Decomposition of Gray-Scale Morphological Templates Using the Rank Method

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Abstract—Convolutions are a fundamental tool in image processing. Classical examples of two dimensional linear convolutions include image correlation, the mean filter, the discrete Fourier transform, and a multitude of edge mask filters. Nonlinear convolutions are used in such operations as the median filter, the medial axis transform, and erosion and dilation as defined in mathematical morphology. For large convolution masks or structuring elements, the computation cost resulting from implementation can be prohibitive. However, in many instances, this cost can be significantly reduced by decomposing the templates representing the masks or structuring elements into a sequence of smaller templates. In addition, such decomposition can often be made architecture specific and, thus, resulting in optimal transform performance. In this paper we provide methods for decomposing morphological templates which are analogous to decomposition methods used in the linear domain. Specifically, we define the notion of the rank of a morphological template which categorizes separable morphological templates as templates of rank one. We establish a necessary and sufficient condition for the decomposability of rank one templates into 3×3 templates. We then use the invariance of the template rank under certain transformations in order to develop template decomposition techniques for templates of rank two.

Index Terms—Morphology, convolution, structuring element, morphological template, template decomposition, template rank.

1 INTRODUCTION

Both linear convolution and morphological methods are widely used in image processing. One of the common characteristics among them is that they both require applying a template to a given image, pixel by pixel, to yield a new image. In the case of convolution, the template is usually called convolution window or mask; while in mathematical morphology, it is referred to as structuring element. Templates used in realizing linear convolutions are often referred to as *linear templates*. Templates can vary greatly in their weights, sizes, and shapes, depending on the specific applications.

Intuitively, the problem of template decomposition is that given a template $\mathbf{t}$, find a sequence of smaller templates $\mathbf{t}_1, \ldots, \mathbf{t}_n$ such that applying $\mathbf{t}$ to an image is equivalent to applying $\mathbf{t}_1, \ldots, \mathbf{t}_n$ sequentially to the image. In other words, $\mathbf{t}$ can be algebraically expressed in terms of $\mathbf{t}_1, \ldots, \mathbf{t}_n$.

One purpose of template decomposition is to fit the support of the template (i.e., the convolution kernel) optimally into an existing machine constrained by its hardware configuration. For example, ERIM's CytoComputer [1] cannot deal with templates of size larger than 3×3 on each pipeline stage. Thus, a large template, intended for image processing on a CytoComputer, has to be decomposed into a sequence of 3×3 or smaller templates.

A more important motivation for template decomposition is to speed up template operations. For large convolution masks, the computation cost resulting from implementation can be prohibitive. However, in many instances, this cost can be significantly reduced by decomposing the masks or templates into a sequence of smaller templates. For instance, the linear convolution of an image with a gray-valued $n \times n$ template requires n^2 multiplications and $n^2 - 1$ additions to compute a new image pixel value; while the same convolution computed with an $1 \times n$ row template followed by an $n \times 1$ column template takes only $2n$ multiplications and $2(n - 1)$ additions for each new image pixel value. This cost saving may still hold for parallel architectures such as mesh connected array processors [2], where the cost is proportional to the size of the template.

The problem of decomposing *morphological templates* has been investigated by a host of researchers. Zhuang and Haralick [3] gave a heuristic algorithm based on tree search that can find an optimal two-point decomposition of a morphological template if such a decomposition exits. A two-point decomposition consists of a sequence of templates each consisting of at most two points. A two-point decomposition may be best suited for parallel architectures with a limited number of local connections since each two-point template can be applied to an entire image in a multiply-shift-accumulate cycle [2]. Xu [4] has developed an algorithm, using chain code information, for the decomposition of convex morphological templates for two-point system configurations. Again using chain-code information, Park and Chin [5] provide an optimal decomposition of convex morphological templates for four-connected meshes. However, all the above decomposition methods work only on binary morphological templates and do not extend to gray-scale morphological templates.

A very successful general theory for the decomposition

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of templates, in both the linear and morphological domain, evolved from the theory of image algebra [6], [7], [8], [9], [10] which provides an algebraic foundation for image processing and computer vision tasks. In this setting, Ritter and Gader [11], [9] presented efficient methods for decomposing discrete Fourier transform templates. Zhu and Ritter [12] employ the general matrix product to provide novel computational methods for computing the fast Fourier transform, the fast Walsh transform, the generalized fast Walsh transform, as well as a fast wavelet transform.

In image algebra, template decomposition problems, for both linear and morphological template operations, can be reformulated in terms of corresponding matrix or polynomial factorization. Manseur and Wilson [13] used matrix as well as polynomial factorization techniques to decompose two-dimensional linear templates of size $m \times n$ into sums and products of 3×3 templates. Li [14] was the first to investigate polynomial factorization methods for morphological templates. He provides a uniform representation of morphological templates in terms of polynomials, thus reducing the problem of decomposing a morphological template to the problem of factoring the corresponding polynomials. His approach provides for the decomposition of one-dimensional morphological templates into factors of two-point templates. Crosby [15] extends Li's method to two-dimensional morphological templates.

Davidson [16] proved that any morphological template has a weak local decomposition for mesh-connected array processors. Davidson s existence theorem provides a theoretical foundation for morphological template decomposition, yet the algorithm conceived in its constructive proof is not very efficient. Takriti and Gader formulate the general problem of template decomposition as optimization problems [17], [18]. Sussner, Pardalos, and Ritter [19] use a similar approach to solve the even more general problem of morphological template approximation. However, since these problems are inherently NP-complete, researchers try to exploit the special structure of certain morphological templates in order to find decomposition algorithms. For example, Li and Ritter [20] provide very simple matrix techniques for decomposing binary as well as gray-scale linear and morphological convex templates. A separable template is a template that can be expressed in terms of two one-dimensional templates consisting of a row and a column template. Gader [21] uses matrix methods for decomposing any gray-scale morphological template into a sum of a separable template and a totally nonseparable template. If the original template is separable, then Gader s decomposition yields a separable decomposition. If the original template is not separable, then his method yields the closest separable template to the original in the mean square sense.

Separable templates are particularly easy to decompose and the decomposition of separable templates into a product of vertical and horizontal strip templates can be used as a first step for the decomposition into a form which matches the neighborhood configuration of a particular parallel architecture. In the linear case, separable templates are also called rank one templates since their corresponding matrices are rank one matrices. O Leary [22] showed that any linear template of rank one can be factored exactly into a product of 3×3 linear templates. Templates of higher rank are usually not as efficiently decomposable. However, the rank of a template determines upper bounds of worst-case scenarios. For example, a linear template of rank two always decomposes into a sum of two separable templates.

In the linear domain, the notion of template rank stems from the well known concept of matrix rank in linear algebra. The purpose of this paper is to develop the notion of a morphological matrix rank similar to the linear matrix rank. By way of bijection, matrices correspond to certain rectangular templates. In analogy to the linear case, we define the rank of a morphological template as the rank of the corresponding matrix. We demonstrate that this notion allows for an elegant and concise formulation of some new results concerning the decomposition of gray-scale morphological templates into separable morphological templates.

