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Decomposition of fuzzy ideal continuity via fuzzy idealization

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Abstract

Recently, El-Naschie has shown that the notion of fuzzy topology may be relevant to quantum paretical physics in connection with string theory and *E-infinity* space time theory. In this paper, we study the concepts of r-fuzzy semi-I-open, r-fuzzy pre-I-open, r-fuzzy α-I-open and r-fuzzy β-I-open sets, which is properly placed between r-fuzzy openness and r-fuzzy α-I-openness (r-fuzzy pre-I-openness) sets regardless the fuzzy ideal topological space in $\hat{S}$ostak sense. Moreover, we give a decomposition of fuzzy continuity, fuzzy ideal continuity and fuzzy ideal α-continuity, and obtain several characterization and some properties of these functions. Also, we investigate their relationship with other types of function.

Key words : r-fuzzy semi-I-open, r-fuzzy pre-I-open, r-fuzzy α-I-open and r-fuzzy β-I-open sets, fuzzy ideal continuity and fuzzy ideal α-continuity.

1. Introduction and Preliminaries

The concept of fuzzy topology was first defined in 1968 by Chang [1] and later redefined in a somewhat different way by Lowen [21] and by Hutton and Reilly [18]. According to $\hat{S}$ostak's [27], in all these definitions, a fuzzy topology is a crisp subfamily of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore $\hat{S}$ostak's introduced a new definition of fuzzy topology in 1985 [28]. Later on, he developed the theory of fuzzy topological spaces in [29]. After that several authors [2,3,5,19,20,23,25] have introduced the smooth definition and studied smooth fuzzy topological spaces being unaware of $\hat{S}$ostak's works. In fuzzy topology, by introducing the notion of ideal, [27], and several other authors [17,22] carried out such analysis.

The notion of continuity is an important concept in fuzzy topology and fuzzy topology in $\hat{S}$ostak sense as well as in all branches of mathematics and quantum physics (see [6,7,10,11,13,14]). We must state that this subject has been researched by physicists [7,1013] as well as by others. El-Naschie has shown that the notion of fuzzy topology in $\hat{S}$ostak sense has very important applications in quantum particle physics especially in relation to both string theory and $\varepsilon^{(\infty)}$ theory [8,9,12,15,16]. In this paper, we give a decomposition of fuzzy continuity, fuzzy ideal continuity and fuzzy ideal α-continuity, and we obtain several characterizations of fuzzy α-I-continuous functions. Moreover, we introduce the concept of fuzzy α-I-open functions in fuzzy ideal topological spaces and obtain their properties

Throughout this paper, let X be a nonempty set $I = [0,1]$ and $I_0 = (0,1]$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X denoted by I^X. For two fuzzy sets we write $\lambda q \mu$ to mean that λ is quasi-coincident (q-coincident, for short) with μ, i.e, there exists at least one point $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Negation of such a statement is denoted as $\lambda \overline{q} \mu$.

Definition 1.1 [27]. A mapping $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$.

(O2) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, for $\{\mu_i\}_{i\in\Gamma} \in I^X$.

(O3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for $\mu_1, \mu_2 \in I^X$.

Definition 1.2 [26]. A mapping $\mathbf{I} : I^X \to I$ is called fuzzy ideal on X iff:

(I_1) $\mathbf{I}(\underline{0}) = 1, \mathbf{I}(\underline{1}) = 0$.

(I_2) If $\lambda \leq \mu$, then $\mathbf{I}(\lambda) \geq \mathbf{I}(\mu)$, for each $\lambda, \mu \in I^X$.

(I_3) For each $\lambda, \mu \in I^X$, $\mathbf{I}(\lambda \vee \mu) \geq \mathbf{I}(\lambda) \wedge \mathbf{I}(\mu)$.

The pair $(X, \tau, \mathbf{I})$ is called fuzzy ideal topological space (fits, for short)

Corollary 1.1. Let $(X, \tau, \mathcal{I})$ be a fits. The simplest fuzzy

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ideal on X are $\mathcal{I}^0, \mathcal{I}^1 : I^X \to I$ where

$$\mathbf{I}^0(\lambda) = \begin{cases} 1, \text{ if } \lambda = \underline{0}, \\ 0, \text{otherwise.} \end{cases} \quad \mathbf{I}^1(\lambda) = \begin{cases} 0, \text{if } \lambda = \underline{1}, \\ 1, \text{ otherwise.} \end{cases}$$

If $\mathbf{I} = \mathbf{I}^0$, for each $\mu \in I^X$ we have $\mu_r^* = C_\tau(\mu, r)$.
If $\mathbf{I} = \mathbf{I}^1$, for each $\mu \in \Theta'$ we have $\mu_r^* = \underline{0}$, where, $\underline{1} \notin \Theta'$ be a subset of I^X.

Definition 1.4 [4]. Let $(X, \tau, \mathbf{I})$ be a fits. Let $\mu, \lambda \in I^X$, the r-fuzzy open local function μ_r^* of μ is the union of all fuzzy points x_t such that if $\rho \in Q(x_t, r)$ and $\mathbf{I}(\lambda) \geq r$ then there is at least one $y \in X$ for which $\rho(y) + \mu(y) - 1 > \lambda(y)$.

Theorem 1.1[3]. Let (X, τ) be a fts. Then for each $r \in I_0,\ \lambda \in I^X$ we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows:

$$C_\tau(\lambda, r) = \bigwedge \{\mu \in I^X :\ \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following conditions:
(1) $C_\tau(\overline{0}, r) = \overline{0}$.
(2) $\lambda \leq C_\tau(\lambda, r)$.
(3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.
(4) $C_\tau(\lambda, r) \leq C_\tau(\lambda, s)$ if $r \leq s$.
(5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Theorem 1.2[24]. Let (X, τ) be a fts. Then for each $r \in I_0,\ \lambda \in I^X$ we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows:

$$I_\tau(\lambda, r) = \bigvee \{\mu \in I^X :\ \lambda \geq \mu,\ \tau(\mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following conditions:
(1) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$
(2) $I_\tau(\overline{1}, r) = \overline{1}$.
(3) $\lambda \geq I_\tau(\lambda, r)$.
(4) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.
(5) $I_\tau(\lambda, r) \leq I_\tau(\lambda, s)$ if $r \geq s$.
(6) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

Theorem 1.3[4]. Let (X, τ) be a fts and $\mathbf{I}_1,\ \mathbf{I}_2$ be two fuzzy ideals of X. Then for each $r \in I_0$ and $\mu, \eta, \rho \in I^X$.
(1) $\mu \leq \eta$, then $\mu_r^* \leq \eta_r^*$.
(2) $\mathbf{I}_1 \leq \mathbf{I}_2,\ \Rightarrow\ \mu_r^*(\mathbf{I}_1, \tau) \leq \eta_r^*(\mathbf{I}_2, \tau)$.
(3) $\mu_r^* = C_\tau(\mu_r^*, r) \leq C_\tau(\mu, r)$.
(4) $(\mu_r^*)^* \leq \mu_r^*$.
(5) $(\mu_r^* \vee \eta_r^*) = (\mu \vee \eta)_r^*$.
(6) If $\mathbf{I}(\rho) \geq r$ then $(\mu \vee \rho)_r^* = \mu_r^* \vee \rho_r^* = \mu_r^*$.
(7) If $\tau(\rho) \geq r$, then $(\rho \wedge \mu_r^*) \leq (\rho \wedge \mu)_r^*$.
(8) $(\mu_r^* \wedge \eta_r^*) \geq (\mu \wedge \eta)_r^*$.

Theorem 1.4[4]. Let $(X, \tau, \mathbf{I})$ be a fits. Then for each $r \in I_0,\ \mu \in I^X$ we define $C^* : I^X \times I_0 \to I^X$ as follows:

$$Cl^*(\mu, r) = \mu \vee \mu_r^*$$

For $\mu, \eta \in I^X$, the Cl^* satisfies the following conditions:
(1) If $\mu \leq \eta$, then $Cl^*(\mu, r) \leq Cl^*(\eta, r)$.
(2) $Cl^*(Cl^*(\mu, r), r) = Cl^*(\mu, r)$.
(3) $Cl^*(\mu \vee \eta, r) = Cl^*(\mu, r) \vee Cl^*(\eta, r)$.
(4) $Cl^*(\mu \wedge \eta, r) \leq Cl^*(\mu, r) \wedge Cl^*(\eta, r)$.

