{11}-2C_{44} & 0 & -2C_{46} & 0 \\ & & C_{11} & 0 & 0 & -2C_{45} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{3.1}$$

We will deduce (3.1) from (2.9) to (2.12). First, the C_{11}^* obtained from (2.9) to (2.12) is

$$C_{11}^* = 2(A_{11}^* + 2B_{11}^*) - (z_2^* + z_3^*). \tag{3.2}$$

Since z_2^* and z_3^* depend on nine $z_k (k = 1, 2, \ldots, 9)$ and $(A_{11}^* + 2B_{11}^*)$ depend on ($\mathbf{A}$ + 2$\mathbf{B}$) that has six independent components, C_{11}^* depends on 15, not 21, independent constants. Explicit expression of the 15 independent constants in terms of $C_{\alpha\beta}$ is given in (**6**, (3.6)). Next, if C_{11}^* is an invariant, independent of the choice of coordinate system, it is necessary that $C_{11}^* = C_{22}^* = C_{33}^*$. From (2.9) to (2.12) we must have

$$\mathbf{A} + 2\mathbf{B} = \tfrac{1}{2}\gamma\,\mathbf{I}, \quad \mathbf{Z} = \mathbf{0}, \tag{3.3}$$

where γ is an invariant. Equation (2.9) now has the expression

$$\mathbf{C} = \begin{bmatrix} \gamma & \gamma - 2(B_{11} + B_{22}) & \gamma - 2(B_{11} + B_{33}) & -2B_{23} & 0 & 0 \\ & \gamma & \gamma - 2(B_{22} + B_{33}) & 0 & -2B_{13} & 0 \\ & & \gamma & 0 & 0 & -2B_{12} \\ & & & B_{22} + B_{33} & B_{12} & B_{13} \\ & & & & B_{33} + B_{11} & B_{23} \\ & & & & & B_{11} + B_{22} \end{bmatrix}. \quad (3.4)$$

It is easily seen that (3.4) and (3.1) are equivalent. Since (3.4) is structurally invariant, so is (3.1). From (3.4) we have

$$C_{12} + C_{23} + C_{31} = 3\gamma - 4(\operatorname{tr} \mathbf{B}), \quad C_{44} + C_{55} + C_{66} = 2(\operatorname{tr} \mathbf{B}). \quad (3.5)$$

They are invariants as reported in (**6**) using a different derivation.

There are seven independent elastic constants in (3.1) and (3.4). Comparison between (3.1) and (3.4) shows that

$$(\operatorname{tr} \mathbf{B})\mathbf{I} - \mathbf{B} = \begin{bmatrix} C_{44} & -C_{45} & -C_{46} \\ & C_{55} & -C_{56} \\ & & C_{66} \end{bmatrix} = \mathbf{M}, \quad \text{say.} \quad (3.6)$$

Since **B** is a second-order symmetric tensor, so is **M**. If we choose the coordinate axes x_i^* along the eigenvectors of **M**, the off-diagonal elements of **M** can be made to vanish. Thus the number of independent constants can be reduced to four. With $C_{45} = C_{46} = C_{56} = 0$, it is easily seen that the material represented by (3.1) is orthotropic if C_{44}, C_{55}, C_{66} are distinct, hexagonal if two of the C_{44}, C_{55}, C_{66} are identical, and isotropic if $C_{44} = C_{55} = C_{66}$. This recovers the results obtained in (**6**).

It should be noted that, using a different approach, Rychlewski (**16**) also studied anisotropic elastic mateirals for which a longitudinal wave can propagate in any direction.

Case C2. *Constant* C_{66}

Related to the longitudinal wave is the question of whether there are anisotropic elastic materials for which a transverse wave with wave speed $\sqrt{C_{66}/\rho}$ can propagate in any direction. This means that the elastic stiffness C_{66}^* is an invariant, independent of the choice of the coordinate system. This problem was also investigated by Ting (**6**) who showed that a transverse wave with wave speed $\sqrt{C_{66}/\rho}$ could propagate in any direction if the elastic stiffness has the structure

$$\mathbf{C} = \begin{bmatrix} C_{11} & \frac{1}{2}(C_{11} + C_{22}) - 2C_{66} & \frac{1}{2}(C_{11} + C_{33}) - 2C_{66} & C_{14} & C_{15} & C_{16} \\ & C_{22} & \frac{1}{2}(C_{22} + C_{33}) - 2C_{66} & C_{14} & C_{15} & C_{16} \\ & & C_{33} & C_{14} & C_{15} & C_{16} \\ & & & C_{66} & 0 & 0 \\ & & & & C_{66} & 0 \\ & & & & & C_{66} \end{bmatrix}. \quad (3.7)$$

