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Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers

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The reduction of a large number of scalar multi-loop integrals to the smaller set of Master Integrals is an integral part of the computation of any multi-loop amplitudes. The reduction is usually achieved by employing the traditional Integral-By-Parts (IBP) relations. However, in case of integrals with large number of scales, this quickly becomes a bottleneck. In this talk, I will show the application of the recent idea, connecting the direct decomposition of Feynman integrals with the Intersection theory. Specifically, we will consider few maximally cut Feynman integrals and show their direct decomposition to the Master Integrals.

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1. Introduction

Feynman integrals are an integral part for the computation of scattering amplitudes, which encapsulates all the physical information of a collision process. Typically, in the evaluation of scattering amplitudes there are $\mathscr{O}(10000)$ or more Feynman integrals, which are not all independent. Rather there exists linear relations, which can be envisaged through the Integration-By-Parts identities [1], thereby helping them to reduce to a smaller set of integrals, known as Master Integrals (MIs). However, this process quickly becomes a bottleneck with the presence of a number of scales. The novel application of Intersection theory to Feynman Integrals was proposed in [2], where the *intersection numbers* [3, 4] of differential forms played the role of a *scalar product* on the *vector space* of Feynman integrals in a given family, thereby obtaining their direct decomposition to MIs. This method avoided the generation of intermediate, auxiliary expressions, needed for applying Gauss' elimination in case of the standard IBP-based approaches. The applications of Intersection theory was further examined on a handful number of Feynman integrals on maximal cuts, mostly admitting a one-fold integral representation[5]. Eventually the algorithm was further extended to obtain the full decomposition of a Feynman integral to the MIs in [6] by employing the multi-variate intersection numbers [7].

In this proceedings, I report on the results obtained from the application of Intersection theory to special mathematical uni-variate functions as well as few maximally cut Feynman integrals, having a one fold integral representation[1]. The proceedings is organized as follows. First in sec. 2, we discuss the basics of Intersection theory and introduce the co-homology group and thereby obtaining the number of MIs and the computation of univariate intersection number. Then in sec. 3, we show the application to special mathematical functions, falling in the class of Lauricella functions and extend the application to maximally cut Feynman integrals in sec. 4. Finally, sec. 5 contains our conclusion and further outlook.

2. Basics of Intersection theory

Let us consider an integral of the form

$$I = \int_{\mathscr{C}} u(\mathbf{z})\, \varphi(\mathbf{z}), \tag{2.1}$$

where $u(\mathbf{z})$ is a multi-valued function and $\varphi(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^m\mathbf{z}$ is a single-valued differential m-form. We assume that $u(\mathbf{z})$ *vanishes on the boundaries* of $\mathscr{C}$, $u(\partial\mathscr{C}) = 0$, thereby demanding the vanishing of a surface-term. Now, assuming $u(\mathbf{z})$ regulates all boundaries and using Stokes' theorem, we obtain

$$0 = \int_{\mathscr{C}} d\,(u\,\xi) = \int_{\mathscr{C}} (du \wedge \xi + u\, d\xi) = \int_{\mathscr{C}} u \left(\frac{du}{u} \wedge + d\right) \xi \equiv \int_{\mathscr{C}} u\, \nabla_\omega \xi, \tag{2.2}$$

where ξ is a differential $(m-1)$ form and $\nabla_\omega \equiv d + \omega\wedge$, with a one form $\omega \equiv d\log u$. So, we obtain

$$\int_{\mathscr{C}} u\,\varphi = \int_{\mathscr{C}} u\,(\varphi + \nabla_\omega \xi)\,. \tag{2.3}$$

[1] For the extension to multivariate case, see the contribution of Hjalte Frellesvig in [8]

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Hence, we can define an equivalence class for the φ as:

$$ {}_{\omega}\langle\varphi| : \ \varphi \sim \varphi + \nabla_{\omega}\xi. \tag{2.4}$$