The paper is organized as follows. In Section 2, we introduce the image algebra notation used throughout this paper and in most of the aforementioned algebraic template decomposition methods. In Section 3, we develop the notions of linear dependence, linear independence and rank pertinent to morphological image processing. In Section 4, we establish general theorems for the separability of matrices in the morphological domain. Finally, in Section 5, we apply the result of the previous sections and establish decomposition criteria, methods, and algorithms for the decomposition of gray-scale morphological templates. Proofs of theorems are given in [23] so as not to obscure the main ideas and results of this paper.

2 Some Image Algebra Background

Image algebra is a *heterogeneous* or *many-valued* algebra in the sense of Birkhoff and Lipson [24], [6], with multiple sets of operands and operators. In a broad sense, image algebra is a mathematical theory concerned with the transformation and analysis of images. Although much emphasis is focused on the analysis and transformation of digital images, the main goal is the establishment of a comprehensive and unifying theory of image transformations, image analysis, and image understanding in the discrete as well as the continuous domain [6], [8], [7]. In this paper, however, we restrict our attention only to the notations and operations that are necessary for establishing the results mentioned in the introduction. Hence, our focus is on morphological image algebra operations.

Henceforth, let $\mathbf{X}$ be a subset of the digital plane $\mathbb{Z}^2 = \{(i, j) : i, j \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of integers. For any set $\mathbb{F}$, we denote the set of all functions from $\mathbf{X}$ into $\mathbb{F}$ by $\mathbb{F}^X$. We use the symbols $\vee$ and $\wedge$ to denote the binary operations of maximum and minimum, respectively.

2.1 Images and Templates

From the image algebra viewpoint, images are considered to be functions and templates are viewed as functions whose values are images. In particular, an $\mathbb{F}$-valued image $\mathbf{a}$ over the point set $\mathbf{X}$ is a function $\mathbf{a} : \mathbf{X} \to \mathbb{F}$ $\left(\text{i.e.}, \mathbf{a} \in \mathbb{F}^{\mathbf{X}}\right)$, while an $\mathbb{F}$-valued template $\mathbf{t}$ on $\mathbf{X}$ is a function

$\mathbf{t}: \mathbf{X} \to \mathbb{F}^{\mathbf{X}} \left(\text{i.e.}, \mathbf{t} \in \left(\mathbb{F}^{\mathbf{X}}\right)^{\mathbf{X}} \right)$. For notational convenience, we define $\mathbf{t_y}$ as $\mathbf{t(y)}$ for all $\mathbf{y} \in \mathbf{X}$. Note that the *image* $\mathbf{t_y}$ has representation

$$\mathbf{t_y} = \left\{ \left(\mathbf{x}, \mathbf{t_y}(\mathbf{x})\right) : \mathbf{x} \in \mathbf{X} \right\} \tag{1}$$

where the pixel values $\mathbf{t_y}(\mathbf{x})$ at location $\mathbf{x}$ of this image are called *template weights* at point $\mathbf{y}$.

Since we are concerned with optimizing morphological convolutions, the set $\mathbb{F}$ of interest will be the real numbers with the symbol $-\infty$ appended. More precisely, $\mathbb{F} = \mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ denotes the set of real numbers. The algebraic system associated with $\mathbb{R}_{-\infty}$ will be the semi-lattice ordered semi-group $\left(\mathbb{R}_{-\infty}, \vee, +\right)$ with the extended arithmetic and logic operations defined as follows:

$$\begin{aligned} a + (-\infty) &= (-\infty) + a = -\infty \quad \forall a \in \mathbb{R}_{-\infty} \\ a \vee (-\infty) &= (-\infty) \vee a = a \qquad \forall a \in \mathbb{R}_{-\infty} \end{aligned} \tag{2}$$

Note that the element $-\infty$ acts as a null element in the system $\left(\mathbb{R}_{-\infty}, \vee, +\right)$ if we view the operation + as multiplication and the operation $\vee$ as addition. The *dual* of this system is the semi-lattice ordered semi-group $\left(\mathbb{R}_{+\infty}, \wedge, +\right)$. The algebraic system $\left(\mathbb{R}_{-\infty}, \vee, +\right)$ provides the mathematical environment for the morphological operation of gray scale dilation, while $\left(\mathbb{R}_{+\infty}, \wedge, +\right)$ provides the environment for the dual operation of gray scale erosion.

Our focus will be on *translation invariant* $\mathbb{R}_{-\infty}$-valued templates over $\mathbf{X}$ since gray-scale structuring elements can be realized by these templates. A template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ is called *translation invariant* if and only if

$$\mathbf{t_{y+z}}(\mathbf{x} + \mathbf{z}) = \mathbf{t_y}(\mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X} = \mathbb{Z}^2 \tag{3}$$

whenever $\mathbf{y} + \mathbf{z}$ and $\mathbf{x} + \mathbf{z}$ are elements of $\mathbf{X}$. The *support* of a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ at a point $\mathbf{y}$ is denoted by $S(\mathbf{t_y})$ and defined as follows:

$$S\left(\mathbf{t_y}\right) = \left\{ \mathbf{x} \in \mathbf{X} : \mathbf{t_y}(\mathbf{x}) \neq -\infty \right\} \tag{4}$$

A translation invariant template $\mathbf{t}$ is called *rectangular*, if $S(\mathbf{t_y})$ forms a rectangular discrete array.

EXAMPLE. Let $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ be the translation invariant template which is determined at each point $\mathbf{y} \in \mathbf{X}$ by the following function values of $\mathbf{x} \in \mathbf{X}$:

$$\mathbf{r_y}(\mathbf{x}) = \begin{cases} 5 + 3 \cdot l & \text{if } \mathbf{x} = \mathbf{y} + (0, -1) + l \cdot (1, 0) \; \textit{for some } l \in \{-1, 0, 1\} \\ 7 + 3 \cdot l & \text{if } \mathbf{x} = \mathbf{y} + l \cdot (1, 0) \quad \textit{for some } l \in \{-1, 0, 1\} \\ 3 \cdot l & \text{if } \mathbf{x} = \mathbf{y} + (0, 1) + l \cdot (1, 0) \; \textit{for some } l \in \{-1, 0, 1\} \\ -\infty & \text{else} \end{cases} \tag{5}$$

If $\mathbf{y} = (x, y)$, we can visualize the rectangular template $\mathbf{r}$ as shown in Fig. 1.

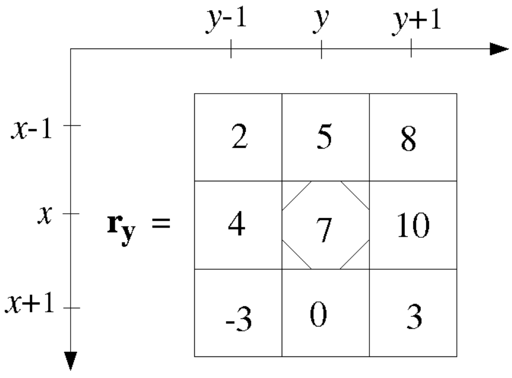

Fig. 1. The support of the template $\mathbf{r}$ at point $\mathbf{y}$. The hashed cell indicates the location of the target point $\mathbf{y} = (x, y)$.