Definition 1.5 [24]. Let (X, τ) be a fts. For $\lambda \in I^X$ and $r \in I_0$.
(1) λ is called r-fuzzy semiopen (**r-FSO**, for short) iff $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$.
(2) λ is called r-fuzzy semiclosed (**r-FSC**, for short) iff $\overline{1} - \lambda$ is r-fuzzy semiopen set of X.
(3) λ is called r-fuzzy β-closed (**r-FβC**, for short) iff $\lambda \leq C_\tau(I_\tau(C_\tau(\lambda, r), r), r)$.

Definition 1.6[4]. Let (X, τ, I) be a fuzzy ideal topological space. For each $\mu \in I^X$ and $r \in I_0$.
(1) μ is called r-fuzzy ideal open (r-**FIO**, for short) iff $\mu \leq I_\tau(\mu_r^*, r)$.
(2) μ is called r-fuzzy ideal closed (r-**FIC**, for short) iff $\overline{1} - \mu$ is r-**FIO**.

Lemma 1.1[4]. Let $(X, \tau, \mathbf{I})$ be a fits.
(1) Any union of r-**FIO** sets is r-**FIO**.
(2) Any intersection of r-**FIC** sets is r-**FIC**

Definition 1.7 [27]. Let (X, τ) and (X, η) be fts's. Let $f : X \to Y$ be a mapping.
(1) f is called fuzzy continuous iff $\eta(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^X$.
(2) f is called fuzzy open iff $\tau(\mu) \leq \eta(f(\mu))$ for each $\mu \in I^X$.
(3) f is called fuzzy closed iff $\tau(\overline{1} - \mu) \leq \eta(f(\overline{1} - \mu))$ for each $\mu \in I^X$.

2. r-fuzzy semi-I-open and r-fuzzy α-I-open sets

Definition 2.1. Let (X, τ, I) be a fuzzy ideal topological space, for each $\mu \in I^X$ and $r \in I_0$.
(1) μ is called r-fuzzy semi-I-open (r-**FSIO**, for short) iff $\mu \leq Cl^*(I_\tau(\mu, r), r)$.
(2) μ is called r-fuzzy pre-ideal open (r-**FPIO**, for short) iff $\mu \leq I_\tau(Cl^*(\mu, r), r)$. The complement of a r-fuzzy pre-ideal open set is said to be r-fuzzy pre-ideal closed (r-**FPIC**, for short.)
(3) μ is called r-fuzzy α-ideal open (r-**FαIO**, for short) iff $\mu \leq I_\tau(Cl^*(I_\tau(\mu, r), r),)$. The complement of a r-fuzzy α-ideal open set is said to be r-fuzzy α-ideal closed (r-**FαIC**, for short.)

(4) μ is called r-fuzzy β-ideal open (r-**F**β**IC**, for short) iff $\mu \leq C_\tau(I_\tau(Cl^*(\mu, r), r), r)$. The complement of a r-fuzzy β-ideal open set is said to be r-fuzzy β-ideal closed (r-**F**β**IC**, for short.)

Theorem 2.1. Let $(X, \tau, \mathbf{I})$ be a fits.
(1) Every r-fuzzy open set is r-**F**α**IO**
(2) Every r-**F**α**IO** set is r-**FSIO**.
(3) Every r-**FSIO** set is r-**F**β**IO**.
(4) Every r-**F**α**IO** set is r-**FPIO**.
(5) Every r-**FPIO** set is r-**F**β**IO**.
(6) Every r-**FPIO** set is **r-FPO**.
(7) Every r-fuzzy open set is r-**FSIO**.
(8) Every r-**FSIO** set is **r-FSO**.
(9) Every r-fuzzy open set is r-**FPIO** .
(10) Every r-**FIO** set is r-**FPIO**.
(11) Every r-**F**β**IO** set is r-**F**β**O**.

Proof. (1) Let μ be r-fuzzy open set. Then

$$\begin{aligned}\mu &= I_\tau(\mu, r)\\ &\leq I_\tau(\mu, r) \vee (I_\tau(\mu, r))^*\\ &= Cl^*(I_\tau(\mu, r), r).\end{aligned}$$

Therefore, $\mu = I_\tau(\mu, r) \leq I_\tau(Cl^*(I_\tau(\mu, r), r), r)$. Implies that μ is r-**F**α**IO**.

(2) Let μ be r-**F**α**IO**. Then by Theorem 1.4(1),

$$\mu \leq I_\tau(Cl^*(I_\tau(\mu, r), r) \leq Cl^*(I_\tau(\mu, r), r).$$

(3) Let μ be r-**FSIO**. Then

$$\begin{aligned}\mu &\leq Cl^*(I_\tau(\mu, r), r)\\ &\leq I_\tau(\mu, r) \vee (I_\tau(\mu, r))^*\\ &\leq \mu \vee \mu_r^*\\ &\leq Cl^*(I_\tau(\mu \vee \mu_r^*, r)\\ &\leq Cl^*(I_\tau(Cl^*(\mu, r), r), r)\\ &\leq C_\tau(I_\tau(Cl^*(\mu, r), r), r).\end{aligned}$$

(4) Let μ be r-**F**α**IO** set. Then

$$\begin{aligned}\mu &\leq I_\tau(Cl^*(I_\tau(\mu, r), r))\\ &= I_\tau(I_\tau(\mu, r) \vee (I_\tau(\mu, r))^*)\\ &\leq I_\tau(\mu \vee \mu_r^*)\\ &= I_\tau(Cl^*(\mu, r), r).\end{aligned}$$

(5-10) This proof is obvious.

Remark 2.1. By Theorem 2.1, we obtain the diagram for a r-fuzzy ideal topological space:

$$\begin{array}{ccccccc}\textbf{r-fuzzy open} & \Rightarrow & r-\mathbf{F}\alpha\mathbf{IO} & \Rightarrow & r-\mathbf{FSIO} & \Rightarrow & \textbf{r-FSO}\\ & & \Downarrow & & \Downarrow & & \Downarrow\\ r-\mathbf{FIO} & \Rightarrow & r-\mathbf{FPIO} & \Rightarrow & r-\mathbf{F}\beta\mathbf{IO} & \Rightarrow & \textbf{r-F}\beta\textbf{O}\end{array}$$

Remark 2.2. r-**FPIO** and r-**FSIO** are independent notions as show by the following Examples 2.1. and 2.2.

Example 2.1. Define two fuzzy topologies and fuzzy ideal $\tau, \mathbf{I} : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, \text{if } \lambda = \overline{1}, \overline{0},\\ \frac{1}{2}, \text{if } \lambda = \overline{0.4},\\ 0, \text{Otherwise.}\end{cases}$$

If we take $\mathbf{I} = \mathbf{I}^0$ for all $r \in I_0$, and let $\mu = \overline{0.3}$, then μ is $\frac{1}{2}$-**FPIO**, but μ is not $\frac{1}{2}$-**FSIO**.

Example 2.2. Let $X = \{a, b, c\}$ be a set and $a_t \in P_t(X)$. Define $\mu_1\mu_2 \in I^X$ as follows:

$$\mu_1(a) = 0.2,\ \mu_1(b) = 0.3,\ \mu_1(c) = 0.7;$$
$$\mu_2(a) = 0.1,\ \mu_2(b) = 0.2,\ \mu_2(c) = 0.2.$$

We define $\tau, \mathbf{I} : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, \text{if} \quad \lambda = \overline{1}, \overline{0},\\ \frac{1}{2}, \text{if} \quad \lambda = \mu_2,\\ 0, \quad \text{otherwise.}\end{cases}$$

If we take $\mathbf{I} = \mathbf{I}^0$ for all $r \in I_0$, then μ_1 is $\frac{1}{2}$-**FSIO**, but μ is not $\frac{1}{2}$-**FPIO**.

Remark 2.3. r-**FIO** and r-**FSIO** are independent notions as show by the following Examples 2.1. and 2.3.