From (2.9) to (2.12),

$$C^*_{66} = B^*_{11} + B^*_{22} + z^*_3. \tag{3.8}$$

Since z^*_3 depends on nine $z_k (k = 1, 2, \ldots, 9)$, and B^*_{11}, B^*_{22} depend on six independent components of **B**, C^*_{66} depends on 15, not 21, independent constants. Explicit expression of the 15 independent constants in terms of $C_{\alpha\beta}$ is given in (**6**, (4.2)). Next, if C^*_{66} is an invariant, it is necessary that $C^*_{44} = C^*_{55} = C^*_{66}$. From (2.9) to (2.12) we must have

$$\mathbf{B} = \tfrac{1}{2}\gamma\,\mathbf{I}, \quad \mathbf{Z} = \mathbf{0}, \tag{3.9}$$

where γ is an invariant. Equation (2.9) then gives

$$\mathbf{C} = \begin{bmatrix} 2(\gamma + A_{11}) & A_{11} + A_{22} & A_{11} + A_{33} & A_{23} & A_{13} & A_{12} \\ & 2(\gamma + A_{22}) & A_{22} + A_{33} & A_{23} & A_{13} & A_{12} \\ & & 2(\gamma + A_{33}) & A_{23} & A_{13} & A_{12} \\ & & & \gamma & 0 & 0 \\ & & & & \gamma & 0 \\ & & & & & \gamma \end{bmatrix}. \tag{3.10}$$

It is easily seen that (3.7) and (3.10) are equivalent. Since (3.10) is structurally invariant, so is (3.7). From (3.10) we obtain the invariants (**6**)

$$C_{11} + C_{22} + C_{33} = 6\gamma + 2(\mathrm{tr}\,\mathbf{A}), \quad C_{12} + C_{23} + C_{31} = 2(\mathrm{tr}\,\mathbf{A}). \tag{3.11}$$

There are seven independent elastic constants in (3.7) and (3.10). Comparison between (3.7) and (3.10) shows that

$$2(\gamma\,\mathbf{I} + \mathbf{A}) = \begin{bmatrix} C_{11} & 2C_{16} & 2C_{15} \\ & C_{22} & 2C_{14} \\ & & C_{33} \end{bmatrix} = \mathbf{M}, \quad \text{say.} \tag{3.12}$$

Since **A** is a second-order symmetric tensor, so is **M**. If we choose the coordinate axes x^*_i along the eigenvectors of **M**, the off-diagonal elements of **M** can be made to vanish. Thus the number of independent constants can be reduced to four. With $C_{14} = C_{15} = C_{16} = 0$, it is easily seen that the material represented by (3.7) is orthotropic if C_{11}, C_{22}, C_{33} are distinct, hexagonal if two of the C_{11}, C_{22}, C_{33} are identical, and isotropic if $C_{11} = C_{22} = C_{33}$. This recovers the results obtained in (**6**).

Again, Rychlewski (**16**) studied the same problem using a different derivation.

Case C3. *Uniform contraction due to uniform pressure*

The elastic stiffness shown in (1.13) can be deduced from (2.9) to (2.12). The condition (1.12) demands that

$$3\mathbf{A} + 4\mathbf{B} = \gamma\,\mathbf{I}, \tag{3.13}$$

where γ is an invariant. Equation (2.9) then becomes

$$\mathbf{C} = \mathbf{Z} + \begin{bmatrix} \gamma - A_{11} & A_{11}+A_{22} & A_{11}+A_{33} & A_{23} & -\frac{1}{2}A_{13} & -\frac{1}{2}A_{12} \\ & \gamma - A_{22} & A_{22}+A_{33} & -\frac{1}{2}A_{23} & A_{13} & -\frac{1}{2}A_{12} \\ & & \gamma - A_{33} & -\frac{1}{2}A_{23} & -\frac{1}{2}A_{13} & A_{12} \\ & & & \frac{1}{2}\gamma - \frac{3}{4}(A_{22}+A_{33}) & -\frac{3}{4}A_{12} & -\frac{3}{4}A_{13} \\ & & & & \frac{1}{2}\gamma - \frac{3}{4}(A_{33}+A_{11}) & -\frac{3}{4}A_{23} \\ & & & & & \frac{1}{2}\gamma - \frac{3}{4}(A_{11}+A_{22}) \end{bmatrix}. \tag{3.14}$$

There are sixteen independent constants, six from **A**, nine from **Z** and one from γ. Thus the number of independent constants for the **C** shown in (3.14) and (1.13) is the same. Moreover, (3.14) satisfies the conditions listed in (1.12). Hence, (3.14) and (1.13) are equivalent. Since (3.14) is structurally invariant, so is (1.13).

It should be noted that the three equations in (1.12a) are invariants while the three equations in (1.12b) are structural invariants. Addition of the three equations in (1.12a) leads to the invariant in (2.19a). Thus the invariant in (2.19a) breaks up into three invariants shown in (1.12a).

4. Decomposition of S_{ijks} and a general three-dimensional structural invariant

The stress–strain relation can be written in terms of the elastic compliance S_{ijks} as

$$\varepsilon_{ij} = S_{ijks}\sigma_{ks}, \tag{4.1a}$$

$$S_{ijks} = S_{jiks} = S_{ksij} = S_{ijsk}. \tag{4.1b}$$

In the contracted notation we have

$$\varepsilon_\alpha = s_{\alpha\beta}\sigma_\beta, \quad s_{\alpha\beta} = s_{\beta\alpha}. \tag{4.2}$$