The integration domain $\mathscr{C}$ along with the integrand defined on the branch is known as the twisted cycle and one can show that there exists an equivalence class for the twisted cycles also which forms the twisted homology group. On the other hand, there exists a twisted cohomology group and the elements of this are the twisted co-cycles. The equivalence classes of φ and $\mathscr{C}$ encapsulate all the information of the IBP identities as well as the contour deformation. Then the integral can be defined as a pairing between the twisted cycle and twisted co-cycle.

$$ I = \int_{\mathscr{C}} u(\mathbf{z})\,\varphi(\mathbf{z}) \equiv \langle\varphi|\mathscr{C}], \tag{2.5}$$

Let us assume that there are ν linearly-independent twisted co-cycles and those can be represented in an arbitrary basis of forms like $|e_i\rangle$ for $i = 1,2,\ldots,\nu$. Additionally, we introduce a *dual* space of twisted co-cycles, whose basis we denote by $|h_i\rangle$ for $i = 1,2,\ldots,\nu$. Using these twisted co-cycles and their dual counterparts, we define the pairing $\langle e_i|h_j\rangle$, known as *intersection number*, and construct a matrix $\mathbf{C}$, where these pairing are the entries. This is known as the metric matrix.

Now, following [5], we can obtain the *master decomposition formula*

$$ \langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|, \tag{2.6}$$

which provides an explicit way of projecting $\langle\varphi|$ onto a basis of $\langle e_i|$. Following [2], we use the eq. (2.6) extensively for the decomposition of maximally cut Feynman integrals in terms of MIs. With the pairing of the twisted cycle $|\mathscr{C}]$ in eq. (2.6), we obtain the linear relation between the integrals:

$$ \int_{\mathscr{C}} u\,\varphi = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \int_{\mathscr{C}} u\,e_i. \tag{2.7}$$

Using eqs.(2.6) and (2.7), the algorithm for decomposition of any integral of the type of eq. (2.5) as linear combinations of MIs follow three steps: (i) Determination of the number ν of MIs, (ii) Choice of the bases of forms $\langle e_i|$ and dual forms $|h_i\rangle$ (One can choose a monomial or dlog basis), and (iii) Evaluation of the individual intersection numbers.

2.1 Number of Master Integrals

The number of master integrals (ν) equals the number of critical points of the multivalued function u (in case of Feynman integral it is the Baikov Polynomial) *i.e.* $\nu = \{$the number of solutions of $\omega = 0\}$. However, it follows some assumptions like, all of the critical points must be isolated and non-degenerate, each critical point is a "saddle point" as well as the exponents of the multi-valued function u are generic enough and in particular not non-positive integers, following [9, 10]

2.2 Computation of Intersection number

Here, we show the evaluation of the Intersection number for 1-forms, in detail. Let us assume that there are ν independent forms. Additionally, we define $\mathscr{P}$ as the set of *poles* of ω ,

$$ \mathscr{P} \equiv \{\, z \mid z \text{ is a pole of } \omega \,\}. \tag{2.8}$$

Let us assume that there are two (univariate) 1-forms φ_L and φ_R, and then the *intersection number* is defined as [11, 3]

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathscr{P}} \mathrm{Res}_{z=p} \Big(\psi_p \, \varphi_R \Big), \tag{2.9}$$

where, ψ_p is a function (0-form), and the solution to the differential equation $\nabla_\omega \psi = \varphi_L$, around the pole p obtained by expanding the differential equation in Laurent series around it, *i.e.*,

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p} \ , \tag{2.10}$$

where $\nabla_\omega \equiv (d + \omega \wedge)$. Although, the above equation can be also solved globally, however only a handful of terms in the Laurent expansion around $z = p$ are needed to evaluate the residue in (2.9). Specifically, after defining $\tau \equiv z - p$, and the *ansatz*,

$$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathscr{O}\left(\tau^{\max+1}\right) \ , \tag{2.11}$$

$$\min = \mathrm{ord}_p(\varphi_L) + 1 \ , \qquad \max = -\mathrm{ord}_p(\varphi_R) - 1 \ , \tag{2.12}$$

the differential equation in eq. (2.10) freezes all unknown coefficients $\psi_p^{(j)}$. In other words, the Laurent expansion of ψ_p around each p, is determined by the Laurent expansion of $\varphi_{L,R}$ and of ω, where we assume that $\mathrm{Res}_{z=p}(\omega)$ is not a non-positive integer.