Example 2.3. Define two fuzzy topologies and fuzzy ideal $\tau, \mathbf{I} : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, \text{if} \quad \lambda = \overline{1}, \overline{0},\\ \frac{1}{2}, \text{if} \quad \lambda = \overline{0.4},\\ \frac{2}{3}, \text{if} \quad \lambda = \overline{0.6},\\ 0, \quad \text{otherwise.}\end{cases}$$

If we take $\mathbf{I} = \mathbf{I}^0$ for all $r \in I_0$, and let $\mu = \overline{0.5}$, then μ is $\frac{1}{2}$-**FIO**, but μ is not $\frac{1}{2}$-**FSIO**.
On the other hand, **In Example 2.1.** If we take $\mathbf{I} = \mathbf{I}^0$ for all $r \in I_0$, and let $\mu = \overline{0.6}$, then μ is $\frac{1}{2}$-**FSIO** but μ is not $\frac{1}{2}$-**FIO**.

Remark 2.4. r-fuzzy open set and r-**FIO** are independent notions as show by the following Example 2.1. and 2.4.

Example 2.4. Define two fuzzy topologies and fuzzy ideal $\tau, \mathbf{I} : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, \text{if} \quad \lambda = \overline{1}, \overline{0},\\ \frac{1}{2}, \text{if} \quad \lambda = \overline{0.4},\\ \frac{1}{3}, \text{if} \quad \lambda = \overline{0.3},\\ 0, \qquad \text{otherwise.}\end{cases}$$

If we take $\mathbf{I} = \mathbf{I}^1$, for all $r \in I_0$, and let $\mu = \overline{0.3}$, then $\tau(\mu) \geq \frac{1}{3}$, but μ is not $\frac{1}{3}$-**FIO**.
On the other hand, **In Example 2.1.** If we take $\mathbf{I} = \mathbf{I}^0$ for all $r \in I_0$, and let $\mu = \overline{0.3}$, then μ is $\frac{1}{2}$-**FIO**, but $\tau(\mu) < \frac{1}{2}$.

Corollary 2.1. Let $(X, \tau, \mathbf{I})$ be a fits. For each $\mu \in I^X$.
(1) If $\mathbf{I} = \mathbf{I}^0$ for all $r \in I_0$, then,
(i) r-**FIO**, r-**FPIO** and r-**FPO** are equivalent,
(ii) μ r-**FSIO** if and only if r-**FSO**,
(iii) μ r-**F**β**IO** if and only if μ is r-**F**β**O**.
(2) If $\mathbf{I} = \mathbf{I}^1$ for all $r \in I_0$, then, μ is r-**F**β**IO** if and only if μ is r-**FSO**.

Proof. (1) If $I = \mathbf{I}^0$ for all $r \in I_0$, then, $\mu^*_r = C_\tau(\mu, r)$ for any $\mu \in I^X$ and hence $Cl^*(\mu, r) = \mu \vee \mu^*_r = C_\tau(\mu, r)$. Therefore, we have $\mu^*_r = C_\tau(\mu, r) = Cl^*(\mu, r)$. Thus, (i), (ii), and (iii) follow immediately.

(2) If $\mathbf{I} = \mathbf{I}^1$ for all $r \in I_0$, then, $\mu^*_r = \overline{0}$. Therefore, we have $C_\tau(I_\tau(Cl^*(\mu, r), r), r) = C_\tau(I_\tau(\mu^*_r \vee \mu, r), r) = C_\tau(I_\tau(\mu, r), r)$. Thus, r-**F**$\beta$**IO** and r-**FSO** are equivalent.

Definition 2.2. Let $(X, \tau, \mathbf{I})$ be a fits. For $\mu, \lambda \in I^X$ and $r \in I_0$.
(1) μ is called r-fuzzy t-I-set if
$I_\tau(Cl^*(\mu, r), r) = I_\tau(\mu, r)$.
(2) μ is called r-fuzzy B-I-set if $\mu = \nu \wedge \lambda$, where $\tau(\nu) \geq r$ and λ is r-fuzzy t-I-set of X.
(3) μ is called r-fuzzy $*$-dense-in-itself if $\mu \leq \mu^*_r$.

Corollary 2.2. Let $(X, \tau, \mathbf{I})$ be a fits and $\lambda \in I^X$, the following properties are holds
(1) Every r-fuzzy t-I-set is r-fuzzy B-I-set.
(2) Every r-fuzzy $*$-dense-in-itself set is r-fuzzy t-I-set.

Proof. (1) Let μ is r-fuzzy t-I-set. Since $\mu = \overline{1} \wedge \mu$ then μ is a r-fuzzy B-I-set.
(2) Let μ is r-fuzzy $*$-dense-in-itself set. Then
$I_\tau(Cl^*(\mu, r), r) = I_\tau((\mu^*_r \vee \mu, r) = I_\tau(\mu, r)$.

Lemma 2.1. Let $(X, \tau, \mathbf{I})$ be a fits, for $\mu \in I^X$. The following statements are equivalent.
(1) μ is r-**F**α**IO**.
(2) μ r-**FSIO** and r-**FPIO**.

Proof. Necessity. This is obvious.
Sufficiency. Let μ be r-**FSIO** and r-**FPIO**. Then, we have

$$\begin{aligned}\mu &\leq I_\tau(Cl^*(\mu, r), r)\\ &\leq I_\tau(Cl^*(Cl^*(I_\tau(\mu, r), r), r), r)\\ &= I_\tau(Cl^*(I_\tau(\mu, r), r), r).\end{aligned}$$

This show that μ is r-**F**α**IO**.

Lemma 2.2. Let $(X, \tau, \mathbf{I})$ be a fits, for $\mu \in I^X$, the following statements are equivalent.
(1) μ is r-**FIO**.
(2) μ are r-**FIPO** and r-fuzzy $*$-dense-in-itself.

Proof. (1⇒2): by Theorem 2.1, every r-**FIO** is r-**FPIO**. On the other hand, $\mu \leq I_\tau(\mu^*_r, r) \leq \mu^*_r$, which show that μ is r-fuzzy $*$-dense-in-itself.
(2⇒1): by the hypothesis, $\mu \leq I_\tau(Cl^*(\mu, r), r) \leq I_\tau(\mu \vee \mu^*_r, r) = I_\tau(\mu^*_r, r)$, then, μ is r-**FIO**.

Lemma 2.3. Let $(X, \tau, \mathbf{I})$ be a fits, for $\mu \in I^X$, the following statements are equivalent.
(1) $\tau(\mu) \geq r$.
(2) μ are r-**FIPO** and r-fuzzy B-I-set.

Proof. Let $\tau(\mu) \geq r$. Then $\mu \wedge \overline{1}$ follows that μ is a r-fuzzy B-I-set. μ is also r-**FPIO** by Theorem 2.1(9). Conversely, Let μ be both r-fuzzy B-I-set and r-**FPIO**. Then, $\mu \leq I_\tau(Cl^*(\mu, r), r)$ and $\mu = \lambda \wedge \omega$ where $\tau(\lambda) \geq r$ and ω is r-fuzzy t-I-set. Therefore,

$$\begin{aligned}\lambda \wedge \omega &\leq I_\tau(Cl^*(\lambda \wedge \omega, r), r)\\ &\leq I_\tau(Cl^*(\lambda, r), r) \wedge I_\tau(Cl^*(\omega, r), r)\\ &= I_\tau(Cl^*(\lambda, r), r) \wedge I_\tau(\omega, r).\end{aligned}$$

Hence,

$$\begin{aligned}\lambda \wedge \omega &\leq (\lambda \wedge \omega) \wedge \lambda\\ &= I_\tau(Cl^*(\lambda, r), r) \wedge I_\tau(\omega, r) \wedge \lambda\\ &= \lambda \wedge I_\tau(\omega, r).\end{aligned}$$

Thus, we obtain $\lambda \wedge \omega = \lambda \wedge I_\tau(\nu, r)$, implies $\tau(\mu) \geq r$.

Lemma 2.4. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space and $\mu, \omega \in I^X$. If $\tau(\omega) \geq r$, then $\omega \wedge Cl^*(\mu, r) \leq Cl^*(\mu \wedge \omega, r)$.

