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ABSTRACT. Given a submodule $J \subset \mathcal{O}_0^{\oplus r}$ and a free resolution of J one can define a certain vector valued residue current whose annihilator is J. We make a decomposition of the current with respect to $\mathrm{A}ss(J)$ that correspond to a primary decomposition of J. As a tool we introduce a class of currents that includes usual residue and principal value currents; in particular these currents admit a certain type of restriction to analytic varieties and more generally to constructible sets.

1. Introduction

Let $(f_1, \ldots, f_q)$ be a holomorphic mapping at $0 \in \mathbb{C}^n$ that forms a complete intersection, that is, the codimension of the common zero set $V^f = \{f_1 = \cdots = f_q = 0\}$ is equal to q. The Coleff-Herrera product

$$\mu^f = \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_q}, \tag{1.1}$$

introduced in [10], is a $\bar{\partial}$-closed $(0, q)$-current with support on V^f, such that $\bar{I}_{V^f}\mu^f = 0$, that is, $\bar{\varphi}\mu^f = 0$ for all holomorphic φ that vanish on V^f. It has turned out to be a good notion of a multivariate residue of f. The duality theorem, [12] and [16], asserts that a holomorphic function φ belongs to the ideal $J = (f_1, \ldots, f_q)$ in $\mathcal{O}_0$ if and only if the current $\varphi\mu^f$ vanishes, in other words the annihilator ideal $\mathrm{ann}\,\mu^f$ equals J. The condition $\varphi\mu^f = 0$ is an intrinsic way of expressing that a certain family of differential operators applied to φ vanishes on V^f. Such a family is referred to as Noetherian operators for J. The fact that $\bar{I}_{V^f}\mu^f = 0$ means that only holomorphic derivatives are involved.

Furthermore μ^f has the so called standard extension property, SEP, which basically means that μ^f has no "mass" concentrated on subvarieties of V^f of codimension $> q$, or equivalently its *restriction*, in a sense that will be defined below, to each subvariety vanishes. Due to the SEP μ^f can be decomposed in a natural way with respect to the irreducible components V_j of V^f: $\mu^f = \sum_j \mu_j$, where μ_j is a current that has the SEP and whose support is contained in V_j; μ_j should be thought of as the restriction of μ^f to V_j. Moreover

$$J = \mathrm{ann}\,\mu^f = \cap_j \mathrm{ann}\,\mu_j. \tag{1.2}$$

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From Proposition 4.1 it follows that $\operatorname{ann}\mu_j$ is an I_{V_j}-primary ideal, where I_{V_j} denotes the ideal associated with V_j, and hence (1.2) gives a minimal primary decomposition of J. For a reference on primary decomposition we refer to [5]. In general, however, μ_j is not $\bar\partial$-closed. It is natural to consider the current μ^f as a geometric object and then $\mu^f = \sum_j \mu_j$ is a geometric decomposition of μ^f that corresponds to a decomposition of the Noetherian operators leading to the algebraic decomposition (1.2) of J.

In [4] we introduced, given a general ideal $J \subset \mathcal{O}_0$ a vector-valued residue current R such that $\operatorname{ann} R = J$. The construction of R is based on a free resolution of J and it also involves a choice of Hermitian metrics on associated vector bundles (see Section 5). In case J is defined by a complete intersection f, then R is just the Coleff-Herrera product μ^f. By means of the currents R we were able to extend several results previously known for complete intersections. Combined with the framework of integral formulas developed in [2] we obtained explicit division formulas realizing the ideal membership, which were used to give for example a residue version of the Ehrenpreis-Palamodov fundamental principle, [13] and [15], generalizing [8].

In this paper we prove that the current R can be decomposed as $R = \sum_{\mathfrak{p}} R^{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all associated prime ideals of J, so that $R_{\mathfrak{p}}$ has support on $V(\mathfrak{p})$ and has the SEP. It is easy to see that this decomposition must be unique. Moreover it turns out that $\operatorname{ann} R^{\mathfrak{p}}$ is $\mathfrak{p}$-primary and

$$J = \bigcap_{\mathfrak{p}} \operatorname{ann} R^{\mathfrak{p}}$$

provides a minimal primary decomposition of J; our main result is Theorem 5.1, which in fact holds also for submodules $J \subset \mathcal{O}_0^{\oplus r}$.

As long as J has no embedded primes the current $R^{\mathfrak{p}}$ is just R restricted to $V(\mathfrak{p})$ as for a complete intersection above, whereas the definition of $R^{\mathfrak{p}}$ in general gets more involved. As a basic tool we introduce a class of currents that admit restrictions to varieties and more generally constructible sets. This class of currents, which we call hypermeromorphic, includes all residue currents in this paper and the definition is modeled on the currents that appear in various works as [1], [4] and [17]; the typical example being the Coleff-Herrera product. The class has many desirable properties; it is closed under $\bar\partial$ and multiplication with smooth forms. If T is hypermeromorphic and has support on the variety V, then T is annihilated by $\bar I_V$ and $\bar\partial \bar I_V$. In particular, (a version of) the SEP follows: if T is of bidegree (p,q) and $q < \operatorname{codim} V$ then T vanishes.

In Section 2 we define hypermeromorphic currents, whereas restrictions to constructible sets are discussed in Section 3. Section 4 deals with annihilators of hypermeromorphic currents. Our main result, the decomposition of R is presented in Section 5 and a corresponding result in the algebraic case is given in Section 6. As an application we get a decomposition of the representation in our version of the fundamental principle.

2. Hypermeromorphic currents

Let X be an n-dimensional complex manifold. Recall that the principal value current $[1/\sigma^a]$ is well-defined in $\mathbb{C}_\sigma$, and that $\bar\partial\,[1/\sigma^a]$ is annihilated by $\bar\sigma$ and $d\bar\sigma$. In $\mathbb{C}^n_\sigma$, therefore, the current

$$\tau = \bar\partial\left[\frac{1}{\sigma_{i_1}^{a_{i_1}}}\right]\wedge\ldots\wedge\bar\partial\left[\frac{1}{\sigma_{i_q}^{a_{i_q}}}\right]\wedge\left[\frac{1}{\sigma_{i_{q+1}}^{a_{i_{q+1}}}}\right]\cdots\left[\frac{1}{\sigma_{i_\nu}^{a_{i_\nu}}}\right]\wedge\alpha, \tag{2.1}$$

where $\{i_1,\ldots,i_\nu\}\subset\{1,\ldots,n\}$, $a_k>0$, and α is a smooth form, is well-defined. If τ is a current on X, and there exists a local chart $\mathcal{U}_\sigma$ such that τ is of the form (2.1) and α has compact support in $\mathcal{U}_\sigma$ we say that τ is *elementary.* Note in particular that this definition, with q equal to 0, includes principal value currents as well as smooth forms.

A current T on X is said to be *hypermeromorphic* if it can be written as a locally finite sum

$$T = \sum \Pi_*\tau_\ell, \tag{2.2}$$

where each τ_ℓ is a an elementary current on some manifold $\widetilde{X}_r$ and $\Pi = \Pi_1\circ\cdots\circ\Pi_r$ is a corresponding composition of resolutions of singularities $\Pi_1 : \widetilde{X}_1 \to X_1 \subset X,\ldots,\Pi_r : \widetilde{X}_r \to X_r \subset \widetilde{X}_{r-1}$. We denote the class of hypermeromorphic currents on X by $\mathcal{HM}(X)$ and we write $\mathcal{HM}^{p,q}(X)$ to denote the elements that have bidegree (p,q).

The Coleff-Herrera-Passare products are typical examples of hypermeromorphic currents. From the proof of Theorem 1.1 in [17] and Theorem 1.1 in [1] it follows that the residue currents of Bochner-Martinelli type are hypermeromorphic, and the arguments in Section 2 in [4] shows that the residue currents introduced there are hypermeromorphic.