2.2 Basic Operations

The basic operations of addition and maximum on $\mathbb{R}_{-\infty}$ induce pixelwise operations on $\mathbb{R}_{-\infty}$-valued images and templates. For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$ and any $\mathbf{s}, \mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$, we set

$$\begin{aligned} (\mathbf{a} + \mathbf{b})(\mathbf{x}) &= \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbf{X} \\ (\mathbf{a} \vee \mathbf{b})(\mathbf{x}) &= \mathbf{a}(\mathbf{x}) \vee \mathbf{b}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbf{X} \\ (\mathbf{s} + \mathbf{t})_y &= \mathbf{s_y} + \mathbf{t_y}, \quad \forall \mathbf{y} \in \mathbf{X} \\ (\mathbf{s} \vee \mathbf{t})_y &= \mathbf{s_y} \vee \mathbf{t_y}, \quad \forall \mathbf{y} \in \mathbf{X} \end{aligned} \tag{6}$$

If $\mathbf{c} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$ denotes the constant image

$$\left\{ \left(\mathbf{x}, \mathbf{c}(\mathbf{x})\right) : \mathbf{c}(\mathbf{x}) = \mathbf{c}, \forall \mathbf{x} \in \mathbf{X} \right\}$$

for some $c \in \mathbb{R}_{-\infty}$, then scalar operations on images and templates can be obtained by defining

$$\begin{aligned} c + \mathbf{a} &= \mathbf{a} + c = \mathbf{a} + \mathbf{c} \\ c \vee \mathbf{a} &= \mathbf{a} \vee c = \mathbf{a} \vee \mathbf{c} \\ (c + \mathbf{t})_{\mathbf{y}} &= (\mathbf{t} + c)_{\mathbf{y}} = \mathbf{t_y} + \mathbf{c}, \forall \mathbf{y} \in \mathbf{X} \\ (c \vee \mathbf{t})_{\mathbf{y}} &= (\mathbf{t} \vee c)_{\mathbf{y}} = \mathbf{t_y} \vee \mathbf{c}, \forall \mathbf{y} \in \mathbf{X} \end{aligned} \tag{7}$$

2.3 Additive Maximum Operations

Forming the *additive maximum* ($*$) of an image $\mathbf{a} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$ and a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ results in the image $\mathbf{a} * \mathbf{t} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$, which is determined by the following function values.

$$(\mathbf{a} * \mathbf{t})(\mathbf{y}) = \bigvee_{\mathbf{x} \in \mathbf{X}} \mathbf{a}(\mathbf{x}) + \mathbf{t_y}(\mathbf{x}) \tag{8}$$

Clearly, each template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ defines a function

$$\begin{aligned} f_{\mathbf{t}} : \mathbb{R}_{-\infty}^{\mathbf{X}} &\to \mathbb{R}_{-\infty}^{\mathbf{X}} \\ \mathbf{a} &\mapsto \mathbf{a} * \mathbf{t} \end{aligned} \tag{9}$$

The *additive maximum* of a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and a template $\mathbf{s} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ is defined as the template $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ which determines $f_{\mathbf{s}} \circ f_{\mathbf{t}}$, the composition of $f_{\mathbf{t}}$ followed by $f_{\mathbf{s}}$. Specifically,

$$(\mathbf{s} * \mathbf{t})_{\mathbf{y}}(\mathbf{z}) = \bigvee_{\mathbf{x} \in \mathbf{X}} \left(\mathbf{t}_{\mathbf{x}}(\mathbf{z}) + \mathbf{s}_{\mathbf{y}}(\mathbf{x})\right) \quad \forall \mathbf{y}, \mathbf{z} \in \mathbf{X} \tag{10}$$

These relationships induce the associative and distributive laws given later. Note that for any constant $c \in \mathbb{R}_{-\infty}$

$$(\mathbf{s} * \mathbf{t}) + c = \mathbf{s} * (\mathbf{t} + c) = (\mathbf{s} + c) * \mathbf{t} \tag{11}$$

EXAMPLE. The following column templates $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ satisfy $\mathbf{r} = \mathbf{s} * \mathbf{t}$.

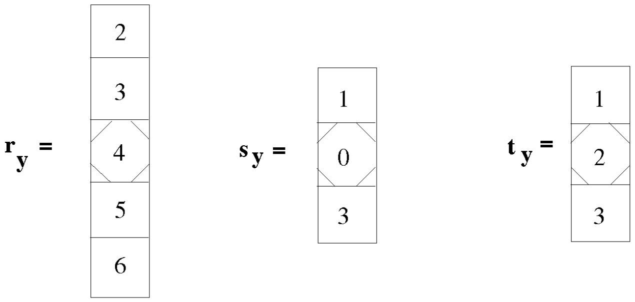

Fig. 2. The template **r** constitutes the additive maximum of the templates **s** and the template **t**.

2.4 Some Properties of Image and Template Operations

The following associative and distributive laws hold for an arbitrary image $\mathbf{a} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$ and arbitrary templates $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and $\mathbf{s} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$:

$$\begin{aligned} \mathbf{a} * (\mathbf{s} * \mathbf{t}) &= (\mathbf{a} * \mathbf{s}) * \mathbf{t} \\ \mathbf{a} * (\mathbf{s} \vee \mathbf{t}) &= (\mathbf{a} * \mathbf{s}) \vee (\mathbf{a} * \mathbf{t}) \end{aligned} \tag{12}$$

These results establish the importance of template decomposition.

2.5 Strong Decompositions of Templates

A sequence of templates $\left(\mathbf{t}^1, \ldots, \mathbf{t}^k\right)$ in $\left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ is called a (*strong*) *decomposition* (with respect to the operation "$*$") of a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ if $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ can be written in the form

$$\mathbf{t} = \mathbf{t}^1 * \mathbf{t}^2 * \ldots * \mathbf{t}^k \tag{13}$$

In the special case where k = 2, we speak of a *separable* template if the support of $\mathbf{t}^1$ is a one dimensional vertical array and the support of $\mathbf{t}^2$ is a one dimensional horizontal array.

EXAMPLE. The template $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ given in Fig. 1 represents a separable template since this template decomposes into a vertical strip template $\mathbf{s} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and a horizontal strip template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$.

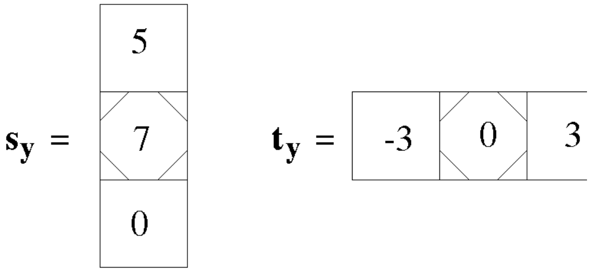

Fig. 3. Pictorial representation of a column template **s** and a row template **t**.