Proof. Let $\tau(\omega) \geq r$, by Theorem 1.3, then we have $(\omega \wedge \mu^*_r) \leq (\omega \wedge \mu)^*_r$ for any $\mu \in I^X$. Thus, we have

$$\begin{aligned}\omega \wedge Cl^*(\mu, r) &= \omega \wedge (\mu \vee \mu^*_r)\\ &= (\omega \wedge \mu) \vee (\omega \wedge \mu^*_r)\\ &\leq (\omega \wedge \mu) \vee (\omega \wedge \mu)^*_r\\ &= Cl^*(\omega \wedge \mu, r).\end{aligned}$$

Theorem 2.2. Let $(X, \tau, \mathbf{I})$ be a fits and $\mu, \omega \in I^X$. Then the following properties hold:
(1) If μ is r-**FSIO** and ω is r-**F**α**IO**, then $\mu \wedge \omega$ is r-**FSIO**.
(2) If μ is r-**FPIO** and ω is r-**F**α**IO**, then $\mu \wedge \omega$ is r-**FPIO**.
(3) If $\tau(\mu) \geq r$ and ω is r-**FPIO**, then $\mu \wedge \omega$ is r-**FPIO**.
(4) If $\tau(\mu) \geq r$ and ω is r-**FSIO**, then $\mu \wedge \omega$ is r-**FSIO**

Proof. (1) Let μ be r-**FSIO** and ω be r-**F**α**IO**. By using Lemma 2.4, we have

$$\begin{aligned}\mu \wedge \omega &\leq Cl^*(I_\tau(\mu, r), r) \wedge I_\tau(Cl^*(I_\tau(\omega, r), r), r)\\ &\leq Cl^*(I_\tau(\mu, r) \wedge Cl^*(I_\tau(\omega, r), r), r)\\ &\leq Cl^*(Cl^*(I_\tau(\mu, r), r) \wedge I_\tau(\omega, r), r)\\ &\leq Cl^*(I_\tau(\mu, r), r).\end{aligned}$$

This show that $\mu \wedge \omega$ is r-**FSIO**.

(2-4) Similarly.

Corollary 2.3. Let $(X, \tau, \mathbf{I})$ be a fits and $\mu, \omega \in I^X$. Then the following properties hold:

(1) If μ is r-**FSIO** and $\tau(\omega) \geq r$, then $\mu \wedge \omega$ is r-**FSIO**.

(2) If μ is r-**FPIO** and $\tau(\omega) \geq r$, then $\mu \wedge \omega$ is r-**FPIO**.

Theorem 2.3. Let $(X, \tau, \mathbf{I})$ be a fits and $\mu, \omega \in I^X$. Then the following properties hold:

(1) If μ and ω are r-**F**α**IO**, then $\mu \wedge \omega$ is r-**F**α**IO**.

(2) If μ_γ is r-**F**α**IO** for $\gamma \in \sigma$, then $\bigvee_{\gamma \in \sigma} \mu_\gamma$ is r-**F**α**IO**.

(3) If μ_γ is r-**FPIO** for $\gamma \in \sigma$, then $\bigvee_{\gamma \in \sigma} \mu_\gamma$ is r-**FPIO**.

Proof. (1) Let μ and ω, be r-**F**α**IO**, by Lemma 2.1, μ is r-**FSIO** and r-**FPIO** and by Theorem 2.2(1,2), $\mu \wedge \omega$ is r-**FSIO** and r-**FPIO**. Therefore, by Lemma 2.1, $\mu \wedge \omega$ is r-**F**α**IO**.

(2) Let μ_γ be a class of r-**F**α**IO**. Then for any $\gamma \in \sigma$,

$$\begin{aligned}\mu_\gamma &\leq I_\tau(Cl^*(I_\tau(\mu_\gamma, r), r), r)\\ &\leq I_\tau(Cl^*(I_\tau(\bigvee_{\gamma \in \sigma} \mu_\gamma, r), r), r).\end{aligned}$$

Hence $\bigvee_{\gamma \in \sigma} \mu_\gamma \leq I_\tau(Cl^*(I_\tau(\bigvee_{\gamma \in \sigma} \mu_\gamma, r), r), r)$. This show that $\bigvee_{\gamma \in \sigma} \mu_\gamma$ is r-**F**α**IO**.

(3) Similarly.

Theorem 2.4. Let $(X, \tau, \mathbf{I})$ be a fits, if μ is r-**FPIC** then $Cl^*(I_\tau(\mu, r), r) \leq \mu$, for each $\mu \in I^X$.

Proof. Let μ be r-**FPIC**. Then $\overline{1} - \mu$ is r-**FPIO**. Hence

$$\begin{aligned}\overline{1} - \mu &\leq I_\tau(Cl^*(\overline{1} - \mu, r), r)\\ &\leq I_\tau(C_\tau(\overline{1} - \mu, r), r)\\ &= \overline{1} - C_\tau(I_\tau(\mu, r), r)\\ &\leq \overline{1} - Cl^*(I_\tau(\mu, r), r).\end{aligned}$$

Therefore, we option $Cl^*(I_\tau(\mu, r), r) \leq \mu$.

Remark 2.5. Let $(X, \tau, \mathbf{I})$ be a fits. For each $\mu \in I^X$, we have $I_\tau(Cl^*(\overline{1} - \mu, r), r) \neq \overline{1} - Cl^*(I_\tau(\mu, r), r)$ as show by the following example.

Example 2.5. In Example 2.4, If we take $\mathbf{I} = \mathbf{I}^0$ for all $r \in I_0$, and let $\mu = \overline{0.7}$, then μ satisfies the above properties.

Corollary 2.4. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space, such that $I_\tau(Cl^*(\overline{1} - \mu, r), r) \neq \overline{1} - Cl^*(I_\tau(\mu, r), r)$. Then μ is r-**FPIC** iff $Cl^*(I_\tau(\mu, r), r) \leq \mu$. for each $\mu \in I^X$ and $r \in I_0$.

Theorem 2.5. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space. For each $\lambda \in I^X$, we define an operator $\mathbf{I}C_\tau : I^X \to I$ as follows:

$$\mathbf{I}C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \ \lambda \leq \mu, \ \mu \text{ is r-}\mathbf{FIC}\}.$$

For each $\lambda, \mu \in I^X$, the following properties are holds:

(1) $\mathbf{I}C_\tau(\overline{0}, r) = \overline{0}$.

(2) $\lambda \leq \mathbf{I}C_\tau(\lambda, r)$.

(3) $\mathbf{I}C_\tau(\lambda, r) \vee \mathbf{I}C_\tau(\mu, r) \leq \mathbf{I}C_\tau(\lambda \vee \mu, r)$.

(4) $\mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r) = \mathbf{I}C_\tau(\lambda, r)$.

(5) If λ is r-**FIC**, iff $\lambda = \mathbf{I}C_\tau(\lambda, r)$.

(6) If $C_\tau(\lambda, r)$ is r-**FIC**, then $C_\tau(\mathbf{I}C_\tau(\lambda, r), r) = \mathbf{I}C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Proof. (1), (2) and (5) are easily proved from the definition of $\mathbf{I}C_\tau$ and Lemma 1.1.

(3) Since $\lambda, \mu \leq \lambda \vee \mu$, we have

$$\mathbf{I}C_\tau(\lambda, r) \vee \mathbf{I}C_\tau(\mu, r) \leq \mathbf{I}C_\tau(\lambda \vee \mu, r)$$

(4) From (2) we have $\mathbf{I}C_\tau(\lambda, r) \leq \mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r)$. Now we show that $\mathbf{I}C_\tau(\lambda, r) \geq \mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r)$. Suppose that

$$\mathbf{I}C_\tau(\lambda, r) \ngeq \mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r).$$

There exist $x \in X$ and $t \in (0, 1)$ such that

$$\mathbf{I}C_\tau(\lambda, r)(x) < t < \mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r)(x). \qquad (\mathbf{B})$$

Since $\mathbf{I}C_\tau(\lambda, r)(x) < t$, by the definition $\mathbf{I}C_\tau$, there exists r-**FIC**, λ_1 with $\lambda \leq \lambda_1$ such that

$$\mathbf{I}C_\tau(\lambda, r)(x) \leq \lambda_1(x) < t.$$

Since $\lambda \leq \lambda_1$, we have $\mathbf{I}C_\tau(\lambda, r) \leq \lambda_1$. Again, by the definition $\mathbf{I}C_\tau$, we have $\mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r) \leq \lambda_1$. Hence $\mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r)(x) \leq \lambda_1(x) < t$. It is a contradiction for (**B**). Thus

$$\mathbf{I}C_\tau(\lambda, r) \geq \mathbf{I}C_\tau(\mathbf{I}C_\tau(\lambda, r), r).$$