Note that if τ is an elementary current, then $\bar\partial\tau$ is a sum of elementary currents and since $\bar\partial$ commutes with pushforwards it follows that $\mathcal{HM}(X)$ is closed under $\bar\partial$. In the same way $\mathcal{HM}(X)$ is closed under ∂. Moreover if T is given by (2.2) and β is a smooth form, then $\beta\wedge T = \sum\Pi_*(\Pi^*\beta\wedge\tau_\ell)$ and thus $\mathcal{HM}(X)$ is closed under multiplication with smooth forms. Furthermore $\mathcal{HM}(X)$ admits a multiplication from the left with meromorphic currents:

Proposition 2.1. *Let $T\in\mathcal{HM}(X)$ and let g be a holomorphic function. Then the analytic continuations*

$$\left[\frac{1}{g}\right]T := \frac{|g|^{2\lambda}}{g}T\Big|_{\lambda=0} \qquad \textit{and} \qquad \bar\partial\left[\frac{1}{g}\right]\wedge T := \frac{\bar\partial|g|^{2\lambda}}{g}\wedge T\Big|_{\lambda=0}$$

exist and are hypermeromorphic currents. The support of the second one is contained in $\{g = 0\} \cap \operatorname{supp} T$. Moreover the products satisfy Leibniz' rule:

(2.3)
$$\bar\partial\left(\left[\frac{1}{g}\right] T\right) = \bar\partial\left[\frac{1}{g}\right]\wedge T + \left[\frac{1}{g}\right]\bar\partial T, \quad \bar\partial\left(\bar\partial\left[\frac{1}{g}\right]\wedge T\right) = -\bar\partial\left[\frac{1}{g}\right]\wedge\bar\partial T.$$

By the first statement in the proposition we mean that the currents $(|g|^{2\lambda}/g)T$ and $(\bar\partial|g|^{2\lambda}/g)\wedge T$, which are clearly well defined if $\operatorname{Re}\lambda$ is large enough, have analytic continuations to $\operatorname{Re}\lambda > -\epsilon$ for some $\epsilon > 0$, and $(|g|^{2\lambda}/g)T|_{\lambda=0}$ and $(\bar\partial|g|^{2\lambda}/g)\wedge T_{\lambda=0}$ denote the values at $\lambda = 0$.

Example 1. In $\mathbb{C}$ the analytic continuations of $(|\sigma^a|^{2\lambda}/\sigma^a)\left[1/\sigma^b\right]$, $(|\sigma^a|^{2\lambda}/\sigma^a)\bar\partial\left[1/\sigma^b\right]$ and $\bar\partial(|\sigma^a|^{2\lambda}/\sigma^a)\left[1/\sigma^b\right]$ to $\operatorname{Re}\lambda > -\epsilon$ exist, which for instance can be seen by integration by parts, and we have

$$\begin{aligned}
\left[\frac{1}{\sigma^a}\right]\left[\frac{1}{\sigma^b}\right] &= |\sigma^a|^{2\lambda}\frac{1}{\sigma^a}\left[\frac{1}{\sigma^b}\right]\Big|_{\lambda=0} = \left[\frac{1}{\sigma^{a+b}}\right]\\
\left[\frac{1}{\sigma^a}\right]\bar\partial\left[\frac{1}{\sigma^b}\right] &= |\sigma^a|^{2\lambda}\frac{1}{\sigma^a}\bar\partial\left[\frac{1}{\sigma^b}\right]\Big|_{\lambda=0} = 0\\
\bar\partial\left[\frac{1}{\sigma^a}\right]\left[\frac{1}{\sigma^b}\right] &= \bar\partial|\sigma^a|^{2\lambda}\frac{1}{\sigma^a}\left[\frac{1}{\sigma^b}\right]\Big|_{\lambda=0} = \bar\partial\left[\frac{1}{\sigma^{a+b}}\right].
\end{aligned}$$

In particular it follows that the products with meromorphic currents in general are not (anti-)commutative. □

Proof. Note that if T is an elementary current and g is a monomial, then, in light of Example 1, the analytic continuations exist and the values at $\lambda = 0$ are elementary.

For the general case, assume that T is of the form (2.2). Locally, due to Hironaka's theorem on resolution of singularities, see [7], for each ℓ, in $\widetilde{X}_r$ we can find a resolution $\Pi^{r+1} : \widetilde{X}_{r+1} \to X_{r+1} \subset \widetilde{X}$ such that for each k, $(\Pi^{r+1})^*\sigma_k$ is a monomial times a non-vanishing factor and moreover $(\Pi^{r+1})^*(\Pi^r)^*\cdots(\Pi^1)^*g$ is a monomial. Thus we may assume that $\Pi^* g$ is a monomial for each ℓ. Now, since $(|g|/^{2\lambda}/g)T = \sum \Pi_*((|\Pi^*g|^{2\lambda}/\Pi^*g)\tau_\ell)$, the analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ exists and the value at $\lambda = 0$ is in $\mathcal{HM}(X)$.

The existence of the analytic continuation of $\bar\partial(|g|^{2\lambda}/g)\wedge T$ follows analogously. If $g \neq 0$ the value at $\lambda = 0$ is clearly zero and hence the support of $\bar\partial[1/g]\wedge T$ is contained in $\{g = 0\} \cap \operatorname{supp} T$.

The last statement (2.3) follows directly from the definition and the uniqueness of analytic continuation. □

If $T \in \mathcal{HM}(X)$ and $V \subset X$ is an analytic variety, we shall now see that the restriction of T to the Zariski-open set $U = V^C$ has a natural (standard) extension to X, which we denote $T|_U$. The current $T - T|_U$, which has support on V, is a kind of residue that we will call the restriction of T to V and denote by $T|_V$.

Proposition 2.2. *Let $T \in \mathcal{HM}(X)$, let $U \subset X$ be a Zariski-open set, and let h a tuple of holomorphic functions such that $\{h = 0\} = U^C$. Then the analytic continuation $T|_U := |h|^{2\lambda}T|_{\lambda=0}$ exists and is independent of the particular choice of h.*

The definition immediately extends to any Zariski-open set on any manifold.

Proof. If T is an elementary current (2.1) and h is a monomial the analytic continuation exists, compare to the proof of Proposition 2.1, and it is easy to see that the value at $\lambda = 0$ is T if none of $\sigma_{i_1}, \ldots, \sigma_{i_q}$ divide h and zero otherwise.

Assume that T is of the form (2.2). Then, for each ℓ we can find resolutions of singularities $\Pi^{r+1} : \widetilde{X}_{r+1} \to X_{r+1} \subset \widetilde{X}_r$ and toric resolutions $\Pi^{r+2} : \widetilde{X}_{r+2} \to X_{r+2} \subset \widetilde{X}_{r+1}$ such that each $(\Pi^{r+2})^*(\Pi^{r+1})^*\sigma_k$ is a monomial times a non-vanishing factor and moreover $(\Pi^{r+2})^*(\Pi^{r+1})^*(\Pi^r)^* \cdots (\Pi^1)^*h$ is a monomial h^0 times a nonvanishing tuple h', see for example [7]. Thus in (2.2) we may assume that each Π^*h is a monomial times a nonvanishing tuple. Now, since $|h|^{2\lambda}T = \sum \Pi_*(\Pi^*|h|^{2\lambda}\tau_\ell)$, the analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ exists. Moreover,

$$|h|^{2\lambda}T|_{\lambda=0} = \sum \Pi_*\tau_{\ell'},$$

where the sum is taken over ℓ' such that none of the factors $\sigma_{i_1}, \ldots, \sigma_{i_q}$ in $\tau_{\ell'}$ divides Π^*h. In particular it follows that $|h|^{2\lambda}T|_{\lambda=0}$ only depends on U and not on the particular choice of h. Indeed, if g is another tuple of functions such that $U^C = \{g = 0\}$, we can find resolutions such that both Π^*h and Π^*g are monomials times nonvanishing tuples. Then clearly Π^*h and Π^*g must be divisible by the same coordinate functions. $\square$