Being a fourth-order tensor like C_{ijks}, S_{ijks} can also be uniquely decomposed in terms of **A**, **B** and **Z** shown in (2.1). All equations presented in section 2 for C_{ijks} apply to S_{ijks} except those equations that are expressed in terms of the contracted notation $C_{\alpha\beta}$. This is so because the conversions between C_{ijks} and $C_{\alpha\beta}$, and between S_{ijks} and $s_{\alpha\beta}$, are different. While (1.2) and (1.6) remain valid, $s_{\alpha\beta} = S_{ijks}$ when both α and β are less than 4, $s_{\alpha\beta} = 2S_{ijks}$ if either α or β is less than 4, and $s_{\alpha\beta} = 4S_{ijks}$ when both α and β are larger than 3. Because of this difference, we denote the contracted notation of S_{ijks} by $s_{\alpha\beta}$, not $S_{\alpha\beta}$. The 6×6 symmetric matrix $s_{\alpha\beta}$ is

$$\mathbf{s} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ & & s_{33} & s_{34} & s_{35} & s_{36} \\ & & & s_{44} & s_{45} & s_{46} \\ & & & & s_{55} & s_{56} \\ & & & & & s_{66} \end{bmatrix}. \tag{4.3}$$

Equations (2.9) to (2.12) are replaced by

$$s_{\alpha\beta} = s_{\alpha\beta}(\mathbf{A}) + s_{\alpha\beta}(\mathbf{B}) + z_{\alpha\beta}, \tag{4.4}$$

where

$$\mathbf{s}(\mathbf{A}) = \begin{bmatrix} 2A_{11} & A_{11}+A_{22} & A_{11}+A_{33} & 2A_{23} & 2A_{13} & 2A_{12} \\ & 2A_{22} & A_{22}+A_{33} & 2A_{23} & 2A_{13} & 2A_{12} \\ & & 2A_{33} & 2A_{23} & 2A_{13} & 2A_{12} \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}, \tag{4.5}$$

$$\mathbf{s}(\mathbf{B}) = \begin{bmatrix} 4B_{11} & 0 & 0 & 0 & 4B_{13} & 4B_{12} \\ & 4B_{22} & 0 & 4B_{23} & 0 & 4B_{12} \\ & & 4B_{33} & 4B_{23} & 4B_{13} & 0 \\ & & & 4(B_{22}+B_{33}) & 4B_{12} & 4B_{13} \\ & & & & 4(B_{33}+B_{11}) & 4B_{23} \\ & & & & & 4(B_{11}+B_{22}) \end{bmatrix}, \tag{4.6}$$

$$\mathbf{z} = \begin{bmatrix} -z_2-z_3 & z_3 & z_2 & 4z_4 & -2(z_8+z_5) & 2(z_9-z_6) \\ & -z_3-z_1 & z_1 & 2(z_7-z_4) & 4z_5 & -2(z_9+z_6) \\ & & -z_1-z_2 & -2(z_7+z_4) & 2(z_8-z_5) & 4z_6 \\ & & & 4z_1 & 8z_6 & 8z_5 \\ & & & & 4z_2 & 8z_4 \\ & & & & & 4z_3 \end{bmatrix}. \tag{4.7}$$

They are structurally invariant, and hence can be applied to any coordinate system. It should be noted that the matrix **z** in (4.7) is different from the matrix **Z** in (2.12).

The two invariants in (2.19) are replaced by

$$S_{iikk} = s_{11} + s_{22} + s_{33} + 2(s_{12} + s_{23} + s_{31}) = 6(\operatorname{tr}\mathbf{A}) + 4(\operatorname{tr}\mathbf{B}), \tag{4.8a}$$

$$S_{ijij} = s_{11} + s_{22} + s_{33} + \tfrac{1}{2}(s_{44} + s_{55} + s_{66}) = 2(\operatorname{tr}\mathbf{A}) + 8(\operatorname{tr}\mathbf{B}). \tag{4.8b}$$

Again, $z_{\alpha\beta}$ does not appear in (4.8).

5. Elastic compliances $s_{\alpha\beta}$ that are structurally invariant

We now employ (4.4) to (4.7) to obtain the elastic compliance of materials that have isotropic-like behaviour for a certain physical property.

Case s1. *Constant Young's modulus* E_1

Young's modulus E_1 for uniaxial tension along the x_1-axis is $1/s_{11}$. Young's modulus E_1^* along the x_1^*-axis is $1/s_{11}^*$. Can s_{11}^* be an invariant independent of the choice of coordinate system? This problem is mathematically similar to case C1 studied in section 3 where C_{11}^* is an invariant. He (**17**) and Ting (**5**) have independently studied this problem. Ting (**5**) showed that s_{11}^* is an invariant if the elastic compliance has the structure

$$\mathbf{s} = \begin{bmatrix} s_{11} & s_{11} - \frac{1}{2}s_{66} & s_{11} - \frac{1}{2}s_{55} & -s_{56} & 0 & 0 \\ & s_{11} & s_{11} - \frac{1}{2}s_{44} & 0 & -s_{46} & 0 \\ & & s_{11} & 0 & 0 & -s_{45} \\ & & & s_{44} & s_{45} & s_{46} \\ & & & & s_{55} & s_{56} \\ & & & & & s_{66} \end{bmatrix}. \tag{5.1}$$

As in case C1, we can deduce (5.1) from (4.4) to (4.7).

