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DECOMPOSITION OF UNBOUNDED DERIVATIONS IN OPERATOR ALGEBRAS

SCHÔICHI ÔTA

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1. Introduction. The theory of unbounded derivations in operator algebras has been recently investigated by many authors (see for complete references, [10]), since the infinitesimal generators of the one-parameter groups of automorphisms in quantum dynamical systems are in general unbounded derivations. There are many examples of derivations which are not generators of dynamical systems and hence it may be important to study the property of unbounded derivations in C^*-algebras. Since a derivation in a C^*-algebra is extended to one in its enveloping von Neumann algebra, we shall mainly study derivations in von Neumann algebras.

In this paper we show that every (unbounded) $*$-derivation in a von Neumann algebra is decomposed into the sum of the normal part and the singular part, by using an algebra on an indefinite inner product space which is induced by the derivation.

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2. Preliminary results. We begin this section by giving the definition of derivations and introducing some notations.

By a *derivation* in a C^*-algebra $\mathfrak{A}$ (resp. a von Neumann algebra $\mathfrak{M}$), we mean a linear mapping δ of the domain $\mathscr{D}(\delta)$, which is a norm-dense (resp. σ-weakly dense) $*$-subalgebra of $\mathfrak{A}$ (resp. $\mathfrak{M}$), into $\mathfrak{A}$ (resp. $\mathfrak{M}$) such that

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for each a, b in $\mathscr{D}(\delta)$. A derivation δ is called a $*$-derivation if $\delta(a^*) = \delta(a)^*$ holds for each a in $\mathscr{D}(\delta)$. Since every derivation can be expressed in the form $\delta_1 + i\delta_2$, where δ_1 and δ_2 are $*$-derivations. We shall only discuss $*$-derivations. It is well known that a derivation δ in a C^*-algebra $\mathfrak{A}$ with $\mathscr{D}(\delta) = \mathfrak{A}$ is necessarily norm-continuous and is also extended to a σ-weakly continuous derivation on the enveloping von Neumann algebra $\mathfrak{A}^{**}$. Let δ be a $*$-derivation in a C^*-algebra $\mathfrak{A}$ and let π be a non-degenerate $*$-representation of $\mathfrak{A}$ on a Hilbert space such that $\delta(\ker \pi \cap \mathscr{D}(\delta)) \subset \ker \pi$; that is, $\delta(a) \in \ker \pi$ for each $a \in \ker \pi \cap \mathscr{D}(\delta)$. Then δ is extended to a $*$-derivation in the weak closure $\overline{\pi(\mathfrak{A})}^{w}$ of $\pi(\mathfrak{A})$. In fact, we define δ_π on $\pi(\mathscr{D}(\delta))$ by $\delta_\pi(\pi(a)) \equiv \pi(\delta(a))$ for each $a \in \mathscr{D}(\delta)$. It is easily seen that δ_π is well-defined and is a $*$-derivation in the weak closure $\overline{\pi(\mathfrak{A})}^{w}$ of $\pi(\mathfrak{A})$.

Suppose that $\mathfrak{M}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and suppose that δ is a $*$-derivation in $\mathfrak{M}$. We denote by $G(\delta)$ the graph of δ in $\mathfrak{M} \oplus \mathfrak{M}$; that is, $G(\delta) = \{\{a, \delta(a)\}; a \in \mathscr{D}(\delta)\}$.

DEFINITION 2.1. A $*$-derivation δ in $\mathfrak{M}$ is said to be *σ-weakly closed* if the graph $G(\delta)$ in $\mathfrak{M} \oplus \mathfrak{M}$ is closed with respect to the σ-weak (operator) topology.

We may analogously introduce the concepts of σ-strong $*$ and σ-strong closedness. Since the graph of δ is a convex set in the von Neumann direct sum $\mathfrak{M} \oplus \mathfrak{M}$, the σ-weak closedness is equivalent to them. It is well known that the infinitesimal generator of a σ-weakly continuous one-parameter group of $*$-automorphisms of a von Neumann algebra is σ-weakly closed. For details, the reader is referred to (cf. [2]).

Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and δ be a $*$-derivation in $\mathfrak{M}$. Suppose that there exists a densely defined symmetric operator h in $\mathscr{H}$ which implements δ, namely $a\mathscr{D}(h) \subset \mathscr{D}(h)$ for all a in $\mathscr{D}(\delta)$ and $\delta(a)\xi = i[h, a]\xi$ $(= i(ha - ah)\xi)$ for all a in $\mathscr{D}(\delta)$ and all ξ in $\mathscr{D}(h)$. It is easily seen that such a derivation is σ-weakly closable.

If a net $\{a_i\}$ converges to a with respect to the σ-weak topology, then we simply denote this convergence by $a_i \to a$ $(\sigma - w)$. The following proposition is more or less known.