2.6 Weak Decompositions of Templates

A sequence of templates $\left(\mathbf{t}^1, \ldots, \mathbf{t}^{k_n}\right)$ in $\left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ together with a strictly increasing sequence of natural numbers $k_1, \ldots, k_n$ is called a (*weak*) *decomposition* (with respect to the operation "$*$") of a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ if the template $\mathbf{t}$ can be represented as follows:

$$\begin{aligned} \mathbf{t} = &\left(\mathbf{t}^1 * \ldots * \mathbf{t}^{k_1}\right) \vee \left(\mathbf{t}^{k_1+1} * \ldots * \mathbf{t}^{k_2}\right) \vee \ldots \\ &\vee \left(\mathbf{t}^{k_{n-1}+1} * \ldots * \mathbf{t}^{k_n}\right) \end{aligned} \tag{14}$$

We say $\left(\mathbf{s}^1, \ldots, \mathbf{s}^k\right)$ is a weak decomposition of a rectangular template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ into separable templates if each $\mathbf{s}^i$, where $i = 1, \ldots, k$, is separable and $\mathbf{t} = \mathbf{s}^1 \vee \ldots \vee \mathbf{s}^k$.

2.7 Correspondence Between Rectangular Templates and Matrices

Note that there is a natural bijection ϕ from the space of all $m \times n$ matrices over $\mathbb{R}_{-\infty}$ into the space of all rectangular $m \times n$ templates in $\left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$.

Let $\mathbf{y} = (x, y) \in \mathbf{X}$ be arbitrary and $\mathbf{x} = \left(x_1, x_2\right) \in \mathbf{X}$ be such that

$$\begin{aligned} x_1 &= x - \min\left\{\frac{m}{2}, \frac{m-1}{2}\right\}, \\ x_2 &= y - \min\left\{\frac{n}{2}, \frac{n-1}{2}\right\} \end{aligned} \tag{15}$$

The image of a matrix $\mathbf{A} \in \left(\mathbb{R}_{-\infty}\right)^{m \times n}$ under ϕ is defined to be the template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ which satisfies

$$\begin{aligned} &\mathbf{t}_{\mathbf{y}}\left(x_1 + i - 1, x_2 + j - 1\right) = a_{ij} \; \forall i = 1, \ldots, m, \; \forall j = 1, \ldots, n \\ &\mathbf{t}_{\mathbf{y}}\left(y_1, y_2\right) = -\infty \; \forall y_1, y_2 \notin \left[x_1, x_1 + m - 1\right] \times \left[x_2, x_2 + n - 1\right] \end{aligned} \tag{16}$$

Henceforth, we restrict our attention to rectangular templates whose target pixel is centered, i.e., rectangular templates of the above form.

The theory of minimax algebra [25] examines the algebraic structures arising from the lattice operations "maximum," "minimum," and "addition," including the space of all matrices over $\mathbb{R}_{-\infty}$ together with the operation additive maximum. The natural correspondence between rectangular templates in $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and matrices over $\mathbb{R}_{-\infty}$ allows us to use a minimax algebra approach in order to study the weak decomposability of rectangular templates into separable templates.

EXAMPLE. Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ be the matrix and $\mathbf{u}$, $\mathbf{v}$ the vectors given below.

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 8 \\ 4 & 7 & 10 \\ -3 & 0 & 3 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \mathbf{v} = (-3, 0, 3) \tag{17}$$

The function ϕ maps $\mathbf{A}$ to the square template $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ in Fig. 1, and it maps the column vector $\mathbf{u}$ to the column template $\mathbf{s} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and the row vector $\mathbf{v}$ to the row template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ in Fig. 3.

3 RANKS OF MATRICES IN MINIMAX ALGEBRA

In this section, we develop a new notion of matrix rank within the mathematical framework of minimax algebra. We relate this concept of matrix rank to the one given by Cuninghame-Green [25] and derive the notion of the rank of a morphological template.

3.1 Algebraic Structures and Operations in Minimax Algebra

The mathematical theory of minimax algebra deals with algebraic structures such as bands, belts, and blogs. For example, $\mathbb{R}_{-\infty}$ together with the operations of maximum ("$\vee$") and addition forms a belt. Cuninghame-Green defines the matrix rank for matrices over certain subsets of the blog $\mathbb{R}_{\pm\infty}$. For our purposes it suffices to consider $\mathbb{R}$, the finite elements of $\mathbb{R}_{\pm\infty}$.

Operations such as the maximum ("$\vee$"), the minimum ("$\wedge$"), and the addition on $\mathbb{R}$ induce entrywise operations on $\mathbb{R}^{m\times n}$, the set of all $m \times n$ matrices over $\mathbb{R}$. Minimax algebra also defines compound operations such as "$*$"—pronounced "additive maximum"—from $\mathbb{R}^{m\times k} \times \mathbb{R}^{k\times n}$ into $\mathbb{R}^{m\times n}$, an operation similar to the regular matrix product known from linear algebra. (An obvious dual of this operation is provided by the "additive minimum" operation.) Given matrices $\mathbf{A} \in \mathbb{R}^{m\times k}$ and $\mathbf{B} \in \mathbb{R}^{k\times n}$, the additive maximum $\mathbf{C} = \mathbf{A} * \mathbf{B} \in \mathbb{R}^{m\times n}$ is determined by

$$c_{ij} = \bigvee_{l=1}^{k}\left(a_{ik} + b_{kj}\right) \ \forall i = 1, \ldots, m, \ \forall j = 1, \ldots, n \tag{18}$$

If $\mathbf{A}$ is a matrix in $\mathbb{R}^{m\times n}$ and if $\mathbf{u}^i$ are column vectors in $\mathbb{R}^{m\times 1}$ and $\mathbf{v}^i$ are row vectors in $\mathbb{R}^{1\times n}$ for $i = 1, \ldots, k$, then the following equivalence holds for the corresponding rectangular template $\phi(\mathbf{A})$, the vertical strip templates $\phi\left(\mathbf{u}^i\right)$, and the horizontal strip templates $\phi\left(\mathbf{v}^i\right)$

$$\mathbf{A} = \bigvee_{i=1}^{k}\left(\mathbf{u}^i * \mathbf{v}^i\right) \Leftrightarrow \phi(\mathbf{A}) = \bigvee_{i=1}^{k}\left(\phi\left(\mathbf{u}^i\right) * \phi\left(\mathbf{v}^i\right)\right) \tag{19}$$

3.2 Linear Dependence of Vectors

A vector $\mathbf{v} \in \mathbb{R}^n$ is said to be *linearly dependent* on the vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \mathbb{R}^n$ if and only if there exist scalars $c_i \in \mathbb{R}$, $i = 1, \ldots, k$, such that

$$\mathbf{v} = \bigvee_{i=1}^{k}\left(c_i + \mathbf{v}^i\right) \tag{20}$$

Otherwise, the vector $\mathbf{v} \in \mathbb{R}^n$ is called *linearly independent* from the vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \mathbb{R}^n$. The vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \mathbb{R}^n$ are *linearly independent* if each one of them is linearly independent from the others.

EXAMPLE. Consider the following elements of $\mathbb{R}^3$:

$$\mathbf{v} = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}, \mathbf{v}^1 = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} -6 \\ 1 \\ 9 \end{pmatrix} \tag{21}$$

Since $\mathbf{v} = \left[2 + \mathbf{v}^1\right] \vee \left[(-2) + \mathbf{v}^2\right]$, the vector $\mathbf{v}$ is linearly dependent on $\mathbf{v}^1$ and $\mathbf{v}^2$.