(6) From (2) and $C_\tau(\lambda, r)$ is a r-**FIC** we have $\mathbf{I}C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.
we only show that

$$C_\tau(\mathbf{I}C_\tau(\lambda, r), r) = C_\tau(\lambda, r).$$

Since $\lambda \leq \mathbf{I}C_\tau(\lambda, r)$

$$C_\tau(\mathbf{I}C_\tau(\lambda, r), r) \geq C_\tau(\lambda, r).$$

Suppose that

$$C_\tau(\mathbf{I}C_\tau(\lambda,r),r)C_\tau(\lambda,r).$$

There exist $x \in X$ and $r \in I_0$ such that

$$C_\tau(\mathbf{I}C_\tau(\lambda,r),r)(x) > C_\tau(\lambda,r)(x).$$

By the definition C_τ, there exists $\nu \in I^X$, with $\lambda \le \nu$ and $\tau(\overline{1}-\nu) \ge r$ such that

$$C_\tau(\mathbf{I}C_\tau(\lambda,r),r)(x) > \nu(x) \ge C_\tau(\lambda,r)(x).$$

On the other hand, since $\nu = C_\tau(\nu,r),\ \lambda \le \nu$, then

$$\mathbf{I}C_{\mathbf{I}}(\lambda,r) \le \mathbf{I}C_\tau(\nu,r) = \mathbf{I}C_\tau(C_\tau(\nu,r),r) = C_\tau(\nu,r) = \nu.$$

Thus $C_\tau(\mathbf{I}C_\tau(\lambda,r),r) \le \nu$.
It is a contradiction. Hence $C_\tau(\mathbf{I}C_\tau(\lambda,r),r) \le C_\tau(\lambda,r)$.

Theorem 2.6. Let $(X,\tau,\mathbf{I})$ be a fits. For each $\lambda \in I^X$, we define an operator $\mathbf{I}I_\tau : I^X \to I$ as follows:

$\mathbf{I}I_\tau(\lambda,r) = \bigvee\{\mu \in I^X :\ \ \mu \le \lambda,\ \ \mu \text{ is } \ r - \mathbf{FIO}\}$.
$For each \mu \in I^X$, it holds the following properties:

(1) $\mathbf{I}I_\tau(\overline{1}-\mu,r) = \overline{1} - (\mathbf{I}C_\tau(\mu,r))$.

(2) $\mathbf{I}I_\tau(\mu,r) \le \mu \le \mathbf{I}C_\tau(\mu,r)$.

(3) If μ is r-**FIO** iff $\mathbf{I}I_\tau(\mu,r) = \mu$.

Proof. (1) It is easily proved form the following:

$$\begin{aligned}&\overline{1} - (\mathbf{I}C_\tau(\lambda,r))\\ &= \overline{1} - \bigwedge\{\mu \in I^X :\ \lambda \le \mu,\ \mu\ is\ r - \mathbf{FIC}\}\\ &= \bigvee\{\mu \in I^X :\ \overline{1}-\lambda \ge \overline{1}-\mu,\ \overline{1}-\mu\ is\ r - \mathbf{FIO}\}\\ &= \mathbf{I}I_\tau(\overline{1}-\lambda,r).\end{aligned}$$

(2) and (3) are easily proved form the definition of $\mathbf{I}I_\tau$ and Lemma 1.1.

3. Decompositions of fuzzy continuity and fuzzy I-continuity

Definition 3.1. A mapping $f : (X,\tau,\mathbf{I}) \to (Y,\eta)$ is called fuzzy I-continuous (resp. fuzzy pre-I-continuous, fuzzy $*$-I-continuous, fuzzy B-I-continuous, fuzzy semi-I-continuous, fuzzy α-I-continuous) if $f^{-1}(\mu)$ is r-**FIO** (resp. r-**FPIO**, r-fuzzy $*$-denes-in-itself, r-fuzzy B-I-set, r-**FSIO**, r-**F**α**IO**) for each $\eta(\mu) \ge r$ and $r \in I_0$.

According to Lemma 2.1–3 we have the following decomposition of fuzzy continuity and decomposition of fuzzy I-continuity.

Theorem 3.1. (1) A mapping $f : (X,\tau,\mathbf{I}) \to (Y,\eta)$ is called fuzzy continuous if and only if it is both fuzzy pre-I-continuous and fuzzy B-I-continuous.

(2) A mapping $f : (X,\tau,\mathbf{I}) \to (Y,\eta)$ is called fuzzy I-continuous if and only if it is both fuzzy pre-I-continuous and fuzzy $*$-I-continuous.

(3) A mapping $f : (X,\tau,\mathbf{I}) \to (Y,\eta)$ is called fuzzy α-I-continuous if and only if it is both fuzzy pre-I-continuous and fuzzy semi-I-continuous.

Theorem 3.2. Let $f : (X,\tau,\mathbf{I}) \to (Y,\eta)$ be a function, then following statements are equivalent.

(1) A map f is fuzzy α-I-continuous.

(2) The inverse image of each r-fuzzy closed set in Y is r-**F**α**IO**.

(3) $C_\tau(Int^*_\tau(C_\tau(f^{-1}(\lambda),r),r),r) \le f^{-1}(C_\eta(\lambda),r)$, for each $\lambda \in I^Y$ and $r \in I_0$.

(4) $f(C_\tau(Int^*_\tau(C_\tau(\mu,r),r),r)) \le C_\eta(f(\mu),r)$, for each $\mu \in I^X$ and $r \in I_0$.

Proof. (1)⇔(2): It easily proved form Definition 3.1, and $f^{-1}(\overline{1}-\mu) = \overline{1} - f^{-1}(\mu)$.

(2)⇔(3): For each $\lambda \in I^Y$ and $r \in I_0$. Since $C_\eta(\lambda,r)$ is r-fuzzy closed set in Y, by (2) $f^{-1}(C_\eta(\lambda,r)$ is r-**F**α**IC** and $\overline{1} - f^{-1}(C_\eta(\lambda,r)$ is r-**F**α**IO**. Therefore,

$$\begin{aligned}\overline{1}-\ \ & f^{-1}(C_\eta(\lambda,r)\\ &\le I_\tau(Cl^*(I_\tau(\overline{1}-f^{-1}(C_\tau(\lambda,r)),r),r)\\ &= \overline{1} - C_\tau(Int^*_\tau(C_\tau(f^{-1}(C_\tau(\lambda,r)),r),r),r).\end{aligned}$$

Hence, we obtain

$$f^{-1}(C_\eta(\lambda,r) \ge C_\tau(Int^*_\tau(C_\tau(f^{-1}(\lambda),r),r),r).$$

(3)⇔(4): For each $\mu \in I^X$ and $r \in I_0$. By (3), we have

$$\begin{aligned}C_\tau(Int^*_\tau(C_\tau(\mu,r),r) &\le C_\tau(Int^*_\tau(C_\tau(f^{-1}f(\mu),r),r)\\ &\le f^{-1}(C_\eta(f(\mu),r)),\end{aligned}$$

and hence

$$f(C_\tau(Int^*_\tau(C_\tau(\mu,r),r),r)) \le C_\eta(f(\mu),r).$$

(4)⇔(1): Let $\eta(\nu) \ge r$. Then by (4),

$$\begin{aligned}f(C_\tau(Int^*_\tau(C_\tau\ \ & (f^{-1}(\overline{1}-\nu)),r),r),r)\\ &\le C_\eta(ff^{-1}(\overline{1}-\nu),r)\\ &\le C_\eta(\overline{1}-\nu),r) = \overline{1}-\nu.\end{aligned}$$

Thus,

$$\begin{aligned}C_\tau(Int^*_\tau(C_\tau(f^{-1}(\overline{1}-\nu),r),r),r)\ \ &\le f^{-1}(\overline{1}-\nu)\\ &\le \overline{1} - f^{-1}(\nu).\end{aligned}$$

Consequently, we have

$$f^{-1}(\nu) \le I_\tau(Cl^*(I_\tau(f^{-1}(\nu),r),r),r).$$

This show that $f^{-1}(\nu)$ is r-**F**α**IO**. Thus, f is fuzzy α-I-continuous.

Theorem 3.3. Let $f : (X,\tau,\mathbf{I}) \to (Y,\eta)$ be fuzzy α-I-continuous, then

(1) $f(Cl^*(\mu,r)) \le C_\eta(f(\mu),r)$, for each $\mu \in I^X$ is r-**FPIO**.