3. Applications of Intersection Theory to Mathematical functions

In the following, we first show the application of the Intersection theory in case of few mathematical functions, admitting an integral representation. Specifically, we consider the Euler-Beta integral and the Gauss-hypergeometric function. Applications of Intersection theory to generalised hypergeometric functions have also been studied in connection with co-action in [12]

3.1 Euler Beta Integrals

Let us start here with the integral relations associated to Euler *beta function*, defined as

$$\beta(a,b) \equiv \int_0^1 dz z^{a-1} (1-z)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \ . \tag{3.1}$$

For simplicity, we consider the following integral

$$I_n \equiv \int_{\mathscr{C}} u \, \phi_{n+1} \equiv {}_\omega\langle \phi_{n+1} | \mathscr{C}] \ , \qquad \phi_{n+1} \equiv z^n dz \ , \tag{3.2}$$

with

$$u = B^\gamma \qquad B = z(1-z) \ , \qquad \omega = d \log u = \gamma \left(\frac{1}{z} + \frac{1}{z-1} \right) dz \ , \tag{3.3}$$

$$\nu = 1 \ , \qquad \mathscr{P} = \{0, 1, \infty\}. \tag{3.4}$$

Monomial Basis. Here, we consider the bases in their monomial form and we choose it as $I_0 = {}_\omega\langle\phi_1|\mathscr{C}]$. Now, any $\langle\phi_n|$ can be decomposed in terms of $\langle\phi_1|$ using the *master decomposition formula* described in eq. (2.6).

Let us consider the decomposition of $I_1 = {}_\omega\langle\phi_2|\mathscr{C}]$ and which boils down to

$$\langle\phi_2| = \langle\phi_2|\phi_1\rangle\langle\phi_1|\phi_1\rangle^{-1}\,\langle\phi_1| \tag{3.5}$$

The next goal is to evaluate the intersection numbers $\langle\phi_1|\phi_1\rangle$, and $\langle\phi_2|\phi_1\rangle$. Following the above method of the computation of intersection numbers, we obtain

$$\langle\phi_1|\phi_1\rangle = = \frac{\gamma}{2(2\gamma-1)(2\gamma+1)}, \qquad \langle\phi_2|\phi_1\rangle = \frac{\gamma}{4(2\gamma-1)(2\gamma+1)}\,. \tag{3.6}$$

Finally, we get the decomposition of I_1 in terms of I_0,

$$I_1 = c_1\,I_0 \qquad \text{with} \qquad c_1 = \langle\phi_2|\phi_1\rangle\langle\phi_1|\phi_1\rangle^{-1} = \frac{1}{2}, \tag{3.7}$$

which can be verified in MATHEMATICA.

3.2 Gauss ${}_2F_1$ Hypergeometric Function

Gauss ${}_2F_1$ Hypergeomeric function is defined as

$$\beta(b,c{-}b)\;{}_2F_1(a,b,c;x) = \int_0^1 z^{b-1}(1-z)^{c-b-1}(1{-}xz)^{-a}\,dz \tag{3.8}$$

$\beta(b,c{-}b)$ is the Euler beta function defined in eq. (3.1). For the purpose of obtaining the Contiguity relations for Gauss Hypergeometric functions employing the idea of the intersection theory, we re-write this integral as:

$$\beta(b,c{-}b)\;{}_2F_1(a,b,c;x) = \int_{\mathscr{C}} u\,\varphi = {}_\omega\langle\varphi|\mathscr{C}]\,, \tag{3.9}$$

where

$$u = z^{b-1}(1-xz)^{-a}(1-z)^{-b+c-1}\,, \tag{3.10}$$

$$\omega = d\log u = \frac{xz^2(c-a-2)+z(ax-c+x+2)-bxz+b-1}{(z-1)z(xz-1)}\,dz\,, \tag{3.11}$$

$$\varphi = dz\,. \tag{3.12}$$

The integration contour $\mathscr{C}$ is $[0,1]$ and along with the specified u on the branch is called as the twisted cycle. The single valued-differential form φ is called as the twisted co-cycle. Now, following the counting of number of critical points we obtain 2 independent integrals.