Assume that T is a hypermeromorphic current with support on the analytic variety V of (pure) codimension k. We say that T has the *standard extension property (SEP) with respect to* V if $T|_W = 0$ for all analytic varieties $W \subset V$ of codimension $> k$. Classically a current T with support on V has the SEP if it is equal to its own standard extension in the sense of Barlet [6], which means that $\lim_{\varepsilon\to 0} \chi(|h|/\varepsilon)T = T$ if h is a holomorphic function that is generically nonvanishing on V, that is, $V \cap \{h = 0\}$ has codimension $> k$, and χ is (a possibly smooth approximand of) the characteristic function for the interval $[1, \infty)$. One can show that $\lim_{\varepsilon\to 0} \chi(|h|/\varepsilon)T$ is indeed equal to $T|_{\{h=0\}^C}$, see [3] (in fact, this holds also for a tuple h of holomorphic functions). Moreover, as we will see below, $T|_{\{h=0\}} = T|_{V\cap\{h=0\}}$ if T has support on V. Thus our definition of the SEP coincides with the classical one.

Proposition 2.3. *Let $T \in \mathcal{HM}(X)$. Suppose that $\operatorname{supp} T$ is contained in the variety Z and Ψ is a holomorphic form that vanishes on Z. Then $\overline{\Psi} \wedge T = 0$.*

Proof. Note that if T is an elementary current and Z is a union of coordinate hyperplanes the result follows from the one-dimensional case. Indeed, each term of Ψ then contains a factor σ_k or $d\sigma_k$ for each σ_k that vanishes on Z, and moreover $\bar\sigma$ as well as $d\bar\sigma$ annihilate $\bar\partial[1/\sigma^a]$.

For the general case assume that T is given by (2.2). Note that $T = T|_Z$ since $\operatorname{supp} T \subset Z$. The crucial point is now that according to the proof of Proposition 2.2 we have $T = \sum \Pi_* \tau_{\ell'}$, where $\tau_{\ell'}$ is an elementary current with support on $(\Pi^L)^{-1}(Z)$, and hence

$$\overline{\Psi} \wedge T = \sum \Pi_* \left((\Pi)^* \overline{\Psi} \wedge \tau_{\ell'} \right).$$

Now, since Ψ vanishes on Z, the holomorphic form $(\Pi)^*\Psi$ vanishes on $(\Pi)^{-1}(Z)$, which however is a union of coordinate planes. Hence $(\Pi)^*\overline{\Psi} \wedge \tau_{\ell'}$ vanishes as noted above and we are done. $\square$

In particular, Proposition 2.3 implies that $d\bar h \wedge T = 0$ if h is holomorphic and vanishes on $\operatorname{supp} T$. Arguing as in the proofs of Theorems III.2.10-11 on normal currents in [11] yields the following.

Corollary 2.4. *Let $T \in \mathcal{HM}^{p,k}(X)$. If $\operatorname{supp} T$ is contained in the analytic variety V of codimension $> k$, then $T = 0$.*

In other words, the corollary says that if $T \in \mathcal{HM}^{p,k}(X)$ has support on V of codimension k, then T has the SEP. Also, Proposition 2.3 implies that T is annihilated by all anti-holomorphic functions that vanish on V. Thus, if in addition $\bar\partial T = 0$, then by definition T is a Coleff-Herrera current, that is, $T \in \mathcal{CH}_V^{p,k}(X)$, see [9].

Conversely, if $T \in \mathcal{CH}_V^{p,k}(X)$, then locally $T = \gamma R$, where R is a residue current and γ is a holomorphic $(0,p)$-form, see for example [3], and so $T \in \mathcal{HM}$. Hence we conclude:

Proposition 2.5. *Suppose that $V \subset X$ is an analytic variety of pure codimension k. Then $\mathcal{CH}_V^{p,k}(X)$ is precisely the set of currents $T \in \mathcal{HM}^{p,k}(X)$ with support on V that are $\bar\partial$-closed.*

Example 2. Let $f = (f_1, \ldots, f_q)$ be a holomorphic mapping at $0 \in \mathbb{C}^n$. By iterated use of Proposition 2.1 one can build up a product like (1.1). If f is a complete intersection the product is anti-commutative with respect to the factors $\bar\partial[1/f_i]$. To see this, when $q = 2$, let $T = [1/f_1]\bar\partial[1/f_2] - \bar\partial[1/f_2][1/f_1]$. Then T has support on $\{f = g = 0\}$ which has codimension 2, and so $T = 0$ by Corollary 2.4. Now (2.3) implies that $0 = \bar\partial T = \bar\partial[1/f_1] \wedge \bar\partial[1/f_2] - \bar\partial[1/f_2] \wedge \bar\partial[1/f_1]$. The general case follows analogously. It is now easy to see that in this case the product indeed coincides with the Coleff-Herrera product, compare to [3]. $\square$

3. Restrictions of hypermeromorphic currents

We will now show that one can give meaning to restrictions of hypermeromorphic currents to all constructible sets. Recall that the set of constructible sets in X, which we will denote by $\mathcal{C}(X)$, is the Boolean algebra generated by the Zariski-open sets in X.

Theorem 3.1. *There exists a unique, linear in $\mathcal{HM}(X)$ and degree-preserving, mapping*

$$\mathcal{HM}(X) \times \mathcal{C}(X) \to \mathcal{HM}(X) : (T, W) \mapsto T|_W$$

such that $T|_U$ coincides with the natural extension across U^C of the restriction of T to U if $U \subset X$ is Zariski-open and moreover for all W and W' in $\mathcal{C}(X)$,

(i) $T|_{W^C} = T - T|_W$
(ii) $T|_{W \cap W'} = T|_W|_{W'}$.

The uniqueness of the restriction mapping $\bullet|_\bullet$, provided it exists, follows from (i) and (ii), since any constructible set can be obtained from a finite number of Zariski-open sets by taking intersections and complements.

Since restriction to Zariski-open sets U is local, that is, the value of $T|_U$ in $\Omega \subset X$ only depends on the values of T in Ω, by (i) and (ii) this holds for any $W \in \mathcal{C}(X)$. In particular $\operatorname{supp} T|_W \subset \operatorname{supp} T$. Moreover $T|_W|_{\overline{W}^C} = 0$ by (ii) and hence $T|_W = 0$ in the Zariski-open set $\overline{W}^C$. Thus

$$\operatorname{supp} T|_W \subset \overline{W} \cap \operatorname{supp} T.$$

Furthermore, it follows from (i) and (ii) that

$$T|_{W \cup W'} = T|_W + T|_{W'} - T|_{W \cap W'}.$$

Theorem 3.1 also implies that

$$(\xi \wedge T)|_W = \xi \wedge T|_W, \quad \xi \in \mathcal{E}_{(*,*)}(X). \tag{3.1}$$

Indeed, (3.1) holds if W is open in light of Proposition 2.2 and by (i) and (ii) it extends to all constructible sets.

As a basis for the proof of Theorem 3.1 we first consider a simple situation where the restriction is defined in a more direct way.

Throughout this section we will use the notation $\mathcal{B}(W_1, \ldots, W_r)$ to denote the Boolean algebra generated by the constructible sets $W_1, \ldots, W_r \subset X$. It is clear that every constructible set lies in $\mathcal{B}(U_1, \ldots, U_r)$ for some choice of Zariski open sets $U_1, \ldots, U_r$.