(2) $Cl^*(f^{-1}(\lambda),r) \le f^{-1}(C_\tau(\lambda,r))$, for each $\lambda \in I^Y$ is r-**FPIO**.

Proof. (1) If $\mu \in I^X$ is r-**FPIO**, then $\mu \leq I_\tau(Cl^*(\mu, r), r)$. Thus, by Theorem 3.2 we have

$$\begin{aligned} f(Cl^*(\mu,r)) &\leq f(C_\tau(\mu,r)) \\ &\leq f(C_\tau(I_\tau(Cl^*(\mu,r),r),r)) \\ &\leq f(C_\tau(Int^*_\tau(C_\tau(\mu,r),r),r)) \\ &\leq C_\tau(f(\mu),r). \end{aligned}$$

(2) If $\lambda \in I^Y$ is r-**FPIO**, then $\lambda \leq I_\tau(Cl^*(\lambda, r), r)$. Therefore, by Theorem 3.2, we have

$$\begin{aligned} &Cl^*(f^{-1}(\lambda),r) \leq C_\tau(f^{-1}(\lambda),r) \\ &\leq C_\tau(f^{-1}(I_\tau(Cl^*(\lambda,r),r),r)) \\ &\leq C_\tau(I_\tau(Cl^*(I_\tau(f^{-1}(I_\tau(Cl^*(\lambda,r),r)),r),r),r),r) \\ &\leq C_\tau(Int^*_\tau(C_\tau(f^{-1}(I_\tau(Cl^*(\lambda,r),r)),r),r),r) \\ &\leq f^{-1}(C_\tau(I_\tau(Cl^*(\lambda,r),r),r)) \\ &\leq f^{-1}(C_\tau(\lambda,r)). \end{aligned}$$

Definition 3.2. A mapping $f : (X, \tau) \to (Y, \eta, \mathbf{I})$ is called fuzzy α-I-open (resp. fuzzy semi-I-open, fuzzy pre-I-open, fuzzy β-I-open) if image of each $\mu \in I^X$ with $\tau(\mu) \geq r$ is r-**F**α**IO** (resp. r-**FSIO**, r-**FPIO**, r-**F**β**IO**) set of Y.

Remark 3.1. By Definition 2.2, and Remark 2.1 we obtain the following diagram:

fuzzy open $\Rightarrow$ **fuzzy** $\alpha-$ **I-open** $\Rightarrow$ **fuzzy pre-I-open**

$\Downarrow$ $\Downarrow$

fuzzy semi-I-open $\Rightarrow$ **fuzzy** β**-I-open**

Theorem 3.4. A mapping $f : (X, \tau) \to (Y, \eta, \mathbf{I})$ is called fuzzy α-I-open if and only if it is fuzzy semi-I-open and fuzzy pre-I-open.

Proof. Form Lemma 2.1, the proof straightforward.

Theorem 3.5. A mapping $f : (X, \tau) \to (Y, \eta, \mathbf{I})$ is fuzzy α-I-open if and only if for each $\mu \in I^Y$ and each $\tau(\overline{1} - \lambda) \geq r$, containing $f^{-1}(\mu)$, there exists $\nu \in I^Y$ r-**F**α**IC** containing μ such that $f^{-1}(\nu) \leq \lambda$.

Proof. Necessity. Let $\nu = \overline{1} - f(\overline{1} - \lambda)$. Since $f^{-1}(\mu) \leq \lambda$, we have $f(\overline{1} - \lambda) \leq \overline{1} - \mu$. Since f is fuzzy α-I-open, then ν is r-**F**α**IC** and

$$f^{-1}(\nu) = \overline{1} - f^{-1}(f(\overline{1} - \lambda)) \leq \overline{1} - (\overline{1} - \lambda) = \lambda.$$

Sufficiency. Obvious.

Corollary 3.1. Let $f : (X, \tau, \mathbf{I}) \to (Y, \eta)$ be fuzzy α-I-open. For each $\mu \in I^Y$, then

(1) $f^{-1}(C_\tau(I^*_\tau(C_\tau(\mu, r), r), r)) \leq C_\eta(f^{-1}(\mu), r)$.

(2) $f^{-1}(Cl^*(\lambda, r)) \leq C_\tau(f^{-1}(\lambda), r)$.

Proof. For each $\mu \in I^Y$, then $C_\tau(f^{-1}(\mu), r)$ is f-fuzzy closed. By Theorem 3.5, there exists $\nu \in I^Y$ r-**F**α**IC** containing μ such that $f^{-1}(\nu) \leq C_\tau(f^{-1}(\mu), r)$. Since $\overline{1} - \nu$ is r-**F**α**IO**, $f^{-1}(\overline{1} - \nu) \leq f^{-1}(I_\tau(Cl^*(I_\tau(\overline{1} - \nu, r), r), r))$ and

$$\begin{aligned} \overline{1} - f^{-1}(\nu) &\leq f^{-1}(\overline{1} - C_\tau(Int^*_\tau(C_\tau(\nu,r),r),r)) \\ &\leq \overline{1} - f^{-1}(C_\tau(Int^*_\tau(C_\tau(\nu,r),r),r)). \end{aligned}$$

Therefore,

$$f^{-1}(C_\tau(Int^*_\tau(C_\tau(\nu,r),r),r)) \leq f^{-1}(\nu) \leq C_\eta(f^{-1}(\mu),r).$$

Thus, $f^{-1}(C_\tau(I^*_\tau(C_\tau(\mu, r), r), r)) \leq C_\eta(f^{-1}(\mu), r)$.

(2) Similarly.

Theorem 3.6. Let $f : (X, \tau, \mathbf{I}) \to (Y, \eta)$ be a mapping. For each $r \in I_0$, then following statements are equivalent.

(1) A map f is called fuzzy I-continuous function.

(2) $f^{-1}(\mu)$ is r-**FIC**, in X for each $\mu \in I^X$, $r \in I_0$, With $\eta(\overline{1} - \mu) \geq r$.

(3) $f(\mathbf{I}C_\tau(\lambda, r)) \leq C_\eta(f(\lambda), r)$, for each $\lambda \in I^X$.

(4) $\mathbf{I}C_\tau(f^{-1}(\mu, r)) \leq f^{-1}(C_\eta(\mu, r))$, for $\mu \in I^Y$.

(5) $f^{-1}(I_\eta(\mu, r)) \leq \mathbf{I}I_\tau(f^{-1}(\mu, r))$, for each $\mu \in I^Y$.

Proof. (1)$\Leftrightarrow$(2): It easily proved form Definition 1.6(2), and $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$.

(2)$\Rightarrow$(3): Suppose there exist $\lambda \in I^X$ and $r \in I_0$ such that

$$f(\mathbf{I}C_\tau(\lambda,r)) \not\leq C_\tau(f(\lambda),r).$$

There exist $y \in Y$ and $t \in I_0$ such that

$$f(\mathbf{I}C_\tau(\lambda,r))(y) > t > C_\eta(f(\lambda),r)(y).$$

If $f^{-1}(\{y\}) = \emptyset$, it is a contradiction because $f(\mathbf{I}C_\tau(\lambda, r))(y) \neq 0$.

If $f^{-1}(\{y\}) \neq \emptyset$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(\mathbf{I}C_\tau(\lambda,r))(y) \geq \mathbf{I}C_\tau(\lambda,r)(x) > t > C_\eta(f(\lambda),r)(f(x)). \quad (\mathbf{A})$$

Since $C_\eta(f(\lambda), r)(f(x)) \leq t$, there exists $\eta(\overline{1} - \mu) \geq r$ with $f(\lambda) \leq \mu$ such that

$$C_\eta(f(\lambda),r)(f(x)) \leq \mu(f(x)) \leq t.$$

Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. Form (2), $f^{-1}(\mu)$ is r-**FIC**. Thus,

$\mathbf{I}C_\tau(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t$. It is a contradiction for (**A**).

(3)$\Rightarrow$(4): For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$. Form (3), we have

$$f(\mathbf{I}C_\tau(f^{-1}(\mu),r)) \leq C_\eta(f(f^{-1}(\mu)),r) \leq C_\eta(\mu,r).$$

It implies

$$\begin{aligned} \mathbf{I}C_\tau(f^{-1}(\mu),r) &\leq f^{-1}(f(C_\eta(f^{-1}(\mu)),r)) \\ &\leq f^{-1}(C_\eta(\mu,r)). \end{aligned}$$

(4)⇒(5): It easily proved form Theorems 2.6(1) and Theorem 1.2(1).