$$\nu = 2\,, \qquad \mathscr{P} = \{0,\,1,\,\tfrac{1}{x},\,\infty\} \tag{3.13}$$

Monomial Basis. We consider the bases in their monomial form and we choose it as $\{\langle\phi_i|\}_{i=1,2}$. Now, any $\langle\phi_n|$ can be decomposed in terms of $\langle\phi_i|$ using the *master decomposition formula*

$$\langle\phi_n| = \sum_{i,j=1}^{2} \langle\phi_n|\phi_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle\phi_i|. \tag{3.14}$$

Let us consider the decomposition of $\beta(b+2,c-b)_2F_1(a,b+2,c+2;x) \equiv \langle\phi_3|\mathscr{C}]$ in terms of $\beta(b,c-b)_2F_1(a,b,c;x) \equiv \langle\phi_1|\mathscr{C}]$ and $\beta(b+1,c-b)_2F_1(a,b+1,c+1;x) \equiv \langle\phi_2|\mathscr{C}]$.

Now the goal is to evaluate all the intersection numbers. First, we compute the entries of the metric matrix $\mathbf{C}$ as well as the $\langle\phi_3|\phi_1\rangle$ and $\langle\phi_3|\phi_2\rangle$, following the computation of intersection numbers as described above. Using the eq. (3.14) and plugging all the individual intersection numbers we obtain

$$\begin{aligned}\beta(b+2,c-b)_2F_1(a,b+2,c+2;x) &= \left(\frac{b}{x(a-c-1)}\right)\beta(b,c-b)_2F_1(a,b,c;x)\\ &+\left(\frac{(b-a+1)x+c}{x(c-a+1)}\right)\beta(b+1,c-b)_2F_1(a,b+1,c+1;x),\end{aligned} \tag{3.15}$$

which can be verified using MATHEMATICA.

4. Applications of Intersection Theory to Feynman Integrals

In the following, we focus on *Feynman integrals*. In order to translate them into the form (2.5) we make use of the Baikov representation in the standard form [13] and the Loop-by-Loop approach developed in [14].

4.1 Bhaba scattering

Here we apply the idea of the intersection theory in case of Bhabha scattering, which is $e^+e^- \to e^+e^-$ scattering in QED. The external electrons are assumed to be on-shell so the integrals are functions of three variables, namely $s=(p_1+p_2)^2$, $t=(p_2+p_3)^2$, and $m^2=p_i^2$. There are three such seven-propagator families, two planar and one non-planar. Here we discuss one planar sector and the non-planar one. The results for the other planar family has already been shown in [5]

4.1.1 Planar family

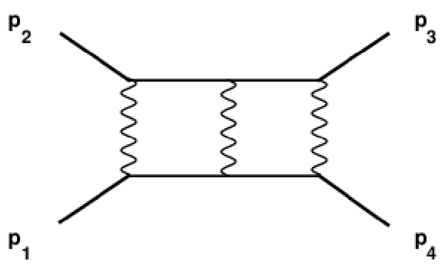

Figure 1: Bhabha - First planar sector.

This planar family in the discussion is depicted on Fig. 1, and was first computed in ref. [15]. We label the seven denominators in the following way

$$D_1 = k_2^2 - m^2, \quad D_2 = (k_2+p_1)^2, \quad D_3 = (k_2+p_1+p_2)^2 - m^2, \quad D_4 = (k_1+p_1+p_2)^2 - m^2,$$
$$D_5 = (k_1+p_1+p_2+p_3)^2, \quad D_6 = k_1^2 - m^2, \quad D_7 = (k_1-k_2)^2. \tag{4.1}$$

as well as the ISP $z = D_8 = (k_1+p_1)^2$. Using the Loop-by-Loop form of the Baikov representation, by first integrating out k_2, we obtain the following baikov polynomial (u), the connection (ω), the number of MIs (ν) and the poles ($\mathscr{P}$):