Example 3. Suppose that T is a sum of elementary currents in $\mathbb{C}^n_\sigma$, that is, $T = \sum \tau_j$, where each τ_j is of the form (2.1), and moreover that $W \in \mathcal{B}(H_1, \ldots, H_n)$, where H_i is the coordinate hyperplane $\{\sigma_i = 0\}$.

The constructible sets in $\mathcal{B}(H_1, \ldots, H_n)$ can be seen to correspond precisely to subsets of the power set $\mathcal{P}([n])$ of $[n] = \{1, \ldots, n\}$. First, identify $\omega \in \mathcal{P}([n])$ with the constructible set

$$W_\omega = \{\sigma_i = 0 \text{ if } i \in \omega, \sigma_i \neq 0 \text{ if } i \notin \omega\};$$

then all W_ω are disjoint and their union is $\mathbb{C}^n$. Next, we claim that to each $W \in \mathcal{B}(H_1, \ldots, H_n)$ there is a unique $\Omega = \Omega(W) \subset \mathcal{P}([n])$ such that $W = \bigcup_{\omega \in \Omega} W_\omega$. To see this first note that if such a Ω exists it is unique since the W_ω are disjoint. Next, observe that $H_i = \bigcup_{\omega \ni i} W_\omega$ and furthermore that if $\Omega(W)$ and $\Omega(W')$ are well defined, then

$$(\Omega(W))^C = \Omega(W^C) \text{ and } \Omega(W) \cap \Omega(W') = \Omega(W \cap W') \tag{3.2}$$

and so $\Omega(W^C)$ and $\Omega(W \cap W')$ are well defined. The claim now follows by induction.

Let d be the mapping from the set of elementary currents on $\mathbb{C}^n_\sigma$ to $\mathcal{P}([n])$ that maps (2.1) to $\{i_1, \ldots, i_q\}$, that is, d sends an elementary current to the subset of $[n]$ corresponding to its residue factors. Then the mapping

$$(T, W) \mapsto T|_W = \sum_{j : d(\tau_j) \in \Omega(W)} \tau_j \tag{3.3}$$

satisfies the requirements in the theorem. It is clear that it is linear in T and that $T|_W$ is in $\mathcal{HM}^{p,q}(\mathbb{C}^n)$ if T is. Next, (i) and (ii) follow because of (3.2).

In the case when $W = H_i^C$ it is clear that (3.3) coincides with the analytic definition in Proposition 2.2 and thus with the natural extension across H_i of the restriction of T to H_i^C. Hence (3.3) must coincide with the natural extension across W^C of the restriction of T to W for all Zariski-open sets W in $\mathcal{B}(H_1, \ldots, H_n)$, provided that the analytic definition satisfies (i) and (ii). This, however, will be clear from the proof of Theorem 3.1 below. $\square$

Example 4. Suppose $n = 2$. Then the four elements in $\mathcal{P}([2])$, $\{1,2\}$, $\{1\}$, $\{2\}$ and $\emptyset$ correspond to the origin, the σ_2-axis H_1 with the origin removed, the σ_1-axis H_2 with the origin removed, and $\mathbb{C}^2$ with the coordinate axes removed, respectively. For example H_2 is given as $W_{\{2\}} \cup W_{\{1,2\}}$. Suppose that

$$T = \alpha \left[\frac{1}{\sigma_1^3}\right] + \beta \left[\frac{1}{\sigma_2}\right] \bar{\partial} \left[\frac{1}{\sigma_1^2}\right] + \gamma\, \bar{\partial} \left[\frac{1}{\sigma_1}\right] \wedge \bar{\partial} \left[\frac{1}{\sigma_2^2}\right] = \tau_1 + \tau_2 + \tau_3,$$

where α, β and γ are just smooth functions with compact support. Then $d(\tau_1) = \emptyset$, $d(\tau_2) = \{1\}$ and $d(\tau_3) = \{1,2\}$. Now $T|_{H_2} = \tau_3$ whereas $T|_W = \tau_1 + \tau_3$ if $W = W_\emptyset \cup W_{\{1,2\}}$. $\square$

Lemma 3.2. *Let $U_1, \ldots, U_r \subset X$ be Zariski-open sets. Then there is a degree-preserving mapping $\mathcal{HM}(X) \times \mathcal{B}(U_1, \ldots, U_r) \to \mathcal{HM}(X)$: $(T, W) \mapsto T|_W$ that is linear in $\mathcal{HM}(X)$, such that $T|_{U_i}$ coincides with the natural extension across U_i^C of the restriction of T to U_i and (i)-(ii) hold.*

Proof. Once we have Proposition 2.2 we can give meaning to $\bullet|_W$ for $W \in \mathcal{B}(U_1, \ldots, U_r)$ (or any constructible set) by inductively using (i) and (ii). However, a priori it is not clear that this definition of $|_W$ is independent of the representation of W in $\mathcal{B}(U_1, \ldots, U_r)$. To show that this is indeed the case we will introduce an auxiliary definition of restriction modeled on Example 3.

We say that a set $\{\Pi^L\}$ of compositions $\Pi^L = \Pi_{\ell_1} \circ \ldots \circ \Pi_{\ell_{r_\ell}}$ of resolutions $\Pi_{\ell_1} : \widetilde{X}_{\ell_1} \to X_{\ell_1} \subset X, \ldots, \Pi_{\ell_{r_\ell}} : \widetilde{X}_{\ell_{r_\ell}} \to X_{\ell_{r_\ell}} \subset \widetilde{X}_{\ell_{r_\ell - 1}}$ is good with respect to the Zariski-open set $U \subset X$ if $(\Pi^L)^{-1}(U^C)$ locally in $\widetilde{X}_{\ell_{r_\ell}}$ is a union of coordinate hyperplanes. Moreover, we say that $T \in \mathcal{HM}(X)$ is good with respect to $\{\Pi^L\}$ if T can be written

$$T = \sum \Pi^L_* \tau_L,$$

where each τ_L is an elementary current (2.1) (with respect to some local coordinate chart in $\widetilde{X}_{\ell_{r_\ell}}$). We denote the set of currents in $\mathcal{HM}(X)$ that are good with respect to $\{\Pi^L\}$ by $\mathcal{G}(\{\Pi^L\})$.

Now, let $\{\Pi^L\}$ be a set of resolutions that is good with respect to $U_1, \ldots, U_r$. Inspired by (3.3), we define a restriction mapping $\mathcal{G}(\{\Pi^L\}) \times \mathcal{B}(U_1, \ldots, U_r\} \to \mathcal{G}(\{\Pi^L\})$,

$$(T, W) \mapsto T|_W^{\{\Pi^L\}} = \sum_{L : d(\tau_L) \in \Omega((\Pi^L)^{-1}(W))} \Pi^L_* \tau_L;$$

here d and Ω are taken with respect to a fixed coordinate chart in $\widetilde{X}_{\ell_{r_\ell}}$.

It is clear that $\bullet|_W^{\{\Pi^L\}}$ is linear and maps $\mathcal{G}(\{\Pi^L\})$ to $\mathcal{G}(\{\Pi^L\})$ and in light of Example 3 it is easily checked that $\bullet|_\bullet^{\{\Pi^L\}}$ satisfies (i)-(ii) using that $(\Pi^L)^{-1}$ commutes with complements and intersections.