(5)⇒(1): Let $\eta(\mu) \geq r$. Then we have by definition I_τ, $\mu = I_\eta(\mu, r)$. By (5) we have

$$f^{-1}(\mu) \leq \mathbf{I}I_\tau(f^{-1}(\mu), r).$$

On the other hand, by Theorem 2.6(2),

$$f^{-1}(\mu) \geq \mathbf{I}I_\tau(f^{-1}(\mu), r).$$

Thus, $f^{-1}(\mu) = \mathbf{I}I_\tau(f^{-1}(\mu), r)$ that is $f^{-1}(\mu)$ is r-**FIO**.

Analogous theorems to Theorem 3.6 can be given for the types of continuity in Definition 3.1.

Definition 3.3. Let $f : (X, \tau, \mathbf{I}) \to (Y, \eta, \mathbf{I})$ be a mapping.

(1) f is called fuzzy I-irresolute if $f^{-1}(\mu)$ is r-**FIO** set of X for each r-**FIO** $\mu \in I^Y$ and $r \in I_0$.

(2) f is called fuzzy I-irresolute open (resp. fuzzy I-open) if $f(\mu)$ is r-**FIO** set of Y for each r-**FIO** $\mu \in I^X$ (resp. $\tau(\mu) \geq r$).

(3) f is called fuzzy I-irresolute closed (resp. fuzzy I-closed) if $f(\mu)$ is r-**FIC** set of Y for each r-**FIC** $\mu \in I^X$ (resp. $\tau(\overline{1} - \mu) \geq r$.

The Following theorem is similarly proved as Theorem 3.6.

Theorem 3.7. Let $f : (X, \tau, \mathbf{I}) \to (Y, \eta, \mathbf{I})$ be a mapping. Then following statements are equivalent.

(1) A map f is fuzzy I-irresolute.

(2) For each r-**FIC** $\mu \in I^Y$, $f^{-1}(\mu)$ is r-**FIC**.

(3) $f(\mathbf{I}C_\tau(\lambda, r)) \leq \mathbf{I}C_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(4) $\mathbf{I}C_\tau(f^{-1}(\mu, r)) \leq f^{-1}(\mathbf{I}C_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(5) $f^{-1}(\mathbf{I}I_\eta(\mu, r)) \leq \mathbf{I}I_\tau(f^{-1}(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

Theorem 3.8. Let $f : (X, \tau, \mathbf{I}) \to (Y, \eta, \mathbf{I})$ be a bijective mapping. The following statements are equivalent.

(1) A map f is fuzzy I-irresolute.

(2) $\mathbf{I}I_\eta(f(\mu), r)) \leq f(\mathbf{I}I_\tau(\mu), r)$, for each $\mu \in I^X$.

Proof. (1)⇒(2): Let f be fuzzy I-irresolute mapping and $\mu \in I^X$. Then $f^{-1}(\mathbf{I}I_\eta(f(\mu), r))$ is r-**FIO**. Form Theorem 3.7(5), and the fact that f is one-to-one we have

$$f^{-1}(\mathbf{I}I_\eta(f(\mu), r)) \leq \mathbf{I}I_\tau(f^{-1}(f(\mu)), r) = \mathbf{I}I_\tau(\mu, r).$$

Again since f is onto we have

$$\mathbf{I}I_\eta(f(\mu), r) = ff^{-1}(\mathbf{I}I_\eta(f(\mu), r)) \leq f(\mathbf{I}I_\tau(\mu, r)).$$

(2)⇒(1): Let μ is r-**FIO** set of Y. Form Theorem 2.6(3), $\mu = \mathbf{I}I_\eta(\mu, r)$. By (2) we have

$$f(\mathbf{I}I_\tau(f^{-1}(\mu), r)) \geq \mathbf{I}I_\eta(ff^{-1}(\mu), r) = \mathbf{I}I_\eta(\mu, r) = \mu$$

and

$$\mathbf{I}I_\tau(f^{-1}(\mu), r) = f^{-1}f(\mathbf{I}I_\tau(f^{-1}(\mu), r)) \leq f^{-1}(\mu).$$

Thus, $f^{-1}(\mu) = \mathbf{I}I_\tau(f^{-1}(\mu), r)$. Thus, f is fuzzy I-irresolute.

Theorem 3.9. Let $f : (X, \tau, \mathbf{I}) \to (Y, \eta, \mathbf{J})$ be fuzzy ideal topological space $f : X \to Y$ be a mapping. Then following statements are equivalent.

(1) f is fuzzy I-irresolute open.

(2) $f(\mathbf{I}I_\tau(\lambda, r)) \leq \mathbf{I}I_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) $\mathbf{I}I_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mathbf{I}I_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(4) For any $\mu \in I^Y$ and any r-**FIC** $\lambda \in I^X$ with $f^{-1}(\mu) \leq \lambda$, there exists a r-**FIC** $\rho \in I^Y$ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.

Proof.

(1)⇒(2): For each $\lambda \in I^X$. Since $\mathbf{I}I_\tau(f(\lambda), r) \leq \lambda$ form Theorem 2.6(2), we have $f(\mathbf{I}I_\tau(\lambda, r)) \leq f(\lambda)$. form (1), $f(\mathbf{I}I_\tau(\lambda, r))$ is r-**FIO**. Therefore $f(\mathbf{I}I_\tau(\lambda, r)) \leq \mathbf{I}I_\eta(f(\lambda), r)$.

(2)⇒(3): For all $\mu \in I^Y$ and $r \in I_0$, put $\lambda = f^{-1}(\mu)$ form (2). Then

$$f(\mathbf{I}I_\tau(f^{-1}(\mu), r)) \leq \mathbf{I}I_\eta(f(f^{-1}(\mu)), r) \leq \mathbf{I}I_\eta(\mu, r).$$

It implies $\mathbf{I}I_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mathbf{I}I_\eta(\mu, r))$.

(3)⇒(4): Let λ be r-**FIC** set of X such that $f^{-1}(\mu) \leq \lambda$. Since $\overline{1} - \lambda \leq f^{-1}(\overline{1} - \mu)$ and $\mathbf{I}I_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda$,

$$\mathbf{I}I_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda \leq \mathbf{I}I_\tau(f^{-1}(\overline{1} - \mu), r).$$

From (3),

$$\overline{1} - \lambda \leq \mathbf{I}I_\tau(f^{-1}(\overline{1} - \mu), r) \leq f^{-1}(\mathbf{I}I_\eta(\overline{1} - \mu, r)).$$

It implies

$$\begin{aligned} \lambda &\geq \overline{1} - f^{-1}(\mathbf{I}I_\eta(\overline{1} - \mu), r) \\ &= f^{-1}(\overline{1} - \mathbf{I}I_\eta(\overline{1} - \mu, r)) \\ &= f^{-1}(\mathbf{I}C_\eta(\mu, r)). \end{aligned}$$

Hence there exists a r-**FIC** $\mathbf{I}C_\eta(\mu, r)$ with $\mu \leq \mathbf{I}C_\eta(\mu, r)$ such that $f^{-1}(\mathbf{I}C_\eta(\mu, r)) \leq \lambda$.

(4)⇒(1) Let ω be r-**FIO** of X. Put $\mu = \overline{1} - f(\omega)$ and $\lambda = \overline{1} - \omega$ such that λ is r-**FIC**. We obtain

$$\begin{aligned} f^{-1}(\mu) &= f^{-1}(\overline{1} - f(\omega)) \\ &= \overline{1} - f^{-1}(f(\omega)) \\ &\leq \overline{1} - \omega = \lambda. \end{aligned}$$

Form (4) there exists a r-**FIC** set ρ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda = \bar{1} - \omega$. It implies $\omega \leq \bar{1} - f^{-1}(\rho) = f^{-1}(\bar{1} - \rho)$. Thus, $f(\omega) \leq f(f^{-1}(\bar{1} - \rho)) = \bar{1} - \rho$. On the other hand, since $\mu \leq \rho$,

$$f(\omega) = \bar{1} - \mu \geq \bar{1} - \rho.$$

Hence $f(\omega) = \bar{1} - \rho$, that is, $f(\omega)$ is r-**FIO**.