The s_{11}^* obtained from (4.4) to (4.7) is

$$s_{11}^* = 2(A_{11}^* + 2B_{11}^*) - (z_2^* + z_3^*). \tag{5.2}$$

Hence s_{11}^* depends on 15, not 21, independent constants. Explicit expression of the 15 independent constants in terms of $s_{\alpha\beta}$ is given in (**5**, (2.6b)). If s_{11}^* is an invariant, it is necessary that $s_{11}^* = s_{22}^* = s_{33}^*$. From (4.4) to (4.7) we must have

$$\mathbf{A} + 2\mathbf{B} = \tfrac{1}{2}\gamma\,\mathbf{I}, \quad \mathbf{z} = \mathbf{0}, \tag{5.3}$$

where γ is an invariant. Equation (4.4) now has the expression

$$\mathbf{s} = \begin{bmatrix} \gamma & \gamma - 2(B_{11} + B_{22}) & \gamma - 2(B_{11} + B_{33}) & -4B_{23} & 0 & 0 \\ & \gamma & \gamma - 2(B_{22} + B_{33}) & 0 & -4B_{13} & 0 \\ & & \gamma & 0 & 0 & -4B_{12} \\ & & & 4(B_{22} + B_{33}) & 4B_{12} & 4B_{13} \\ & & & & 4(B_{33} + B_{11}) & 4B_{23} \\ & & & & & 4(B_{11} + B_{22}) \end{bmatrix}. \tag{5.4}$$

It is easily seen that (5.1) and (5.4) are equivalent. Since (5.4) is structurally invariant, so is (5.1). From (5.4) we obtain the invariants

$$s_{12} + s_{23} + s_{31} = 3\gamma - 4(\operatorname{tr}\mathbf{B}), \quad s_{44} + s_{55} + s_{66} = 8(\operatorname{tr}\mathbf{B}), \tag{5.5}$$

as reported in (**6**) using a different derivation.

There are seven independent elastic constants in (5.1) and (5.4). Comparison between (5.1) and (5.4) shows that

$$4[(\mathrm{tr}\,\mathbf{B})\mathbf{I} - \mathbf{B}] = \begin{bmatrix} s_{44} & -s_{45} & -s_{46} \\ & s_{55} & -s_{56} \\ & & s_{66} \end{bmatrix} = \mathbf{N}, \quad \text{say.} \tag{5.6}$$

Since $\mathbf{B}$ is a second-order symmetric tensor, so is $\mathbf{N}$. If we choose the coordinate axes x_i^* along the eigenvectors of $\mathbf{N}$, the off-diagonal elements of $\mathbf{N}$ can be made to vanish. Thus the number of independent constants can be reduced to four. With $s_{45} = s_{46} = s_{56} = 0$, it is easily seen that the material represented by (5.1) is orthotropic if s_{44}, s_{55}, s_{66} are distinct, hexagonal if two of the s_{44}, s_{55}, s_{66} are identical, and isotropic if $s_{44} = s_{55} = s_{66}$. This recovers the results obtained in (**6**, **17**).

Case s2. *Constant shear modulus* G_{12}

An analogue to constant Young's modulus is the question of constant shear modulus in an anisotropic elastic material. The question of whether there are anisotropic elastic materials for which the shear modulus $G_{12}^* = 1/s_{66}^*$ is an invariant was studied independently by He (**16**) and Ting (**5**). Ting (**5**) showed that s_{66}^* is an invariant if

$$\mathbf{s} = \begin{bmatrix} s_{11} & \frac{1}{2}(s_{11}+s_{22}-s_{66}) & \frac{1}{2}(s_{11}+s_{33}-s_{66}) & s_{14} & s_{15} & s_{16} \\ & s_{22} & \frac{1}{2}(s_{22}+s_{33}-s_{66}) & s_{14} & s_{15} & s_{16} \\ & & s_{33} & s_{14} & s_{15} & s_{16} \\ & & & s_{66} & 0 & 0 \\ & & & & s_{66} & 0 \\ & & & & & s_{66} \end{bmatrix}. \tag{5.7}$$

From (4.4) to (4.7),

$$s_{66}^* = 4(B_{11}^* + B_{22}^*) + 4z_3^*. \tag{5.8}$$

Hence s_{66}^* depends on 15, not 21, independent constants. Explicit expression of the 15 independent constants in terms of $s_{\alpha\beta}$ is given in (**5**, (2.7b)). Next, if s_{66}^* is an invariant, it is necessary that $s_{44}^* = s_{55}^* = s_{66}^*$. From (4.4) to (4.7) we must have

$$\mathbf{B} = \tfrac{1}{8}\gamma\,\mathbf{I}, \quad \mathbf{z} = \mathbf{0}, \tag{5.9}$$

where γ is an invariant. Equation (4.4) then gives

$$\mathbf{s} = \begin{bmatrix} \frac{1}{2}\gamma + 2A_{11} & A_{11}+A_{22} & A_{11}+A_{33} & 2A_{23} & 2A_{13} & 2A_{12} \\ & \frac{1}{2}\gamma + 2A_{22} & A_{22}+A_{33} & 2A_{23} & 2A_{13} & 2A_{12} \\ & & \frac{1}{2}\gamma + 2A_{33} & 2A_{23} & 2A_{13} & 2A_{12} \\ & & & \gamma & 0 & 0 \\ & & & & \gamma & 0 \\ & & & & & \gamma \end{bmatrix}. \tag{5.10}$$