$$u = (z+s-4m^2)^{(4-d)/2}(z-t)^{d-5}z^{(d-6)/2}, \tag{4.2}$$
$$\omega = \frac{(d-6)(4m^2-s)t + (2t-(3d-16)(4m^2-s))z + 2(d-6)z^2}{2z(z+s-4m^2)(z-t)}\,dz, \tag{4.3}$$
$$\nu = 2, \qquad \mathscr{P} = \{0, 4m^2-s, t, \infty\}. \tag{4.4}$$

Mixed Bases. Here we employ mixed bases for the decomposition, where the form is chosen as monomial base compared to the dual one defined in dlog base. Let us consider the decomposition of $I_{1,1,1,1,1,1,1;-1} = \langle \phi_2 | \mathscr{C}]$.
We choose the MIs as:

$$J_1 = I_{1,1,1,1,1,1,1;0} = \langle e_1 | \mathscr{C}], \quad J_2 = I_{1,1,1,1,1,1,2;0} = \langle e_2 | \mathscr{C}], \tag{4.5}$$

where:

$$\hat{e}_1 = 1, \quad \hat{e}_2 = \frac{5-d}{z}. \tag{4.6}$$

The dlog differential-stripped co-cycles are defined as:

$$\hat{\varphi}_1 = \frac{1}{z} - \frac{1}{z-t}, \quad \hat{\varphi}_2 = \frac{1}{z-t} - \frac{1}{z+s-4m^2}. \tag{4.7}$$

The $\mathbf{C}$ matrix is built as:

$$\mathbf{C}_{ij} = \langle e_i | \varphi_j \langle, \quad i,j = 1,2, \tag{4.8}$$

where each entries are as follows:

$$\langle e_1|\varphi_1\rangle = \frac{t}{d-5}, \qquad \langle e_1|\varphi_2\rangle = \frac{4m^2-s-t}{d-5}, \tag{4.9}$$
$$\langle e_2|\varphi_1\rangle = \frac{-2(d-5)}{d-6}, \qquad \langle e_2|\varphi_2\rangle = 0. \tag{4.10}$$

The other necessary intersection numbers are:

$$\langle \phi_2|\varphi_1\rangle = \frac{(4m^2-s)t}{2(d-5)}, \quad \langle \phi_2|\varphi_2\rangle = \frac{(3d-14)(4m^2-s)(4m^2-s-t)}{2(d-5)(d-4)}. \tag{4.11}$$

The final reduction can be obtained by using the *master decomposition formula* given in eq. (2.6):

$$I_{1,1,1,1,1,1,1;-1} = c_1 J_1 + c_2 J_2 \tag{4.12}$$

where:

$$c_1 = \frac{(3d-14)(4m^2-s)}{2(d-4)}, \quad c_2 = \frac{(d-6)(4m^2-s)t}{2(d-5)(d-4)}, \tag{4.13}$$

which is in agreement with FIRE.

4.1.2 Non-Planar Sector

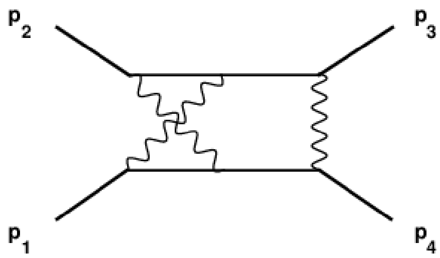

Figure 2: Bhabha non-planar sector.