We will now show that $T|_W^{\{\Pi^L\}}$ is independent of the particular choice of $\{\Pi^L\}$ (as long as $T \in \mathcal{G}(\{\Pi^L\})$). First, we claim that if $\{\Pi^L\}$ is good with respect to the Zariski-open set U and T is good with respect to $\{\Pi^L\}$, then $T|_U^{\{\Pi^L\}}$ is equal to $T|_U$. In this case $(\Pi^L)^{-1}(U^C)$ is a union of coordinate hyperplanes, say $H_{i_1}, \ldots, H_{i_q}$, and so $\Omega((\Pi^L)^{-1}(U^C))$ is the set of subsets of $[n]$ that contains at least one of the elements $i_1, \ldots, i_q$, (that is, the dual order ideal generated by $\{i_1, \ldots, i_q\}$). However, from the proof of Proposition 2.2 we know that $|h|^{2\lambda} \Pi^L_* \tau_L|_{\lambda=0}$ is equal to $\Pi^L_* \tau_L$ if $d(\tau_L)$ does not contain any of $i_1, \ldots, i_q$, in other words if $d(\tau_L) \in \Omega((\Pi^L)^{-1}(U^C))^C = \Omega((\Pi^L)^{-1}(U))$, and zero otherwise. Hence the claim follows.

Furthermore, assume that $W, W' \in \mathcal{B}(U_1, \ldots, U_r)$ are such that $T|_W^{\{\Pi^L\}}$ and $T|_{W'}^{\{\Pi^L\}}$ if are independent of $\{\Pi^L\}$ for all $T \in \mathcal{G}(\{\Pi^L\})$. Then

$$\begin{aligned} T|_{W^C}^{\{\Pi^L\}} &= T - T|_W^{\{\Pi^L\}} \\ T|_{W \cap W'}^{\{\Pi^L\}} &= T|_W^{\{\Pi^L\}}|_{W'}^{\{\Pi^L\}}. \end{aligned}$$

are independent of $\{\Pi^L\}$ since the right hand side expressions are, and it follows by induction that $T|_W^{\{\Pi^L\}}$ is independent of $\{\Pi^L\}$ for all $W \in \mathcal{B}(U_1, \ldots, U_r)$.

Now for $(T, W) \in \mathcal{HM}(X) \times \mathcal{B}(U_1, \ldots, U_r)$ we define $T|_W = T|_W^{\{\Pi^L\}} \in \mathcal{HM}(X)$, where $\{\Pi^L\}$ is some set of resolutions that is good with respect $U_1, \ldots, U_r$ and such that T is good with respect to $\{\Pi^L\}$. By Hironaka's theorem and the definition of hypermeromorphic currents such a $\{\Pi^L\}$ always exists, and thus, since $T|_W^{\{\Pi^L\}}$ is independent of $\{\Pi^L\}$, $T|_W$ is well-defined. It is clear that $\bullet|_\bullet$ fulfills the requirements in the lemma since $\bullet|_\bullet^{\{\Pi^L\}}$ does for a fixed $\{\Pi^L\}$. $\square$

Proof of Theorem 3.1. For $(T, W) \in \mathcal{HM}(X) \times \mathcal{C}(X)$ let $T|_W = T|_W^{(U_1,\ldots,U_r)} \in \mathcal{HM}(X)$, where $\bullet|_\bullet^{(U_1,\ldots,U_r)}$ denotes the restriction mapping $\mathcal{HM}(X) \times \mathcal{B}(U_1, \ldots, U_r) \to \mathcal{HM}(X)$ from Lemma 3.2 and $U_1, \ldots, U_r \subset X$ are Zariski-open sets such that $W \in \mathcal{B}(U_1, \ldots, U_r)$. To see that this definition is independent of the choice of $U_1, \ldots, U_r$, suppose that W also is in $\mathcal{B}(O_1, \ldots, O_s)$. Then according to Lemma 3.2, $T|_W^{(U_1,\ldots,U_r)} = T|_W^{(U_1,\ldots,U_r,O_1,\ldots,O_s)} = T|_W^{(O_1,\ldots,O_s)}$. It is clear that $\bullet|_\bullet$ is linear and degree preserving, coincides with the natural restriction of to open sets, and fulfills (i) and (ii), since this holds for $\bullet|_\bullet^{(U_1,\ldots,U_r)}$. $\square$

Example 5. The restriction $\bullet|_\bullet$ does not commute with $\bar\partial$. Let

$$T = \bar\partial \left[\frac{1}{\sigma\tau}\right] = \left(\bar\partial \left[\frac{1}{\sigma}\right]\right) \left[\frac{1}{\tau}\right] + \left[\frac{1}{\sigma}\right] \bar\partial \left[\frac{1}{\tau}\right].$$

Then T is clearly hypermeromorphic and since $\bar\partial T = 0$ it follows that $(\bar\partial T)|_{\{\tau=0\}} = 0$. However,

$$T|_{\{\tau=0\}} = \left[\frac{1}{\sigma}\right] \bar\partial \left[\frac{1}{\tau}\right],$$

and consequently

$$\bar\partial(T|_{\{\tau=0\}}) = \bar\partial \left[\frac{1}{\sigma}\right] \wedge \bar\partial \left[\frac{1}{\tau}\right] \neq 0.$$

$\square$

4. Annihilators of hypermeromorphic currents

Let $\mathcal{HM}_x$ denote the $\mathcal{E}_x$-module of germs of hypermeromorphic currents at $x \in X$. For $T \in \mathcal{HM}_x$ let $\operatorname{ann} T$ denote the annihilator ideal $\{\varphi \in \mathcal{O}_x; \varphi T = 0\}$ in $\mathcal{O}_x$.

Example 6. Assume $T \in \mathcal{HM}_x$ and let W be a germ of a constructible set at x. Then if $\varphi \in \mathcal{O}_x$

$$\varphi T = (\varphi T)|_W + (\varphi T)|_{W^C} = \varphi T|_W + \varphi T|_{W^C},$$

where the first equality follows using (i) and the second one from (3.1). Hence if $\varphi \in \operatorname{ann} T|_W \cap \operatorname{ann} T|_{W^C}$, then $\varphi \in \operatorname{ann} T$. On the other hand if $\varphi T = 0$, then by (3.1) $\varphi T|_W = (\varphi T)|_W = 0$ and analogously $\varphi T|_{W^C} = 0$. Thus

$$\operatorname{ann} T = \operatorname{ann} T|_W \cap \operatorname{ann} T|_{W^C}.$$

□

For a germ Z at x of a variety let I_Z denote the prime ideal in $\mathcal{O}_x$ of germs of holomorphic functions that vanish on Z and for an ideal $J \subset \mathcal{O}_x$ let $V(J)$ denote the (germ of the) variety of J.

Proposition 4.1. *Suppose that Z is an irreducible germ at x of a variety of codimension k. If $T \in \mathcal{HM}_x^{p,k}$ has its support in Z then either $T = 0$ or ann T is an I_Z-primary ideal.*

Proof. Suppose $\varphi \in \mathcal{O}_x$ vanishes on Z. Then, since T has finite order, $\varphi^m T = 0$ for m large enough. It follows that $V(\operatorname{ann} T) \subset Z$. If $h \in \operatorname{ann} T$, that is, $hT = 0$, then $\operatorname{supp} T \subset Z \cap \{h = 0\}$. Since Z is irreducible, $Z \cap \{h = 0\}$ is either equal to Z or has codimension $\geq k+1$. In the latter case $T = 0$ according to Corollary 2.4. Hence $V(\operatorname{ann} T) = Z$ if $T \neq 0$.