It is easily seen that (5.7) and (5.10) are equivalent. Since (5.10) is structurally invariant, so is (5.7). From (5.10) we have the invariants (**5**)

$$s_{11} + s_{22} + s_{33} = \tfrac{3}{2}\gamma + 2(\operatorname{tr}\mathbf{A}), \quad s_{12} + s_{23} + s_{31} = 2(\operatorname{tr}\mathbf{A}). \tag{5.11}$$

There are seven independent elastic constants in (5.7) and (5.10). Comparison between (5.7) and (5.10) shows that

$$\tfrac{1}{2}\gamma\,\mathbf{I} + 2\mathbf{A} = \begin{bmatrix} s_{11} & s_{16} & s_{15} \\ & s_{22} & s_{14} \\ & & s_{33} \end{bmatrix} = \mathbf{N}, \quad \text{say.} \tag{5.12}$$

Since $\mathbf{A}$ is a second-order symmetric tensor, so is $\mathbf{N}$. If we choose the coordinate axes x_i^* along the eigenvectors of $\mathbf{N}$, the off-diagonal elements of $\mathbf{N}$ can be made to vanish. Thus the number of independent constants can be reduced to four. With $s_{14} = s_{15} = s_{16} = 0$, it is easily seen that the material represented by (5.7) is orthotropic if s_{11}, s_{22}, s_{33} are distinct, hexagonal if two of the s_{11}, s_{22}, s_{33} are identical, and isotropic if $s_{11} = s_{22} = s_{33}$. This recovers the results obtained in (**5**).

Case s3. *Constant area modulus*

Intermediate between Young's modulus and bulk modulus is the area modulus $\eta(\mathbf{n})$ introduced by Scott (**18**). It is the ratio of an equibiaxial stress to the area change in the plane with the unit normal $\mathbf{n}$ in which the stress acts. Thus

$$\frac{1}{\eta(\mathbf{n})} = S_{ijks}(\delta_{ij} - n_i n_j)(\delta_{ks} - n_k n_s). \tag{5.13}$$

He (**16**) has shown that there are anisotropic elastic materials for which $\eta(\mathbf{n})$ is an invariant, independent of $\mathbf{n}$. We will deduce from (4.4) to (4.7) that this is indeed the case.

When $\mathbf{n}$ is along the x_3-axis, (5.13) simplifies to

$$\frac{1}{\eta(\mathbf{n})} = s_{11} + s_{22} + 2s_{12}. \tag{5.14}$$

If $\mathbf{n}$ is along the x_3^*-axis we have

$$\frac{1}{\eta(\mathbf{n})} = s_{11}^* + s_{22}^* + 2s_{12}^* = 4[(A_{11}^* + B_{11}^*) + (A_{22}^* + B_{22}^*)] - (z_1^* + z_2^*), \tag{5.15}$$

where the second equality follows from (4.4) to (4.7). Since $(A_{ij}^* + B_{ij}^*)$ depends on six components of $(\mathbf{A} + \mathbf{B})$ and z_k^* depends on nine z_k, $\eta(\mathbf{n})$ depends on 15 independent constants. In order that $\eta(\mathbf{n})$ be independent of $\mathbf{n}$, it is necessary that

$$s_{11} + s_{22} + 2s_{12} = s_{22} + s_{33} + 2s_{23} = s_{11} + s_{33} + 2s_{13} = \gamma, \tag{5.16}$$

where γ is an invariant. From (4.4) to (4.7), (5.16) holds if

$$\mathbf{A} + \mathbf{B} = \tfrac{1}{8}\gamma\,\mathbf{I}, \quad \mathbf{z} = \mathbf{0}. \tag{5.17}$$

Equation (4.4) then yields

$$\mathbf{s}=\begin{bmatrix} \frac{1}{4}\gamma+2B_{11} & \frac{1}{4}\gamma-(B_{11}+B_{22}) & \frac{1}{4}\gamma-(B_{11}+B_{33}) & -2B_{23} & 2B_{13} & 2B_{12} \\ & \frac{1}{4}\gamma+2B_{22} & \frac{1}{4}\gamma-(B_{22}+B_{33}) & 2B_{23} & -2B_{13} & 2B_{12} \\ & & \frac{1}{4}\gamma+2B_{33} & 2B_{23} & 2B_{13} & -2B_{12} \\ & & & 4(B_{22}+B_{33}) & 4B_{12} & 4B_{13} \\ & & & & 4(B_{33}+B_{11}) & 4B_{23} \\ & & & & & 4(B_{11}+B_{22}) \end{bmatrix}, \tag{5.18}$$

or

$$\mathbf{s}=\begin{bmatrix} s_{11} & \frac{1}{2}(\gamma-s_{11}-s_{22}) & \frac{1}{2}(\gamma-s_{11}-s_{33}) & -\frac{1}{2}s_{56} & \frac{1}{2}s_{46} & \frac{1}{2}s_{45} \\ & s_{22} & \frac{1}{2}(\gamma-s_{22}-s_{33}) & \frac{1}{2}s_{56} & -\frac{1}{2}s_{46} & \frac{1}{2}s_{45} \\ & & s_{33} & \frac{1}{2}s_{56} & \frac{1}{2}s_{46} & -\frac{1}{2}s_{45} \\ & & & 2(s_{22}+s_{33})-\gamma & s_{45} & s_{46} \\ & & & & 2(s_{33}+s_{11})-\gamma & s_{56} \\ & & & & & 2(s_{11}+s_{22})-\gamma \end{bmatrix}. \tag{5.19}$$

It has seven independent elastic constants. Equation (5.19) is structurally invariant because (5.18) is.