The non-planar family in the discussion is depicted on Fig. 2. We label the seven denominators in the following way:

$$\begin{aligned} &D_1 = k_2^2 - m^2,\ D_2 = (k_2+p_1)^2,\ D_3 = (k_1+p_1+p_2)^2 - m^2,\ D_4 = (k_1+p_1+p_2+p_3)^2, \\ &D_5 = k_1^2 - m^2,\ D_6 = (k_1-k_2)^2,\ D_7 = (k_1-k_2+p_2)^2 - m^2\,, \end{aligned} \tag{4.14}$$

as well as the ISP as:

$$z = D_8 = (k_1+p_1)^2. \tag{4.15}$$

Using the Loop-by-Loop form of the Baikov representation, we obtain the following baikov polynomial (u)and the connection (ω):

$$u = (z-t)^{d-5}\, z^{(d-6)/2}\,(z+s)^{(d-5)/2}\,(z+s-4m^2)^{-1/2} \tag{4.16}$$

$$\omega = \frac{c_0 + c_1 z + c_2 z^2 + c_3 z^3}{2z(z+s)(s+z-4m^2)(z-t)}\, dz \tag{4.17}$$

where in eq. (4.17):

$$\begin{aligned} c_0 &= (d-6)(4m^2-s)st\,, \\ c_1 &= (3d-16)s(s-4m^2) + (8d-44)m^2 t - 3(d-6)st\,, \\ c_2 &= (84-16d)m^2 + (7d-38)s - 2(d-6)t\,, \\ c_3 &= (4d-22)\,, \end{aligned} \tag{4.18}$$

We obtain the number of MIs (ν) and the poles ($\mathscr{P}$) as:

$$\nu = 3\,, \quad \mathscr{P} = \{0, -s, 4m^2-s, t\}\,. \tag{4.19}$$

Mixed Bases: Here also, we employ the mixed bases for the decomposition, where the form is chosen as a monomial one compared to the dual one defined in dlog base. Let us consider the decomposition of $I_{1,1,1,1,1,1,1;-1} = \langle \phi_2 | \mathscr{C}]$.
We choose the MIs as:

$$J_1 = I_{1,1,1,1,1,1,1;0} = \langle e_1 | \mathscr{C}]\,, \quad J_2 = I_{1,1,1,1,1,2,1;0} = \langle e_2 | \mathscr{C}]\,, \quad J_3 = I_{1,1,1,1,1,1,2;0} = \langle e_3 | \mathscr{C}]\,, \tag{4.20}$$

where:

$$\hat{e}_1 = 1\,, \quad \hat{e}_2 = \frac{5-d}{z}\,, \quad \hat{e}_3 = \frac{d-5}{z+s}\,. \tag{4.21}$$

We define the following dlog differential-stripped bases for the dual co-cycles:

$$\hat{\varphi}_1 = \frac{1}{z} - \frac{1}{z+s-4m^2}\,, \tag{4.22}$$

$$\hat{\varphi}_2 = \frac{1}{z+s-4m^2} - \frac{1}{z+s}\,, \tag{4.23}$$

$$\hat{\varphi}_3 = \frac{1}{z+s} - \frac{1}{z-t} \tag{4.24}$$

The $\mathbf{C}$ matrix is defined as follows:

$$\mathbf{C}_{ij} = \langle e_i|\varphi_j\rangle\,, \quad 1 \le i,j \le 3, \tag{4.25}$$

where the entries are:

$$\langle e_1|\varphi_1\rangle = \frac{4m^2-s}{2(d-5)}\,, \qquad \langle e_1|\varphi_2\rangle = \frac{2m^2}{5-d}\,, \qquad \langle e_1|\varphi_3\rangle = \frac{s+t}{2(d-5)}\,, \tag{4.26}$$

$$\langle e_2|\varphi_1\rangle = \frac{2(5-d)}{d-6}\,, \qquad \langle e_2|\varphi_2\rangle = 0\,, \qquad \langle e_2|\varphi_3\rangle = 0\,, \tag{4.27}$$

$$\langle e_3|\varphi_1\rangle = 0\,, \qquad \langle e_3|\varphi_2\rangle = -2\,, \qquad \langle e_3|\varphi_3\rangle = 2\,. \tag{4.28}$$

The other necessary intersection numbers can be computed:

$$\langle \phi_2|\varphi_1\rangle = \frac{(4m^2-s)(4(4d-19)m^2+(14-3d)s-2(d-5)t)}{4(d-5)(2d-9)}\,, \tag{4.29}$$

$$\langle \phi_2|\varphi_2\rangle = \frac{m^2((76-16d)m^2+(7d-34)s+2(d-5)t)}{(d-5)(2d-9)}\,, \tag{4.30}$$

$$\langle \phi_2|\varphi_3\rangle = \frac{(s+t)(4m^2+(14-3d)s+2(d-5)t)}{4(d-5)(2d-9)}\,. \tag{4.31}$$