3.3 Strong Linear Independence

Vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \mathbb{R}^n$ are called *strongly linearly independent* (SLI) if and only if there exists a vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v}$ has a *unique* representation

$$\mathbf{v} = \bigvee_{i=1}^{k}\left(c_i + \mathbf{v}_i\right) \tag{22}$$

Since this definition does not provide a suitable criterion for testing a collection of vectors in $\mathbb{R}^n$ for strong linear independence, we choose to provide an alternative equivalent definition based on the following theorem.

THEOREM 1. *Vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \mathbb{R}^n$ are SLI if and only if the following inequalities hold.*

$$\bigvee_{\substack{i=1 \\ i\neq j}}^{k} \tilde{\mathbf{v}}^i < \bigvee_{i=1}^{k} \tilde{\mathbf{v}}^i \quad \forall j = 1, \ldots, k \tag{23}$$

where $\tilde{\mathbf{v}} = \mathbf{v} - \bigvee_{j=1}^{n} v_j$ *for any vector*

$$\mathbf{v} = \left(v_1, \ldots, v_n\right) \in \mathbb{R}^n$$

COROLLARY. *There are k vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k \in \mathbb{R}^n$ which are SLI if and only if $k \in \{1, \ldots, n\}$.*

3.4 Rank of a Matrix

Cuninghame-Green defines the *rank* of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ as the maximal number of SLI row vectors or, equivalently, the maximum number of SLI column vectors. The rank of a finite matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is less than or equal to min $\{m, n\}$.

3.5 Remarks on the Rank of a Matrix in Minimax Algebra

The notion of (regular) linear independence is not suited to define a rank in minimax algebra because certain dimensional abnormalities would occur. For example, it is possible to find k linearly independent vectors in $\mathbb{R}^n$ for any number $k \in \mathbb{N}$. The notion of strong linear independence gives rise to a satisfactory theory of rank and dimension (although certain equivalences known from linear algebra do not hold).

3.6 The Separable Rank of a Matrix

The *separable rank* of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is denoted by $\text{rank}_{\text{sep}}(\mathbf{A})$ and defined as the minimal number r of column vectors $\mathbf{u}^1, \ldots, \mathbf{u}^r \in \mathbb{R}^{m\times 1}$ and row vectors $\mathbf{v}^1, \ldots\ \mathbf{v}^r \in \mathbb{R}^{1\times n}$ which permit a representation of $\mathbf{A}$ in the following form:

$$\mathbf{A} = \bigvee_{i=1}^{r}\left(\mathbf{u}^i * \mathbf{v}^i\right) \tag{24}$$

A representation of this form is called a (*weak*) *separable decomposition* of $\mathbf{A}$. We say $\mathbf{A}$ is a *separable matrix* (with respect to the operation $*$) if $\text{rank}_{\text{sep}}(\mathbf{A}) = 1$.

3.7 The Rank of a Rectangular Template

If $\mathbf{t} = \phi(\mathbf{A})$ for some real valued matrix $\mathbf{A}$, then we define the *rank of the template* $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ as the separable rank of $\mathbf{A}$.

Our interest in the rank of a morphological template is motivated by the problem of morphological template decomposition since the rank of a morphological template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ represents the minimal number of separable templates whose maximum is $\mathbf{t}$ or, equivalently, the minimal number r of column templates $\mathbf{r}^i \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and row templates $\mathbf{s}^i \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ such that $\mathbf{t} = \bigvee_{i=1}^{r}\left(\mathbf{r}^i * \mathbf{s}^i\right)$.

4 General Results About the Separable Matrix Rank

In this section, we derive some theorems concerning the separable rank of matrices which translate directly into results about the rank of rectangular templates. These theorems greatly simplify the proofs of the decomposition results which we will present in the next section.

THEOREM 2. *If a matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *has a representation* $\mathbf{A} = \bigvee_{l=1}^{k}\left(\mathbf{u}^l * \mathbf{v}^l\right)$ *in terms of column vectors* $\mathbf{u}^l \in \mathbb{R}^{m\times 1}$ *and row vectors* $\mathbf{v}^l \in \mathbb{R}^{1\times n}$, *where* $l = 1, \ldots, k$, *then* $\mathbf{A}$ *can be expressed in the following form*:

$$A = \bigvee_{l=1}^{k}\left(\mathbf{w}^l * \mathbf{v}^l\right) \tag{25}$$

where $\mathbf{w}^l \in \mathbb{R}^{m\times 1}$ *is given by*

$$w_i^l = \bigwedge_{j=1}^{n}\left(a_{ij} - v_j^l\right)\ \left(i = 1, \ldots, m\right) \tag{26}$$

REMARK. Theorem 2 implies that, for any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ of separable rank k, it suffices to know the row vectors $\mathbf{v}^l \in \mathbb{R}^{1\times n}$, $l = 1, \ldots, k$ which permit a weak decomposition of $\mathbf{A}$ into k separable matrices in order to determine a representation of $\mathbf{A}$ in the form:

$$\mathbf{A} = \bigvee_{l=1}^{k}\left(\mathbf{w}^l * \mathbf{v}^l\right)$$
$$\mathbf{w}^l \in \mathbb{R}^{m\times 1}\quad \forall l = 1, \ldots, k \tag{27}$$

Like most of the theorems established in this paper, Theorem 2 has an obvious dual in terms of column vectors which we choose to omit.

We now are going to introduce certain transforms which preserve the separable matrix rank. These transforms are suited to simplify the task of determining the separable rank of a given matrix.

4.1 Column Permutations of Matrices

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and ρ be a permutation of $\{1, \ldots, n\}$. The *associated column permuted matrix* $\rho_c(\mathbf{A})$ of $\mathbf{A}$ with respect to ρ is defined as follows:

$$\rho_c(\mathbf{A}) = \begin{pmatrix} a_{1\rho(1)} & a_{1\rho(2)} & \cdots & a_{1\rho(n)} \\ a_{2\rho(1)} & a_{2\rho(2)} & \cdots & a_{2\rho(n)} \\ \vdots & \vdots & & \vdots \\ a_{m\rho(1)} & a_{m\rho(2)} & \cdots & a_{m\rho(n)} \end{pmatrix} \tag{28}$$

4.2 Row Permutations

If ρ is a permutation of $\{1, \ldots, m\}$, then we define the *associated row permuted matrix* $\rho_r(\mathbf{A})$ of $\mathbf{A} \in \mathbb{R}^{m\times n}$ with respect to ρ as follows:

$$\rho_r(\mathbf{A}) = \begin{pmatrix} a_{\rho(1)1} & a_{\rho(1)2} & \cdots & a_{\rho(1)n} \\ a_{\rho(2)1} & a_{\rho(2)2} & \cdots & a_{\rho(2)n} \\ \vdots & \vdots & & \vdots \\ a_{\rho(m)1} & a_{\rho(m)2} & \cdots & a_{\rho(m)n} \end{pmatrix} \tag{29}$$

The *multiplication of a matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *by a scalar* $c \in \mathbb{R}$ is defined as usual. In this case, $-\mathbf{A}$ stands for $(-1)\cdot\mathbf{A}$.

THEOREM 3. *The following transformations preserve the separable rank of a matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$:

- *column and row permutations*;
- *additions of separable matrices*;
- *scalar multiplications*.