There are three invariants shown in (5.16). Additional invariants obtained from (5.18) are

$$\begin{gathered} s_{11}+s_{22}+s_{33}=\tfrac{3}{4}\gamma+2(\operatorname{tr}\mathbf{B}), \\ s_{12}+s_{23}+s_{31}=\tfrac{3}{4}\gamma-2(\operatorname{tr}\mathbf{B}), \\ s_{44}+s_{55}+s_{66}=8(\operatorname{tr}\mathbf{B}), \\ s_{44}+4s_{23}=s_{55}+4s_{31}=s_{66}+4s_{12}=\gamma, \\ s_{11}-2s_{23}=s_{22}-2s_{31}=s_{33}-2s_{12}=-\tfrac{1}{4}\gamma+2(\operatorname{tr}\mathbf{B}), \\ 2s_{11}+s_{44}=2s_{22}+s_{55}=2s_{33}+s_{66}=\tfrac{1}{2}\gamma+4(\operatorname{tr}\mathbf{B}). \end{gathered} \tag{5.20}$$

Of course, not all of them are independent of each other.

Comparison between (5.18) and (5.19) suggests that

$$4[(\operatorname{tr}\mathbf{B})\mathbf{I}-\mathbf{B}]=\begin{bmatrix} 4(B_{22}+B_{33}) & -4B_{12} & -4B_{13} \\ & 4(B_{33}+B_{11}) & -4B_{23} \\ & & 4(B_{11}+B_{22}) \end{bmatrix}$$

$$=\begin{bmatrix} s_{44} & -s_{45} & -s_{46} \\ & s_{55} & -s_{56} \\ & & s_{66} \end{bmatrix}=\mathbf{N}, \quad \text{say.} \tag{5.21}$$

Since $\mathbf{B}$ is a second-order symmetric tensor, so is $\mathbf{N}$. If we choose the coordinate axes x_i^* along the eigenvectors of $\mathbf{N}$, the off-diagonal elements of $\mathbf{N}$ can be made to vanish. Thus the number of independent constants can be reduced to four. With $s_{45}=s_{46}=s_{56}=0$, it is easily seen that the material represented by (5.19) is orthotropic if s_{44}, s_{55}, s_{66} are distinct, hexagonal if two of the s_{44}, s_{55}, s_{66} are identical, and isotropic if $s_{44} = s_{55} = s_{66}$. This recovers the results obtained in (**17**).

Case s4. *Constant traction-associated bulk modulus*

When the material is under a uniaxial stress σ in the direction of a unit vector $\mathbf{n}$, the volume change ε_{ii} is

$$\varepsilon_{ii} = S_{iiks} n_k n_s \sigma. \tag{5.22}$$

He (**17**) defined the traction-associated bulk modulus $\hat{\kappa}(\mathbf{n})$ as

$$\frac{1}{\hat{\kappa}(\mathbf{n})} = \frac{\varepsilon_{ii}}{\sigma} = S_{iiks} n_k n_s. \tag{5.23}$$

Written in full using the contracted notation we have

$$[\hat{\kappa}(\mathbf{n})]^{-1} = t_1 n_1^2 + t_2 n_2^2 + t_3 n_3^2 + 2(t_4 n_2 n_3 + t_5 n_3 n_1 + t_6 n_1 n_2), \tag{5.24}$$

where

$$t_\alpha = s_{\alpha 1} + s_{\alpha 2} + s_{\alpha 3} \quad (\alpha = 1, 2, \ldots, 6). \tag{5.25}$$

Thus $\hat{\kappa}(\mathbf{n})$ depends on the six constants $t_1, t_2, \ldots, t_6$. It is independent of $\mathbf{n}$ if $t_1 = t_2 = t_3$ and $t_4 = t_5 = t_6 = 0$, which means that

$$s_{\alpha 1} + s_{\alpha 2} + s_{\alpha 3} = \hat{\kappa}^{-1} \qquad \text{for } \alpha = 1, 2, 3, \tag{5.26a}$$

$$s_{\alpha 1} + s_{\alpha 2} + s_{\alpha 3} = 0 \qquad \text{for } \alpha = 4, 5, 6, \tag{5.26b}$$

where $\hat{\kappa}$ is an invariant. The elastic compliance that satisfies (5.26) has the structure

$$\mathbf{s} = \begin{bmatrix} \hat{\kappa}^{-1} - s_{12} - s_{13} & s_{12} & s_{13} & -s_{24} - s_{34} & s_{15} & s_{16} \\ & \hat{\kappa}^{-1} - s_{12} - s_{23} & s_{23} & s_{24} & -s_{15} - s_{35} & s_{26} \\ & & \hat{\kappa}^{-1} - s_{13} - s_{23} & s_{34} & s_{35} & -s_{16} - s_{36} \\ & & & s_{44} & s_{45} & s_{46} \\ & & & & s_{55} & s_{56} \\ & & & & & s_{66} \end{bmatrix}. \tag{5.27}$$

It has sixteen independent elastic constants.