Then, following the eq. (2.6) we obtain:

$$I_{1,1,1,1,1,1,1;-1} = c_1 J_1 + c_2 J_2 + c_3 J_3\,, \tag{4.32}$$

where:

$$c_1 = \frac{4(4d-19)m^2+(14-3d)s+2(d-5)t}{4d-18}\,, \tag{4.33}$$

$$c_2 = \frac{(d-6)(4m^2-s)t}{2(d-5)(2d-9)}\,, \tag{4.34}$$

$$c_3 = \frac{-2m^2(s+t)}{2d-9}\,, \tag{4.35}$$

is in agreement with KIRA.

5. Conclusion

We provide a systematic presentation of the novel method for the decomposition of Feynman integrals onto a basis of master integrals by projections [2, 5, 6], through the intersection numbers of differential forms [3, 7], thereby avoiding the computationally expensive system-solving strategy in case of the standard IBP based approaches.

It would be interesting to study the application of the ideas of intersection theory to other representations of Feynman integrals other than the Baikov one as well as finding an optimal method for the computation of these intersection numbers, thereby providing new insights in the computation of scattering amplitudes.

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References

[1] K. G. Chetyrkin and F. V. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, *Nucl. Phys.* **B192** (1981) 159–204.

[2] P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, *JHEP* **02** (2019) 139, [arXiv:1810.03818].

[3] K. Matsumoto, *Intersection numbers for logarithmic k-forms*, *Osaka J. Math.* **35** (1998), no. 4 873–893.

[4] S. Mizera, *Scattering Amplitudes from Intersection Theory*, *Phys. Rev. Lett.* **120** (2018), no. 14 141602, [arXiv:1711.00469].

[5] H. Frellesvig, F. Gasparotto, S. Laporta, M. K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, *Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers*, *JHEP* **05** (2019) 153, [arXiv:1901.11510].

[6] H. Frellesvig, F. Gasparotto, M. K. Mandal, P. Mastrolia, L. Mattiazzi, and S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*, *Phys. Rev. Lett.* **123** (2019), no. 20 201602, [arXiv:1907.02000].

[7] S. Mizera, *Aspects of Scattering Amplitudes and Moduli Space Localization*. PhD thesis, Perimeter Inst. Theor. Phys., 2019. arXiv:1906.02099.

[8] H. Frellesvig, *Intersection Theory and Higgs Physics*, in *14th International Symposium on Radiative Corrections: Application of Quantum Field Theory to Phenomenology (RADCOR 2019) Avignon, France, September 8-13, 2019*, 2019.

[9] K. Aomoto and M. Kita, *Theory of Hypergeometric Functions*. Springer Monographs in Mathematics. Springer Japan, 2011.

[10] J. Milnor, *Morse Theory. (AM-51)*. No. v. 51 in Annals of Mathematics Studies. Princeton University Press, 2016.

[11] K. Cho and K. Matsumoto, *Intersection theory for twisted cohomologies and twisted Riemann's period relations I*, *Nagoya Math. J.* **139** (1995) 67–86.

[12] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew, *From positive geometries to a coaction on hypergeometric functions*, arXiv:1910.08358.

[13] P. A. Baikov, *Explicit solutions of the multiloop integral recurrence relations and its application*, *Nucl. Instrum. Meth.* **A389** (1997) 347–349, [hep-ph/9611449].

[14] H. Frellesvig and C. G. Papadopoulos, *Cuts of Feynman Integrals in Baikov representation*, *JHEP* **04** (2017) 083, [arXiv:1701.07356].

[15] J. M. Henn and V. A. Smirnov, *Analytic results for two-loop master integrals for Bhabha scattering I*, *JHEP* **11** (2013) 041, [arXiv:1307.4083].