REMARK. Column and row permutations as well as additions of separable matrices also preserve the rank of a matrix, as defined by Cuninghame-Green. This invariance property follows directly from the definition of this matrix rank as the minimal number of SLI row vectors or column vectors.

THEOREM 4. *The separable rank of a finite matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *is bounded from below by the rank of* $\mathbf{A}$ *and bounded from above by the minimal number* l *of linearly independent row vectors or column vectors of* $\mathbf{A}$.

5 WEAK SEPARABLE DECOMPOSITIONS

At this point, we are finally ready to tackle the problem of determining weak decompositions of matrices in view of their separable ranks. The reader should bear in mind the consequences for the corresponding rectangular templates.

For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we use the notation $\mathbf{a}(i)$, where $i = 1, \ldots, m$ to denote the ith row vector of $\mathbf{A}$ and we use the notation $\mathbf{a}[j]$, where $j = 1, \ldots, n$, to denote the jth column vector of $\mathbf{A}$.

THEOREM 5 [20]. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *be a separable matrix and* $\{\mathbf{a}(i) : i = 1, \ldots, m\}$ *be the collection of row vectors of* $\mathbf{A}$. *For each arbitrary row vector* $\mathbf{a}(i_0)$, *where* $1 \le i_0 \le m$, *there exist scalars* $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, m$, *such that the following equations are satisfied*:

$$\lambda_i + \mathbf{a}(i_0) = \mathbf{a}(i), \quad \forall i = 1, \ldots, m \tag{30}$$

In other words, given an arbitrary index $1 \le i_0 \le m$, *each row vector* $\mathbf{a}(i)$ *is linearly dependent on the* i_0th *row vector* $\mathbf{a}(i_0)$ *of* $\mathbf{A}$.

Clearly, Li and Ritter's theorem gives rise to the following straightforward algorithm which tests if a given matrix over $\mathbb{R}$ is separable. In the separable case, the algorithm computes a vector pair into which the given matrix can be decomposed.

ALGORITHM 1. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be given and let $\mathbf{a}(i)$ denote the ith row vector of $\mathbf{A}$. The algorithm proceeds as follows for all $i = 2, \ldots, m$:

STEP 1:

Form $c_i \in \mathbb{R}$, where $c_i = a_{i1} - a_{1,1}$.

STEP $2(j-1)$, $j = 2, \ldots, n$:

Subtract a_{1j} from a_{ij}.

STEP $2(j-1)+1$, $j = 2, \ldots, n$:

Compare c_i with $a_{ij} - a_{1j}$. If $c_i \ne a_{ij} - a_{1j}$, the matrix $\mathbf{A}$ is not separable and the algorithm stops. If step $2(n-1)+1$ has been successfully completed, then $\mathbf{A}$ is separable and $\mathbf{A}$ is given by $\mathbf{c} * \mathbf{a}(1)$.

Note that this algorithm only involves $(m-1)n$ subtractions and $(m-1)(n-1)$ comparisons. Hence, the number of operations adds up to $2(m-1)n - m + 1$, which implies that the algorithm has order $O(2mn)$.

REMARK. As mentioned earlier, the given image processing hardware often calls for the decomposition of a given template into 3×3 templates. The theorem below shows that, in the case of a separable square template, this problem reduces to the problem of decomposing a column template into 3×1 templates as well as decomposing a row template into 1×3 templates. Suppose that the original template is of size $(2n+1) \times (2n+1)$. Clearly, $4n^2 + 4n + 1$ operations per pixel are needed when applying this template to an image. If the template decomposes into n 3×3 templates, this number of operations reduces to $9n$. However, the simple strong decomposition of the original separable template into a row and a column template of length $2n+1$ is often more appropriate especially when using a sequential machine since only $4n+2$ operations per pixel are required when taking advantage of this decomposition.

THEOREM 6. *Let* $\mathbf{t}$ *be a square morphological template of rank* 1, *given by* $\mathbf{t} = \mathbf{r} * \mathbf{s}$, *where* $\mathbf{r}$ *is a column template and* $\mathbf{s}$ *is a row template. The template* $\mathbf{t}$ *is decomposable into* $m \times m$ *templates if and only if* $\mathbf{r}$ *is decomposable into* $m \times 1$ *templates and* $\mathbf{s}$ *is decomposable into* $1 \times m$ *templates.*

EXAMPLE. Let $\mathbf{A}$ be the real valued 5×5 matrix given below.

$$\mathbf{A} = \begin{pmatrix} 3 & 7 & 4 & 2 & 1 \\ 11 & 15 & 12 & 10 & 9 \\ 5 & 9 & 6 & 4 & 3 \\ 6 & 10 & 7 & 5 & 4 \\ 12 & 16 & 13 & 11 & 10 \end{pmatrix} = \mathbf{u} * \mathbf{v} \tag{31}$$

where

$$\mathbf{u} = \begin{pmatrix} 0 \\ 8 \\ 2 \\ 3 \\ 9 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} 3 & 7 & 4 & 2 & 1 \end{pmatrix} \tag{32}$$

The template $\mathbf{t} = \phi(A)$ is not decomposable into two 3×3 templates since the template $\mathbf{r} = \phi(\mathbf{u})$ is not decomposable into two 3×1 templates.

REMARK. Of course, Theorem 6 does not preclude the existence of templates $\mathbf{t}$ of rank ≥ 2 which are strongly decomposable into 3×3 templates.

EXAMPLE. The following template $\mathbf{t}$ of rank > 1 can be written as a $*$-product of two 3×3 templates $\mathbf{t}^1$ and $\mathbf{t}^2$. See Fig. 4 and Fig. 5.

COROLLARY. *Let* $\mathbf{t}$ *be a square morphological template of rank* k, *given by* $\mathbf{t} = (\mathbf{r}^1 * \mathbf{s}^1) \vee (\mathrm{r}^2 * \mathrm{s}^2) \vee \ldots \vee (\mathrm{r}^k * \mathbf{s}^k)$, *where* $\mathbf{r}^1, \ldots, \mathbf{r}^k$ *are column templates and* $\mathbf{s}^1, \ldots, \mathbf{s}^k$ *are row templates. If the templates* $\mathbf{r}^1, \ldots, \mathbf{r}^k$ *are decomposable into* $m \times 1$ *templates and the templates* $\mathbf{s}^1, \ldots, \mathbf{s}^k$ *are decomposable into* $1 \times m$ *templates, then* $\mathbf{t}$ *has representation* $\mathbf{t} = \mathbf{t}^1 \vee \ldots \vee \mathbf{t}^k$ *in terms of templates* $\mathbf{t}^1, \ldots, \mathbf{t}^k$ *which are strongly decomposable into* $m \times m$ *templates.*

LEMMA 1. *If a matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *has separable rank two, then there exits a transform* T—*consisting of only row permutations, column permutations, and additions of row or column vectors—as well as vectors* $\mathbf{u} \in \mathbb{R}^{m\times 1}$ *and* $\mathbf{v} \in \mathbb{R}^{1\times n}$ *such that* $T(\mathbf{A})$ *can be written in the following form*:

$$T(\mathbf{A}) = \mathbf{0}_{m\times n} \vee (\mathbf{u} * \mathbf{v}) \tag{33}$$

where

$$u_1 \le u_2 \le \ldots \le u_m \tag{34}$$

and $\mathbf{0}_{m\times n}$ *denotes the* $m \times n$ *zero matrix.*

$\mathbf{t_y} =$

3	4	5	4	5
7	8	9	10	9
6	5	8	7	7
4	6	5	12	11
4	3	10	9	7

Fig. 4. Example of a 5×5 template $\mathbf{t}$ of rank > 1 which is decomposable into 3×3 templates $\mathbf{t}^1$ and $\mathbf{t}^2$.