Equation (5.27) can be deduced from (4.4) to (4.7). When $\mathbf{n}$ is in the x_1-direction (5.24) gives

$$[\hat{\kappa}(\mathbf{n})]^{-1} = t_1 = s_{11} + s_{21} + s_{31} \tag{5.28}$$

or, using (4.4) to (4.7),

$$[\hat{\kappa}(\mathbf{n})]^{-1} = 3A_{11} + 4B_{11} + \operatorname{tr}\mathbf{A}. \tag{5.29}$$

Similar expression can be obtained when **n** is in the x_2- or the x_3-direction. If $\hat{\kappa}(\mathbf{n})$ is independent of **n**, we must have

$$3\mathbf{A} + 4\mathbf{B} = \gamma\,\mathbf{I}, \tag{5.30}$$

where γ is an invariant. Thus the **s** obtained from (4.4) to (4.7) is

$$\mathbf{s} = \mathbf{z} + \begin{bmatrix} \gamma - A_{11} & A_{11} + A_{22} & A_{11} + A_{33} & 2A_{23} & -A_{13} & -A_{12} \\ & \gamma - A_{22} & A_{22} + A_{33} & -A_{23} & 2A_{13} & -A_{12} \\ & & \gamma - A_{33} & -A_{23} & -A_{13} & 2A_{12} \\ & & & 2\gamma - 3(A_{22} + A_{33}) & -3A_{12} & -3A_{13} \\ & & & & 2\gamma - 3(A_{33} + A_{11}) & -3A_{23} \\ & & & & & 2\gamma - 3(A_{11} + A_{22}) \end{bmatrix}. \tag{5.31}$$

There are sixteen independent constants, six from **A**, nine from **z** and one from γ. It is easily shown that (5.26) is satisfied so that (5.31) and (5.27) are equivalent. Moreover, (5.27) is structurally invariant because (5.31) is.

The two invariants given in (4.8a,b) apply here. However, invariant (4.8a) breaks up into three invariants given in (5.26a).

It is interesting to note that (1.13) and (5.27) are similar; so are (3.14) and (5.31). The problem of uniform pressure leading to uniform contraction discussed in section 1 can be expressed in terms of the elastic compliance. If we insert (1.10) into (4.1a) we have

$$S_{ijkk} = (3\kappa)^{-1}\delta_{ij}, \tag{5.32}$$

where $\kappa = p/v$. This is (5.26) if we let $3\kappa = \hat{\kappa}$. Thus anisotropic elastic materials for which a uniform pressure leads to a uniform contraction also have the property that the traction-associated bulk modulus is independent of the choice of coordinate system, and vice versa. Since (1.13) and (5.27) apply to the same physical problem of a uniform contraction under a uniform pressure, the matrix **C** in (1.13) and the matrix **s** in (5.27) must be the inverse of each other.

6. General structural invariants when Z = 0 or z = 0

The elastic stiffness **C** for a constant C_{11} (or C_{66}) presented in (3.4) (or (3.10)) is obtained from (2.9) by setting $\mathbf{Z} = \mathbf{0}$ and imposing a relation between **A** and **B**. A general stiffness matrix **C** with $\mathbf{Z} = \mathbf{0}$ but with **A** and **B** being arbitrary is

$$\mathbf{C} = \begin{bmatrix} 2(A_{11} + 2B_{11}) & A_{11} + A_{22} & A_{11} + A_{33} & A_{23} & A_{13} + 2B_{13} & A_{12} + 2B_{12} \\ & 2(A_{22} + 2B_{22}) & A_{22} + A_{33} & A_{23} + 2B_{23} & A_{13} & A_{12} + 2B_{12} \\ & & 2(A_{33} + 2B_{33}) & A_{23} + 2B_{23} & A_{13} + 2B_{13} & A_{12} \\ & & & B_{22} + B_{33} & B_{12} & B_{13} \\ & & & & B_{33} + B_{11} & B_{23} \\ & & & & & B_{11} + B_{22} \end{bmatrix}, \tag{6.1}$$

or

$$\mathbf{C} = \begin{bmatrix} C_{11} & \frac{1}{2}(C_{11}+C_{22})-2C_{66} & \frac{1}{2}(C_{11}+C_{33})-2C_{55} & C_{14} & C_{25}+2C_{46} & C_{36}+2C_{45} \\ & C_{22} & \frac{1}{2}(C_{22}+C_{33})-2C_{44} & C_{14}+2C_{56} & C_{25} & C_{36}+2C_{45} \\ & & C_{33} & C_{14}+2C_{56} & C_{25}+2C_{46} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{6.2}$$

The **C** in (6.2) is structurally invariant because (6.1) is. It has the property that

$$C_{11}+C_{22}+C_{33}, \quad C_{12}+C_{23}+C_{31} \text{ and } C_{44}+C_{55}+C_{66}$$

are invariants. The **C** in (3.1) and (3.7) are special cases of (6.2). In reducing (3.1) or (3.7) from (6.2) it is not sufficient to simply set $C_{11}=C_{22}=C_{33}$ or $C_{44}=C_{55}=C_{66}$. The reason is that (6.2) with $C_{11}=C_{22}=C_{33}$ or $C_{44}=C_{55}=C_{66}$ is not structurally invariant so that C_{11} or C_{66} changes its value when referred to a rotated coordinate system.