If $\varphi\psi \in \operatorname{ann} T$, then $\varphi \in \operatorname{ann}(\psi T)$. Since ψT satisfies the assumptions of the proposition, the first part of the proof implies that if $\psi \notin \operatorname{ann} T$, then $\varphi \in I_Z = \sqrt{\operatorname{ann} T}$. Thus $\operatorname{ann} T$ is I_Z-primary. □

Remark 1. Note that the proof only uses that T has the SEP with respect to Z. Thus $\operatorname{ann} T$ is Z-primary whenever $T \in \mathcal{HM}_x$ has support on Z, has the SEP with respect to Z and does not vanish identically. □

5. Decomposition of R with respect to A$ss(J)$

We will now use the results from the the previous sections to make the decomposition of R. Let us start by briefly recalling the construction of residue currents from [4]. Let J be a submodule of $\mathcal{O}_x^{\oplus r_0}$, in particular if $r_0 = 1$, then J is an ideal in $\mathcal{O}_x$, and let

$$0 \to \mathcal{O}_x^{\oplus r_N} \xrightarrow{F_N} \dots \xrightarrow{F_2} \mathcal{O}_x^{\oplus r_1} \xrightarrow{F_1} \mathcal{O}_x^{\oplus r_0}, \tag{5.1}$$

be a free resolution of $\mathcal{O}_x$-modules of $\mathcal{O}_x^{\oplus r_0}$, where $J = \operatorname{Im}(\mathcal{O}_x^{\oplus r_1} \to \mathcal{O}_x^{\oplus r_0})$. Now (5.1) induces a holomorphic complex

$$0 \to E_N \xrightarrow{F_N} \dots \xrightarrow{F_2} E_1 \xrightarrow{F_1} E_0, \tag{5.2}$$

of (trivial) r_k-bundles E_k over some neighborhood Ω of $x \in X$ that is exact outside $Z = V(J)$ and such that $\mathcal{O}_x(E_k) \simeq \mathcal{O}_x^{\oplus r_k}$. Equipping the bundles E_k with Hermitian metrics we construct a current R that has support on Z, is annihilated by $\bar{I}_Z$, and

$$R = R_p + \dots + R_\mu, \tag{5.3}$$

where $p = \operatorname{codim} Z$, $\mu = \min(n, N)$, and R_j is a $(0, j)$-current that takes values in $\operatorname{Hom}(E_0, E_j)$.

Moreover, if φ is a germ of a holomorphic section of E_0 at x, that is, an element in $\mathcal{O}_x^{\oplus r_0}$, then $\varphi \in J$ if and only if $R\varphi = 0$ and φ lies generically in $\operatorname{Im} F_1$. If F_1 is generically surjective, that is, $\operatorname{codim} \mathcal{O}_x^{\oplus r_0}/J > 0$, in particular if $r_0 = 1$ and $F_1 \not\equiv 0$, the latter condition is automatically satisfied and $J = \operatorname{ann} R$. In general, one can extend the complex (5.2) with a mapping $F_0 : E_0 \to E_{-1}$ so that the extended complex is generically exact. Then $J = \ker F_0 \cap \operatorname{ann} R$.

Recall that a proper submodule J of the $\mathcal{O}_x$-module $\mathcal{O}_x^{\oplus r}$ is primary if $\varphi\xi \in J$ implies that $\xi \in J$ or $\varphi^\nu \in \operatorname{ann}(\mathcal{O}_x^{\oplus r}/J)$ for some $\nu > 0$. If $J \subset \mathcal{O}_x^{\oplus r}$ is primary then $\operatorname{ann}(\mathcal{O}_x^{\oplus r}/J)$ is a primary ideal and so $\mathfrak{p} = \sqrt{\operatorname{ann}(\mathcal{O}_x^{\oplus r}/J)}$ is a prime ideal. We say that J is $\mathfrak{p}$-primary. As for ideals in $\mathcal{O}_x$, a submodule of $\mathcal{O}_x^{\oplus r}$ always admits a primary decomposition; that is, $J = \bigcap J_k$, where J_k are $\mathfrak{p}_k$-primary modules. If all $\mathfrak{p}_k$ are different and no intersectands can be removed, then the primary decomposition is said to be minimal and the $\mathfrak{p}_k$ are said to be associated prime ideals of $\mathcal{J}$. The set of associated prime ideals is unique and we denote it by $\mathrm{A}ssJ$.

Example 7. If $F_0 : \mathcal{O}_x^{\oplus r_0} \to \mathcal{O}_x^{\oplus r_{-1}}$ is a non-zero $\mathcal{O}_x$-homomorphism, then $J = \ker F_0$ is a $\mathfrak{p}$-primary module, with $\mathfrak{p} = (0)$. Indeed, we have that $\sqrt{\operatorname{ann}\ \ker F_0} = (0)$, and moreover if $\varphi \in \mathcal{O}_x$ and $\xi \in \mathcal{O}_x^{\oplus r_0}$ are such that $F_0(\varphi\xi) = 0$, then $\varphi F_0 \xi = 0$ and so $\xi \in \ker F_0$ or $\varphi = 0$. $\square$

Let $R^{(0)} = F_0$ so that $\operatorname{ann} R^{(0)} = \ker F_0$. For each associated prime ideal $\mathfrak{p} \neq (0)$ of J let

$$R^{\mathfrak{p}} = R|_{V(\mathfrak{p}) \setminus \bigcup_{\mathfrak{q} \supset \mathfrak{p}} V(\mathfrak{q})}. \tag{5.4}$$

In view of (5.3) (and Corollary 2.4) we have that $R^{\mathfrak{p}} = R_k^{\mathfrak{p}} + \ldots + R_\mu^{\mathfrak{p}_i}$, where $k = \operatorname{codim} \mathfrak{p}$ and $R_j^{\mathfrak{p}}$ is of bidegree $(0, j)$ and takes values in $\operatorname{Hom}(E_0, E_j)$.

Theorem 5.1. *Let J be a submodule of $\mathcal{O}_x^{\oplus r_0}$, and let R be the residue current associated with* (5.1) *(and the choice of Hermitian metrics on the bundles E_k in* (5.2)*). Then for each $\mathfrak{p} \in AssJ$, $R^{\mathfrak{p}}$ has support on $V(\mathfrak{p})$ and has the SEP with respect to $V(\mathfrak{p})$, $ann\, R^{\mathfrak{p}} \subset \mathcal{O}_x^{\oplus r_0}$ is $\mathfrak{p}$-primary,*

$$R = \sum_{\mathfrak{p} \in AssJ,\ \mathfrak{p} \neq (0)} R^{\mathfrak{p}}, \tag{5.5}$$

and

$$J = ann\, R^{(0)} \cap ann\, R = \bigcap_{\mathfrak{p} \in AssJ} ann\, R^{\mathfrak{p}} \tag{5.6}$$

yields a minimal primary decomposition of J.

The decomposition (5.5) is unique once the $R^{\mathfrak{p}}$ are required to have support on $V(\mathfrak{p})$ and the SEP with respect to $V(\mathfrak{p})$. Indeed, suppose that $\mathfrak{p}$ is of minimal codimension, say q, among the associated primes. Then $R^{\mathfrak{p}} = R$ outside a set of codimension $\geq q+1$ and so, because of the SEP, $R^{\mathfrak{p}}$ is uniquely determined. Consequently $R' = \sum_{\text{codim}\,\mathfrak{p}>q} R^{\mathfrak{p}}$, whose support is of codimension $\geq q+1$, is uniquely determined. By the same argument applied to R' all $R^{\mathfrak{p}}$ with $\text{codim}\, p = q+1$ are unique. The general statement follows by induction. In the same way, as soon as we have the decomposition (5.5) with the above assumptions on $R^{\mathfrak{p}}$, then (5.6) must hold.

We first show a lemma which asserts that $R^{\mathfrak{p}}$ has the SEP.