Theorem 3.10 is similarly proved from Theorem 3.9.

Theorem 3.10. Let $(X, \tau, \mathbf{I})$ and $(Y, \eta, \mathbf{I})$ be fuzzy ideal topological spaces $f : X \to Y$ be a mapping. Then following statements are equivalent.

(1) f is fuzzy I-irresolute closed.

(2) $f(\mathbf{I}C_\tau(\lambda, r)) \leq \mathbf{I}C_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) For any $\mu \in I^Y$ and any r-**FIO** $\lambda \in I^X$ with $f^{-1}(\mu) \leq \lambda$, there exists a r-**FIO** $\rho \in I^Y$ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.

Theorem 3.11. Let $(X, \tau, \mathbf{I})$ and (Y, η) be fuzzy ideal topological space. A mapping $f : X \to Y$ be a fuzzy I-open. Then the following statements are holed.

(1) $f(I_\tau(\lambda, r)) \leq \mathbf{I}I_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(2) $I_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mathbf{I}I_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(3) For any $\mu \in I^Y$ and $\tau(\bar{1} - \lambda) \geq r$ such that $f^{-1}(\mu) \leq \lambda$, there exists a r-**FIC** set $\rho \in I^Y$
with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.

Proof. (1) For each $\lambda \in I^X$ since $I_\tau(\lambda, r) \leq \lambda$, by Theorem 1.2(3). Then $f(I_\tau(\lambda, r)) \leq f(\lambda)$. From (1), $f(I_\tau(\lambda, r))$ is r-**FIO**. Therefore

$$f(I_\tau(\lambda, r)) \leq \mathbf{I}I_\eta(f(\lambda), r)$$

.

(2) For all $\mu \in I^Y$ and $r \in I_0$, put $\lambda = f^{-1}(\mu)$ form (2). Then

$$f(I_\tau(f^{-1}(\mu), r)) \leq \mathbf{I}I_\eta(f(f^{-1}(\mu)), r) = \mathbf{I}I_\eta(\mu, r).$$

It implies $I_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mathbf{I}I_\eta(\mu, r))$.

(3) Let $\tau(\bar{1} - \lambda) \geq r$ set of X such that $f^{-1}(\mu) \leq \lambda$. Since $\underline{1} - \lambda \leq f^{-1}(\underline{1} - \mu)$ and $I_\tau(\underline{1} - \lambda, r) = \underline{1} - \lambda$.

$$I_\tau(\underline{1} - \lambda, r) = \underline{1} - \lambda \leq I_\tau(f^{-1}(\underline{1} - \mu), r).$$

Form (2), we have

$$\underline{1} - \lambda \leq I_\tau(f^{-1}(\underline{1} - \mu), r) \leq f^{-1}(\mathbf{I}I_\eta(\underline{1} - \mu, r)).$$

It implies

$$\begin{aligned} \lambda &\geq \underline{1} - f^{-1}(\mathbf{I}I_\eta(\underline{1} - \mu), r) \\ &= f^{-1}(\underline{1} - \mathbf{I}I_\eta(\underline{1} - \mu, r)) \\ &= f^{-1}(\mathbf{I}C_\eta(\mu, r)). \end{aligned}$$

Hence there exists a r-**FIC** $\mathbf{I}C_\eta(\mu, r) \in I^Y$ with $\mu \leq \mathbf{I}C_\eta(\mu, r)$ such that $f^{-1}(\mathbf{I}C_\eta(\mu, r)) \leq \lambda$.

Theorem 3.12 is similarly proved from Theorem 3.11.

Theorem 3.12. Let (X, τ) and $(Y, \eta, \mathbf{I})$ be fuzzy ideal topological spaces. A mapping $f : X \to Y$ be a fuzzy I-closed. Then following statements are holed.

(1) $f(C_\tau(\lambda, r)) \leq \mathbf{I}C_\eta(f(\lambda), r)$, for $\lambda \in I^{X,}$ $r \in I_0$.

(2) For any $\lambda \in I^Y$ and $\tau(\mu) \geq r$ such that $f^{-1}(\lambda) \leq \mu$, there exists a r-**FIO** with $\lambda \leq \rho$ such that $f^{-1}(\rho) \leq \mu$.

Theorem 3.13. Let $(X, \tau, \mathbf{I})$ and $(Y, \eta, \mathbf{J})$ be fuzzy ideal topological space and $f : X \to Y$ be a bijective mapping

(1) f is a fuzzy I-irresolute closed iff $f^{-1}(\mathbf{I}C_\eta(\mu, r)) \geq \mathbf{I}C_\tau(f^{-1}(\mu), r)$, for each $\mu \in I^Y$.

(2) f is a fuzzy I-irresolute closed iff fuzzy I-irresolute open for each $\mu \in I^X$ and $r \in I_0$.

Proof. 1(⇒) Let f be a fuzzy I-irresolute closed. Form Theorem 3.10(2), for each $\mu \in I^X$ and $r \in I_0$.

$$f(\mathbf{I}C_\tau(\lambda, r)) \leq \mathbf{I}C_\eta(f(\lambda), r).$$

For all $\mu \in I^Y$ and $r \in I_0$ put $\lambda = f^{-1}(\mu)$. Since f is onto, $ff^{-1}(\mu) = \mu$. Thus

$$\begin{aligned} f(\mathbf{I}C_\tau(f^{-1}(\mu), r)) &\leq \mathbf{I}C_\eta(f(f^{-1}(\mu)), r) \\ &= \mathbf{I}C_\eta(\mu, r). \end{aligned}$$

It implies

$$\begin{aligned} \mathbf{I}C_\tau(f^{-1}(\mu), r) &= f^{-1}(f(\mathbf{I}C_\tau(f^{-1}(\mu), r))) \\ &\leq f^{-1}(\mathbf{I}C_\eta(\mu, r)). \end{aligned}$$

1(⇐) Put $\mu = f(\lambda)$. Since f is injective

$$f^{-1}(\mathbf{I}C_\eta(f(\lambda), r)) \leq \mathbf{I}C_\tau(f^{-1}(f(\lambda)), r) = \mathbf{I}C_\tau(\lambda, r)$$

Since f is onto $\mathbf{I}C_\eta(f(\lambda), r) \leq f(\mathbf{I}C_\tau(\lambda, r))$.

(2) It easily proved from:

$f^{-1}(\mathbf{I}C_\eta(\mu, r)) \leq \mathbf{I}C_\tau(f^{-1}(\mu), r)$

$\Leftrightarrow \bar{1} - f^{-1}(\mathbf{I}I_\eta(\bar{1} - \mu, r)) \leq \bar{1} - \mathbf{I}I_\tau(\bar{1} - f^{-1}(\mu), r)$.

$\Leftrightarrow f^{-1}(\mathbf{I}I_\eta(\bar{1} - \mu, r)) \geq \mathbf{I}I_\tau(f^{-1}(\bar{1} - \mu), r)$.

Form above theorems we have the following theorem.

Theorem 3.14. Let $(X, \tau, \mathbf{I})$ and $(Y, \eta, \mathbf{I})$ be fuzzy ideal topological spaces and $f : X \to Y$ be mappings. Then following statements are equivalent.

(1) f is fuzzy I-irresolute and fuzzy I-irresolute open.

(2) f is fuzzy I-irresolute and fuzzy I-irresolute closed.

(3) $f(\mathbf{I}I_\tau(\lambda, r)) \leq \mathbf{I}I_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(4) $f(\mathbf{I}C_\tau(\lambda, r)) \leq \mathbf{I}C_\eta(f(\lambda), r)$, for each $\lambda \in I^X$, $r \in I_0$.

(5) $\mathbf{I}I_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mathbf{I}I_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(6) $\mathbf{I}C_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mathbf{I}C_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

Theorem 3.15. Let $f : (X, \tau, \mathbf{I}) \rightarrow (Y, \eta, \mathbf{I})$ and $g : (Y, \eta, \mathbf{I}) \rightarrow (Z, \gamma)$ be a mapping. the following statements are hold.

(1) If f and g is fuzzy I-irresolute, then $g \circ f$ is fuzzy I-irresolute.

(2) If f is fuzzy I-irresolute and g is fuzzy I-continuous, then $g \circ f$ is is fuzzy I-continuous.

(3) If f and g is fuzzy I-irresolute open, then $g \circ f$ is fuzzy I-irresolute open.

Proof. Obvious.

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