Similarly, a general compliance matrix **s** with $\mathbf{z}=\mathbf{0}$ but with **A** and **B** being arbitrary is

$$\mathbf{s} = \begin{bmatrix} 2(A_{11}+2B_{11}) & A_{11}+A_{22} & A_{11}+A_{33} & 2A_{23} & 2(A_{13}+2B_{13}) & 2(A_{12}+2B_{12}) \\ & 2(A_{22}+2B_{22}) & A_{22}+A_{33} & 2(A_{23}+2B_{23}) & 2A_{13} & 2(A_{12}+2B_{12}) \\ & & 2(A_{33}+2B_{33}) & 2(A_{23}+2B_{23}) & 2(A_{13}+2B_{13}) & 2A_{12} \\ & & & 4(B_{22}+B_{33}) & 4B_{12} & 4B_{13} \\ & & & & 4(B_{33}+B_{11}) & 4B_{23} \\ & & & & & 4(B_{11}+B_{22}) \end{bmatrix}, \tag{6.3}$$

or

$$\mathbf{s} = \begin{bmatrix} s_{11} & \frac{1}{2}(s_{11}+s_{22}-s_{66}) & \frac{1}{2}(s_{11}+s_{33}-s_{55}) & s_{14} & s_{25}+s_{46} & s_{36}+s_{45} \\ & s_{22} & \frac{1}{2}(s_{22}+s_{33}-s_{44}) & s_{14}+s_{56} & s_{25} & s_{36}+s_{45} \\ & & s_{33} & s_{14}+s_{56} & s_{25}+s_{46} & s_{36} \\ & & & s_{44} & s_{45} & s_{46} \\ & & & & s_{55} & s_{56} \\ & & & & & s_{66} \end{bmatrix}. \tag{6.4}$$

Again, (6.4) is structurally invariant because (6.3) is. It has the property that

$$s_{11}+s_{22}+s_{33}, \quad s_{12}+s_{23}+s_{31} \text{ and } s_{44}+s_{55}+s_{66}$$

are invariants. The **s** in (5.1) and (5.7) are special cases of (6.4). Although not obvious by inspection, it can be shown that (5.19) is also a special case of (6.4). As before, it is not sufficient to recover (5.1) or (5.7) by simply setting $s_{11}=s_{22}=s_{33}$ or $s_{44}=s_{55}=s_{66}$.

7. Concluding remarks

The decomposition of the elasticity tensor presented by Backus (**9**) and Spencer (**10**) is rewritten in the form (2.9) to (2.12) for the elastic stiffness $C_{\alpha\beta}$ and in the form (4.4) to (4.7) for the elastic compliance $s_{\alpha\beta}$. It is shown that, in these forms, they are structurally invariant in three dimensions. There are anisotropic elastic materials that behave like isotropic materials for a certain physical property such as Young's modulus, shear modulus, area modulus or traction-associated modulus. For these isotropic-like materials the physical property concerned is an invariant, independent of the choice of the coordinate system. Several such isotropic-like materials have been found in the literature (**5** to **7**, **17**), and the elastic stiffness or compliance was proved to be structurally invariant. We show here that all of them can be deduced from (2.9) to (2.12) or (4.4) to (4.7) with very little effort. Moreover, invariants inherited by the structural invariants are easily obtained. In most cases it is easily identified if the material deduced belongs to certain symmetry groups.

While the structural invariance of the elastic stiffness or compliance is a sufficient condition for a certain physical property to be independent of the choice of coordinate system, it might not be a necessary condition. For instance, the elastic stiffness **C** given in (3.7) is structurally invariant for which a transverse wave can propagate in any direction with the same wave speed. It is shown in (**20**, **22**) that there are other anisotropic elastic materials for which a transverse wave can propagate in any direction with the same wave speed but the elastic stiffness **C** is not structurally invariant. For the elastic stiffness **C** given in (3.1) for which a longitudinal wave can propagate in any direction with the wave speed $\sqrt{C_{11}/\rho}$, the structural invariance of **C** is a necessary and sufficient condition (**21**).

The three-dimensional structural invariants discussed here are all *linear*. There are nonlinear two-dimensional structural invariants. For instance, for a two-dimensional deformation in which the displacement u_i and the stress σ_{ij} depend on x_1 and x_2 only, there are anisotropic elastic materials for which the inplane stresses σ_{11}, σ_{22} and σ_{12} can be non-zero when the inplane displacements u_1 and u_2 vanish. For these materials the relations (**19**)

$$\begin{aligned} C_{15}C_{44} = C_{46}C_{55}, \quad (C_{14} + C_{56})C_{44} = 2C_{46}C_{45}, \\ C_{24}C_{55} = C_{56}C_{44}, \quad (C_{25} + C_{46})C_{55} = 2C_{56}C_{45} \end{aligned} \tag{7.1}$$

hold and are structurally invariants under the rotation of the coordinate system about the x_3-axis. It remains to be seen if there are three-dimensional structural invariants that are nonlinear and, if they exist, what they mean physically.

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