$\mathbf{t}^1_{\mathbf{y}} =$

3	4	5
0	-3	2
1	-1	7

$\mathbf{t}^2_{\mathbf{y}} =$

0	-2	0
4	5	4
3	2	0

Fig. 5. Templates $\mathbf{t}^1$ and $\mathbf{t}^2$ which satisfy $\mathbf{t} = \mathbf{t}^1 * \mathbf{t}^2$.

LEMMA 2. *Let T be a (separable) rank preserving transform as in Theorem 3 and* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *The transform T maps row vectors of* $\mathbf{A}$ *to row vectors of* $T(\mathbf{A})$, *and column vectors of* $\mathbf{A}$ *to column vectors of* $T(\mathbf{A})$.

THEOREM 7. *A matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *has separable rank two if and only there are two row vectors of* $\mathbf{A}$ *on which all other row vectors depend (linearly).*

A similar theorem does not hold for matrices of separable rank $k \geq 3$. This fact is expressed by Theorem 8.

THEOREM 8. *For every natural number* $k \geq 3$, *there are matrices over* $\mathbb{R}$ *which are weakly* $*$*-decomposable into a product of k vector pairs, but not all of whose row vectors are linearly dependent on a single k tuple of their row vectors.*

REMARK. By Theorem 6, a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ has separable rank two if and only if there exist two row vectors of $\mathbf{A}$—$\mathbf{a}(o)$, $\mathbf{a}(p)$ where $o, p \in \{1, \ldots, m\}$—which allow for a weak decomposition of $\mathbf{A}$. In this case, an application of Theorem 2 yields the following representation of $\mathbf{A}$:

$$A = \left[\mathbf{u} * \mathbf{a}(o)\right] \vee \left[\mathbf{v} * \mathbf{a}(p)\right] \tag{35}$$

where

$$\begin{aligned} &u, v \in \mathbb{R}^{m\times 1} \\ &u_i = \bigwedge_{j=1}^{m}\left(a_{ij} - a_{oj}\right) \ (i = 1, \ldots, m) \\ &v_i = \bigwedge_{j=1}^{m}\left(a_{ij} - a_{pj}\right) \ (i = 1, \ldots, m) \end{aligned} \tag{36}$$

Hence, in order to test an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ for weak decomposability into two vector pairs, it is enough to compare $\mathbf{A}$ with $(\mathbf{b}[i] * \mathbf{a}(i)) \vee (\mathbf{b}[j] * \mathbf{a}(j))$ for all indices $i, j \in \{1, \ldots, m\}$. Here $\mathbf{B} \in \mathbb{R}^{m\times m}$ is computed as follows:

$$\begin{aligned} &b_{ij} = \bigwedge_{s=1}^{n}\left(a_{is} - a_{js}\right) \\ &\forall i = 1, \ldots, m, \ \forall j = 1, \ldots, m \end{aligned} \tag{37}$$

ALGORITHM 2. Assume a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ needs to be decomposed into a weak $*$-product of two vector pairs if such a decomposition is possible. Considering the preceding remarks, we are able to give a polynomial time algorithm for solving this problem. For each step, we include the number of operations involved in square brackets.

STEP 1:
Compute $\mathbf{B} \in \mathbb{R}^{m\times m}$
[m^2n subtractions; $m^2(n-1)$ comparisons].

STEP 2:
Form $\mathbf{C}^i = \mathbf{b}[i] * \mathbf{a}(i)$ for all $i = 1, \ldots, m$
[$m(mn)$ additions].

STEP 3:
Form $\mathbf{C}^i \vee \mathbf{C}^j$ for all $i, j = 1, \ldots, m$ and compare the result with $\mathbf{A}$. If $\mathbf{C}^o \vee \mathbf{C}^p = \mathbf{A}$ for certain $o, p = 1, \ldots, m$, then the algorithm stops yielding the following result:

$$A = (\mathbf{b}[o] * \mathbf{a}(o)) \vee (\mathbf{b}[p] * \mathbf{a}(p)) \tag{38}$$

If $\mathbf{C}^i \vee \mathbf{C}^j < \mathbf{A}$ for all $i, j = 1, \ldots, m$, then $\mathbf{A}$ does not have a weak decomposition into two vector pairs.
[At most $2\binom{m}{2}(mn)$ comparisons].

This algorithm involves at most a total number of $m^2(3n - 1 + (m - 1)n)$ operations which amounts to order $O(m^3n)$.

EXAMPLE. Let us apply Algorithm 2 to the following matrix $\mathbf{A} \in \mathbb{R}^{4\times 5}$.

$$\mathbf{A} = \begin{pmatrix} 9 & 5 & 5 & 6 & 5 \\ 7 & 4 & 4 & 3 & 4 \\ 12 & 7 & 7 & 9 & 7 \\ 11 & 3 & 5 & 8 & 3 \end{pmatrix} \tag{39}$$

We compute matrices $\mathbf{B} \in \mathbb{R}^{4\times 4}$ and $\mathbf{C}^i \in \mathbb{R}^{4\times 5}$ for all $i = 1, \ldots, 4$.

$$\mathbf{B}=\begin{pmatrix}0&1&-3&-2\\-3&0&-6&-5\\2&3&0&1\\-2&-1&-4&0\end{pmatrix}$$

$$\mathbf{C}^1=\mathbf{b}[1]*\mathbf{a}(1)=\begin{pmatrix}9&5&5&6&5\\7&4&4&3&4\\12&7&7&9&7\\11&3&5&8&3\end{pmatrix}$$

$$\mathbf{C}^2=\mathbf{b}[2]*\mathbf{a}(2)=\begin{pmatrix}8&5&5&4&5\\7&4&4&3&4\\10&7&7&6&7\\6&3&3&2&3\end{pmatrix}$$

$$\mathbf{C}^3=\mathbf{b}[3]*\mathbf{a}(3)=\begin{pmatrix}9&4&4&6&4\\6&1&1&2&1\\12&7&7&9&7\\7&3&3&5&3\end{pmatrix}$$

$$\mathbf{C}^4=\mathbf{b}[4]*\mathbf{a}(4)=\begin{pmatrix}9&1&3&6&1\\6&-2&0&3&-2\\12&4&6&9&4\\11&3&5&8&3\end{pmatrix} \tag{40}$$

Comparing the matrices $\mathbf{C}^i \vee \mathbf{C}^j$ with $\mathbf{A}$ for all $i, j = 1, \ldots, m$ reveals that $\mathbf{A} = \mathbf{C}^2 \vee \mathbf{C}^4$. Thus,

$$A = (\mathbf{b}[2] * \mathbf{a}(2)) \vee (\mathbf{b}[4] * \mathbf{a}(4)) \tag{41}$$