Lemma 5.2. *Suppose that $\mathfrak{p} \in AssJ$ is of codimension $k > 0$. Then $ann\, R^{\mathfrak{p}} = ann\, R^{\mathfrak{p}}_k$. Moreover, suppose that W is a variety of codimension $> k$. Then $R^{\mathfrak{p}}|_W = 0$.*

Proof. Let Z_k denote the set where the mapping F_k in (5.2) does not have optimal rank. The key observation is that $R^{\mathfrak{p}}_{k+\ell}|_{Z_{k+\ell}} = 0$ for $\ell \geq 1$. To see this let Z' be one of the irreducible components of $Z_{k+\ell}$. If $\text{codim}\, Z' > k+\ell$, then $R^{\mathfrak{p}}_{k+\ell}|_{Z'} = 0$ due to Corollary 2.4. On the other hand if $\text{codim}\, Z' = k+\ell$, then $I_{Z'} \in \text{A}ssJ$ according to Corollary 20.14 in [14]. Thus $R^{\mathfrak{p}}_{k+\ell}|_{Z'} = R^{\mathfrak{p}}_{k+\ell}|_{Z'\cap(V(\mathfrak{p})\setminus\bigcup_{\mathfrak{q}\supset\mathfrak{p}} V(\mathfrak{q}))} = 0$, since either $I_{Z'} \supset \mathfrak{p}$ or $\text{codim}\, Z' \cap V(\mathfrak{p}) > k+\ell$, in which case the current vanishes according to Corollary 2.4.

To prove the first statement take $\varphi \in \text{ann}\, R^{\mathfrak{p}}_k$. Outside Z_{m+1} it holds that $R_{m+1} = \alpha_m R_m$, where α_m is a smooth $\text{Hom}\,(E_m, E_{m+1})$-valued $(0,1)$-form, see for example the proof of Theorem 4.4 in [4]. Now, by (i),

$$R^{\mathfrak{p}}_{k+1}\varphi = \alpha_k R^{\mathfrak{p}}_k \varphi|_{X\setminus Z_{k+1}} + R^{\mathfrak{p}}_{k+1}\varphi|_{Z_{k+1}} = 0.$$

By induction it follows that $R^{\mathfrak{p}}_{k+\ell}\varphi = 0$ for $\ell > 0$ and so $R^{\mathfrak{p}}\varphi = 0$. Thus $\text{ann}\, R^{\mathfrak{p}} = \text{ann}\, R^{\mathfrak{p}}_k$.

For the second statement, note that $R^{\mathfrak{p}}_k|_W = 0$ according to Corollary 2.4. It follows that $R^{\mathfrak{p}}_{k+\ell}|_W = 0$ by the same induction as above. □

We also need the following module version of Proposition 4.1.

Proposition 5.3. *Suppose that Z is an irreducible germ at x of a variety that has codimension k. If $T \in \mathcal{HM}^{p,k}_x(E_0^*)$ has its support in Z, then either $T \equiv 0$ or $ann\, T \subset \mathcal{O}_x(E_0)$ is a I_Z-primary module.*

Proof. For each $\xi \in \mathcal{O}_x(E_0)$, the scalar-valued current $T\xi$ satisfies the assumption of Proposition 4.1. Now, $\text{ann}\,(\mathcal{O}_x(E_0)/\text{ann}\, T) = \bigcap_{\xi\in\mathcal{O}_x(E_0)} \text{ann}\,(T\xi)$. If $T \neq 0$, then $T\xi \neq 0$ for some $\xi \in \mathcal{O}_x(E_0)$ and hence it follows from Proposition 4.1 that $V(\text{ann}\,(\mathcal{O}_x(E_0)/\text{ann}\, T)) = Z$.

Moreover, suppose that $\varphi \in \mathcal{O}_x$ and $\xi \in \mathcal{O}_x(E_0)$ are such that $\varphi\xi \in \text{ann}\, T$. Since the scalar-valued current $T\xi$ satisfies the assumptions of Proposition 4.1 it follows that if $\xi \notin \text{ann}\, T$, that is, $T\xi \neq 0$, then $\varphi \in I_Z$ and thus $\text{ann}\, T$ is I_Z-primary.

$\square$

Proof of Theorem 5.1. Clearly $R^{\mathfrak{p}}$ has support on $V(\mathfrak{p})$ and Lemma 5.2 asserts that it has the SEP. Throughout this proof we will repeatedly use (i) and (ii). From Example 7 we know that $\operatorname{ann} R^{(0)} = \ker F_0$ is (0)-primary. Now, suppose that $\mathfrak{p} \neq (0)$ and let $k = \operatorname{codim} \mathfrak{p}$. Since $R_k^{\mathfrak{p}}$ is a current of bidegree $(0,k)$ and $V(\mathfrak{p})$ is an irreducible variety of codimension k, it follows from Proposition 4.1 that $\operatorname{ann} R_k^{\mathfrak{p}}$ is $\mathfrak{p}$-primary. Hence by Lemma 5.2, $\operatorname{ann} R^{\mathfrak{p}}$ is $\mathfrak{p}$-primary. This could also be seen using Remark 1.

Next, we show (5.5). By Lemma 5.2, for $\mathfrak{p} \neq (0)$ we have that $R^{\mathfrak{p}} = R^{\mathfrak{p}}|_{V(\mathfrak{p}) \setminus \bigcup_{\operatorname{codim} \mathfrak{r} > \operatorname{codim} \mathfrak{p}} V(\mathfrak{r})}$, which by the definition of $R^{\mathfrak{p}}$ is equal to $R|_{V(\mathfrak{p}) \setminus \bigcup_{\operatorname{codim} \mathfrak{r} > \operatorname{codim} \mathfrak{p}} V(\mathfrak{r})}$. Suppose that $\mathfrak{p}$ and $\mathfrak{q}$ are two associated prime ideals of the same codimension $k > 0$. Then, by Lemma 5.2, $R^{\mathfrak{p}} = R^{\mathfrak{p}}|_{V(\mathfrak{p}) \setminus V(\mathfrak{q})}$, since $\operatorname{codim}(V(\mathfrak{p}) \cap V(\mathfrak{q})) > k$. Moreover, in light of (5.4), this is equal to $R|_{\left(V(\mathfrak{p}) \setminus \bigcup_{\operatorname{codim} \mathfrak{r} > k} V(\mathfrak{r})\right) \setminus V(\mathfrak{q})}$. Hence,

$$R^{\mathfrak{p}} + R^{\mathfrak{q}} = R|_{\left(V(\mathfrak{p}) \setminus \bigcup_{\operatorname{codim} \mathfrak{r} > k} V(\mathfrak{r})\right) \setminus V(\mathfrak{q})} + R|_{V(\mathfrak{q}) \setminus \bigcup_{\operatorname{codim} \mathfrak{r} > k} V(\mathfrak{r})} = R|_{(V(\mathfrak{p}) \cup V(\mathfrak{q})) \setminus \bigcup_{\operatorname{codim} \mathfrak{r} > k} V(\mathfrak{r})},$$

and so

$$\sum_{\mathfrak{p} \in \operatorname{Ass} J,\ \mathfrak{p} \neq (0)} R^{\mathfrak{p}} = \sum_{k>0} \sum_{\operatorname{codim} \mathfrak{p} = k} R^{\mathfrak{p}} = \sum_{k>0} R|_{\bigcup_{\operatorname{codim} \mathfrak{p} = k} V(\mathfrak{p}) \setminus \bigcup_{\operatorname{codim} \mathfrak{r} > k} V(\mathfrak{r})} = R|_{\bigcup_{\mathfrak{p} \in \operatorname{Ass} J,\ \mathfrak{p} \neq (0)} V(\mathfrak{p})} = R,$$

since R has support on $V(J) = \bigcup_{\mathfrak{p} \in \operatorname{Ass} J,\ \mathfrak{p} \neq (0)} V(\mathfrak{p})$.