EXAMPLE. Let $\mathbf{A} \in \mathbb{R}^{9\times 9}$ be the following matrix, which constitutes the maximum of a matrix in pyramid form and a matrix in paraboloid form.

$$A=\begin{pmatrix}-18&-14&-10&-6&-2&-6&-10&-14&-18\\-14&-9&-4&-1&2&-1&-4&-9&-14\\-10&-4&-1&4&6&4&1&-4&-10\\-6&-1&4&7&10&7&4&-1&-6\\-2&2&6&10&14&10&6&2&-2\\-6&-1&1&4&6&7&4&-1&-6\\-10&-4&1&4&6&4&1&-4&-10\\-14&-9&-4&-1&2&-1&-4&-9&-14\\-18&-14&-10&-6&-2&-6&-10&-14&-18\end{pmatrix} \tag{42}$$

Since matrices in paraboloid and in pyramid form are separable, Algorithm 2 should yield a weak decomposition of $\mathbf{A}$ in the form $[\mathbf{u} * \mathbf{a}(o)] \vee [\mathbf{v} * \mathbf{a}(p)]$ for some vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^9$ and some indices $o, p \in \{1, \ldots, 9\}$. Indeed, if $\mathbf{B} \in \mathbb{R}^{9\times 9}$ denotes the matrix computed by Algorithm 2,

$$\mathbf{A} = [\mathbf{b}[1] * \mathbf{a}(1)] \vee [\mathbf{b}[3] * \mathbf{a}(3)] \tag{43}$$

where

$$\mathbf{b}[1]=\begin{pmatrix}0\\4\\8\\12\\16\\12\\8\\4\\0\end{pmatrix},\quad \mathbf{b}[3]=\begin{pmatrix}-11\\-5\\0\\3\\10\\3\\0\\-5\\-11\end{pmatrix}$$

$$\begin{aligned}\mathbf{a}(1)&=\begin{pmatrix}-18&-14&-10&-6&-2&-6&-10&-14&-18\end{pmatrix}\\ \mathbf{a}(3)&=\begin{pmatrix}-10&-4&1&4&6&4&1&-4&-9\end{pmatrix}\end{aligned} \tag{44}$$

This weak decomposition of $\mathbf{A}$ can be used to further decompose the square templates $\phi(\mathbf{b}[1] * \mathbf{a}(1))$ and $\phi(\mathbf{b}[3] * \mathbf{a}(3))$ into 3×3 templates. By Theorem 6, the templates $\phi(\mathbf{b}[1] * \mathbf{a}(1))$ and $\phi(\mathbf{b}[3] * \mathbf{a}(3))$ are decomposable into 3×3 templates if and only if the column templates $\phi(\mathbf{b}[1])$ and $\phi(\mathbf{b}[3])$ are decomposable into 3×1 templates and the row templates $\phi(\mathbf{a}[1])$ and $\phi(\mathbf{a}[3])$ are decomposable into 1×3 templates. It is fairly easy to choose 3×1 templates $\mathbf{r}^i \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and 1×3 templates $\mathbf{s}^i \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$, $i = 1, \ldots, 4$, such that $\phi(\mathbf{b}[1]) = ((\mathbf{r}^1 * \mathbf{r}^2) * \mathbf{r}^3) * \mathbf{r}^4$ and $\phi(\mathbf{a}[1]) = ((\mathbf{s}^1 * \mathbf{s}^2) * \mathbf{s}^3) * \mathbf{s}^4$. For more complicated examples, we recommend using one of the integer programming approaches suggested in [17], [18], [19]. See Fig. 6. Hence, we obtain a representation of $\phi(\mathbf{b}[1] * \mathbf{a}(1))$ in the form of (45).

$$\phi(\mathbf{b}[1] * \mathbf{a}(1)) = [((\mathbf{r}^1 * \mathbf{r}^2) * \mathbf{r}^3) * \mathbf{r}^4 * [((\mathbf{s}^1 * \mathbf{s}^2) * \mathbf{s}^3) * \mathbf{s}^4]. \tag{45}$$

By rearranging the templates $\mathbf{r}^i$ and $\mathbf{s}^i$ for $i = 1, \ldots, 4$, we can achieve a decomposition of $\phi(\mathbf{b}[1] * \mathbf{a}(1))$ into four 3×3 templates $\mathbf{t}^i = \mathbf{r}^i * \mathbf{s}^i$. See Fig. 7.

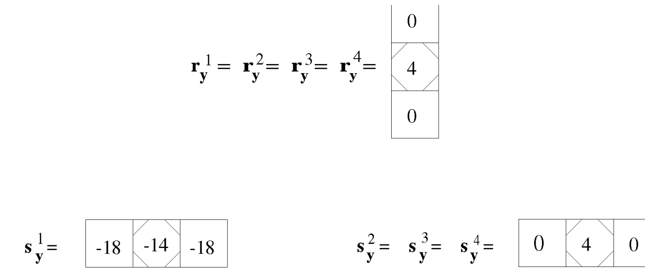

Fig. 6. Templates of size 3×1 and size 1×3 providing a decomposition of the template $\phi(\mathbf{b}[1] * \mathbf{a}(1))$.

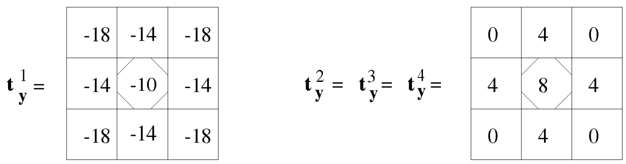

Fig. 7. The 3×3 templates, providing a decomposition of the template $\phi(\mathbf{b}[1] * \mathbf{a}(1))$.

In a similar fashion, we are able to decompose the template $\phi(\mathbf{b}[3] * \mathbf{a}(3))$ into four 3×3 templates.

REMARK. The methods for decomposing rectangular morphological templates presented in this paper can be easily generalized to include arbitrary invariant morphological templates which correspond to matrices over $\mathbb{R}_{-\infty}$.

6 CONCLUSIONS

We introduced the new theory of the separable matrix rank within minimax algebra, which we compared to the theory of matrix rank provided by Cuninghame-Green. The definition of the separable rank of a matrix leads to the concept of the rank of a rectangular morphological template, a notion which has significance for the problem of morphological template decomposition.

Using this terminology, the class of separable templates represents the class of templates of rank one. A separable template can be strongly decomposed into a product of a column template and a row template. Generalizing this decomposition of separable templates, we developed a polynomial time algorithm for the weak decomposition of a rectangular template of rank two into horizontal and vertical strip templates. We are currently working on an improved version of this algorithm.

In an upcoming paper, we will show that determining the rank of an arbitrary rectangular template is an NP complete problem, and we will discuss the consequences for morphological template decomposition problems in general. Moreover, we will present a heuristic algorithm for solving the rank problem and for finding an optimal weak decomposition into strip templates.

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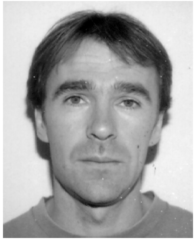

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