We need to show that $\operatorname{ann} R = \bigcap_{\mathfrak{p} \in \operatorname{Ass} J,\ \mathfrak{p} \neq (0)} \operatorname{ann} R^{\mathfrak{p}}$. Clearly if $R\varphi = 0$ then $R^{\mathfrak{p}}\varphi = 0$ if $\mathfrak{p} \neq (0)$ and so $\operatorname{ann} R \subset \bigcap_{\mathfrak{p} \in \operatorname{Ass} J,\ \mathfrak{p} \neq (0)} \operatorname{ann} R^{\mathfrak{p}}$. On the other hand if $R^{\mathfrak{p}}\varphi = 0$ for all associated prime ideals $\mathfrak{p} \neq (0)$ then by (5.5) $R\varphi = \sum_{\mathfrak{p} \in \operatorname{Ass} J, \mathfrak{p} \neq (0)} R^{\mathfrak{p}}\varphi = 0$ and we are done. $\square$

Example 8. Consider the ideal (z^2, zw) with the associated prime ideals $\mathfrak{p} = (z)$ and $\mathfrak{q} = (z, w)$, where $\mathfrak{q}$ is embedded. It is easy to see that

$$0 \to \mathcal{O}_x \xrightarrow{F_2} \mathcal{O}_x^2 \xrightarrow{F_1} \mathcal{O}_x,$$

where $F_1 = \begin{bmatrix} z^2 & zw \end{bmatrix}$ and $F_2 = \begin{bmatrix} w \\ -z \end{bmatrix}$ is (minimal) resolution of $\mathcal{O}_x/J$. Assume that the vector bundles in the corresponding complex (5.2) are equipped with trivial metrics. Then $R^{\mathfrak{p}} = [1/w]\bar\partial[1/z]$ and $R^{\mathfrak{q}} = \bar\partial[1/z^2] \wedge \bar\partial[1/w]$, see Example 2 in [4] or [19]. Thus we get the minimal primary decomposition

$$J = \operatorname{ann} R^{\mathfrak{p}} \cap \operatorname{ann} R^{\mathfrak{q}} = (z) \cap (z^2, w).$$

Let us also point out that the primary decomposition (5.6) in general depends on the choice of Hermitian metrics. Notice that $J = (z^2, z(w - az))$ for $a \in \mathbb{C}$. Thus if we make the same resolution and choice of metrics with respect to the coordinates $\zeta = z, \omega = w - az$, we obtain a residue current that gives the primary decomposition $J = (z) \cap (z^2, w - az)$, which is clearly different for different values of a. Now, since all minimal resolutions are isomorphic this new resolution is obtained from the original resolution with a choice of metrics. $\square$

Example 9. If J has no embedded primes, it is well known that the minimal primary decomposition is unique. In particular $R^{\mathfrak{p}}$ must be independent of the choice of metrics. This can be verified directly, since outside an exceptional set $\mathcal{O}_x^{\oplus r}/J$ is Cohen-Macaulay and in that case R is essentially canonical, compare to [4], Section 4. $\square$

Remark 2. [The semi-global case] Let $K \subset X$ be a Stein compact set, (that is, K admits a neighborhood basis in X consisting of Stein open subsets of X) and let J be a submodule of $\mathcal{O}(K)^{r_0}$, where $\mathcal{O}(K)$ is the ring of germs of holomorphic functions on K. Due to Proposition 3.3 in [4] J can be represented as the annihilator of a residue current as above. The ring $\mathcal{O}(K)$ is Noetherian precisely when $Z \cap K$ has a finite number of topological components for every analytic variety Z defined in a neighborhood of K, see [18]. In this case the arguments in this and the previous section go through and so we get a decomposition of the residue current analogous to the one in Theorem 5.1. $\square$

Example 10. Let $\mathcal{J}$ be a coherent subsheaf of a locally free analytic sheaf $\mathcal{O}(E_0)$ over a complex manifold X. From a locally free resolution of $\mathcal{O}(E_0)/\mathcal{J}$ we constructed in [4] a residue current R, whose annihilator sheaf is precisely $\mathcal{J}$. Let Z_k be the (intrinsically defined) set where the kth mapping in the resolution does not have optimal rank (compare to the proof of Lemma 5.2). Then R can be decomposed as $R = \sum_k {}^kR$, where ${}^kR = R|_{Z_k \setminus Z_{k+1}}$, so that $\mathcal{J} = \bigcap_k \operatorname{ann} {}^kR$ and $\operatorname{ann} {}^kR$ is of pure codimension k (meaning that all its associated primes in each stalk are of codimension k). To see this it is enough to show that the germ of ${}^kR$ at $x \in X$ satisfies that ${}^kR = \sum_{\operatorname{codim} \mathfrak{p} = k} R^{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all associated prime ideals of $\mathcal{J}_x$. However, this can be verified following the proofs of Theorem 5.1 and Lemma 5.2. $\square$

6. The algebraic case

Let J be a submodule of $\mathbb{C}[z_1, \ldots, z_n]^r$ and suppose for simplicity that $\operatorname{codim} \mathbb{C}[z_1, \ldots, z_n]^r/J > 0$, that is, $(0) \notin \mathrm{A}ssJ$. From a free resolution of the corresponding homogeneous modules over the graded ring $\mathbb{C}[z_0, \ldots, z_n]$ we defined in [4] a residue current on $\mathbb{P}^n$ whose restriction R to $\mathbb{C}^n_z$ satisfies that $\varphi \in \mathbb{C}[z_1, \ldots, z_n]^r$ is in J if and only if $R\varphi = 0$. Propositions 4.1 and 5.3 hold with the same proof if Z is an irreducible algebraic variety in $\mathbb{C}^n$ and T is a current on $\mathbb{C}^n$ of finite order (in particular if it has an extension to $\mathbb{P}^n$). If we define the currents $R^{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{A}ssJ$ as in the local case, the proofs of Lemma 5.2 and Theorem 5.1 go through and we obtain the following version of Theorem 5.1.

Theorem 6.1. *Suppose that J is a submodule of $\mathbb{C}[z_1,\dots,z_n]^r$ such that $\mathbb{C}[z_1,\dots,z_n]^r/J$ has positive codimension and let R a residue current associated with J as above. Then for each $\mathfrak{p} \in AssJ$, $R^{\mathfrak{p}}$ has support on $V(\mathfrak{p})$ and has the SEP with respect to $V(\mathfrak{p})$, $ann\, R^{\mathfrak{p}} \subset \mathbb{C}[z_1,\dots,z_n]^r$ is $\mathfrak{p}$-primary,*

$$R = \sum_{\mathfrak{p}\in AssJ} R^{\mathfrak{p}},$$

and

$$J = ann\, R = \bigcap_{\mathfrak{p}\in AssJ} ann\, R^{\mathfrak{p}}$$

yields a minimal primary decomposition of J.

In [4] the residue currents for polynomial modules were used to obtain the following version of the Ehrenpreis-Palamodov fundamental principle: any smooth solution to the system of equations

$$\eta(i\partial/\partial t)\cdot\xi(t) = 0, \eta \in J \subset \mathbb{C}[z_1,\dots,z_n]^r \tag{6.1}$$

on a smoothly bounded convex set in $\mathbb{R}^n$ can be written

$$\xi(t) = \int_{\mathbb{C}^n} R^T(\zeta)A(\zeta)e^{-i\langle t,\zeta\rangle},$$

for an appropriate explicitly given matrix of smooth functions A. Here R^T is (the transpose of) the residue current associated with J as above. Conversely, any $\xi(t)$ given in this way is a homogeneous solution since $J = \operatorname{ann} R$. Now, for each $\mathfrak{p} \in \mathrm{A}ssJ$ let

$$\xi^{\mathfrak{p}}(t) = \int_{\mathbb{C}^n} (R^{\mathfrak{p}})^T(\zeta)A(\zeta)e^{-i\langle t,\zeta\rangle},$$

where $R^{\mathfrak{p}}$ is defined above. Then by Theorem 6.1 $\xi = \sum \xi^{\mathfrak{p}}$. Moreover $\xi^{\mathfrak{p}}$ satisfies $\eta(i\partial/\partial t)\cdot\xi^{\mathfrak{p}} = 0$ for each $\eta \in \operatorname{ann} R^{\mathfrak{p}}$. Hence we get a decomposition of the space of solutions to (6.1) with respect to $\mathrm{A}ssJ$.

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