PROPOSITION 2.2. *Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and let δ be a σ-weakly closable $*$-derivation in $\mathfrak{M}$. Then the σ-weak closure (the minimal σ-weakly closed extension) of δ as a linear mapping is also a $*$-derivation in $\mathfrak{M}$.*

PROOF. Let $\tilde{\delta}$ be the σ-weak closure of δ. Take elements a in $\mathscr{D}(\tilde{\delta})$ and b in $\mathscr{D}(\delta)$. Then there exists a net $\{a_i\}$ in $\mathscr{D}(\delta)$ such that $a_i \to a$ $(\sigma - w)$ and $\delta(a_i) \to \tilde{\delta}(a)$ $(\sigma - w)$. Since $ba_i \to ba$ $(\sigma - w)$ and $\delta(ba_i) = \delta(b)a_i + b\delta(a_i) \to \delta(b)a + b\tilde{\delta}(a)$ $(\sigma - w)$, we have $ba \in \mathscr{D}(\tilde{\delta})$ and $\tilde{\delta}(ba) = \delta(b)a + b\tilde{\delta}(a)$ for each $a \in \mathscr{D}(\tilde{\delta})$ and $b \in \mathscr{D}(\delta)$. Next for each $a, b \in \mathscr{D}(\tilde{\delta})$, there exists a net $\{b_\lambda\}$ in $\mathscr{D}(\delta)$ such that $b_\lambda \to b$ $(\sigma - w)$ and $\delta(b_\lambda) \to \tilde{\delta}(b)$ $(\sigma - w)$. By the same argument, it is easily seen that $ab \in \mathscr{D}(\tilde{\delta})$ and $\tilde{\delta}(ab) = \tilde{\delta}(a)b + a\tilde{\delta}(b)$, which implies our proposition.

DEFINITION 2.3. Let δ be a $*$-derivation in a von Neumann algebra $\mathfrak{M}$. We define the set $\mathscr{I}(\delta; \sigma - w)$ as follows:

$$\mathscr{I}(\delta; \sigma - w) = \{a \in \mathfrak{M}: \{0, a\} \in \overline{G(\delta)}^{\sigma - w}\},$$

where $\overline{G(\delta)}^{\sigma - w}$ denotes the σ-weak closure of $G(\delta)$.

We remark that a $*$-derivation δ in $\mathfrak{M}$ is σ-weakly closable if and only if $\mathscr{I}(\delta; \sigma - w) = \{0\}$.

LEMMA 2.4. *The set $\mathscr{I}(\delta; \sigma - w)$ is a σ-weakly closed two sided ideal of $\mathfrak{M}$.*

PROOF. Since $\mathscr{I}(\delta; \sigma - w)$ is obviously σ-weakly closed, we have only to show that $\mathscr{I}(\delta; \sigma - w)$ is a two sided ideal of $\mathfrak{M}$. Take elements a in $\mathfrak{M}$ and b in $\mathscr{I}(\delta; \sigma - w)$. Since $\mathscr{D}(\delta)$ is σ-weakly dense in $\mathfrak{M}$ there is a net $\{a_\lambda\}$ in $\mathscr{D}(\delta)$ with $a_\lambda \to a$ $(\sigma - w)$. By the definition of $\mathscr{I}(\delta; \sigma - w)$ there is a net $\{b_i\}$ in $\mathscr{D}(\delta)$ such that $b_i \to 0$ $(\sigma - w)$ and $\delta(b_i) \to b$ $(\sigma - w)$. For each a_{λ_0}, we have $a_{\lambda_0} b_i \to 0$ $(\sigma - w)$ and $\delta(a_{\lambda_0} b_i) = a_{\lambda_0}\delta(b_i) + \delta(a_{\lambda_0})b_i \to a_{\lambda_0} b$ $(\sigma - w)$ since $a_{\lambda_0} b_i \in \mathscr{D}(\delta)$, we have $a_{\lambda_0} b \in \mathscr{I}(\delta; \sigma - w)$. It follows from the σ-weak closedness of $\mathscr{I}(\delta; \sigma - w)$ that $ab \in \mathscr{I}(\delta; \sigma - w)$. Similarly, we have $ba \in \mathscr{I}(\delta; \sigma - w)$, which completes the proof.

DEFINITION 2.5. Let δ be a $*$-derivation in a von Neumann algebra $\mathfrak{M}$. A $*$-derivation δ in $\mathfrak{M}$ is said to be *σ-singular* if for every a in $\overline{\mathscr{R}(\delta)}^{\sigma - w}$ (the σ-weak closure of the range $\mathscr{R}(\delta)$ of δ) there exists a net $\{a_\lambda\}$ in $\mathscr{D}(\delta)$ such that $a_\lambda \to 0$ $(\sigma - w)$ and $\delta(a_\lambda) \to a$ $(\sigma - w)$.

It is easy to see that a $*$-derivation δ in $\mathfrak{M}$ is σ-singular if and only if $\mathscr{I}(\delta; \sigma - w) = \overline{\mathscr{R}(\delta)}^{\sigma - w}$. We may define the singularity of $*$-derivations by using the σ-strong $*$ (resp. σ-strong) topology analogously. As we mentioned in the remark to Definition 2.1, the σ-singularity is equivalent to them. Lemma 2.4 gives the following:

COROLLARY 2.6. *Suppose that a von Neumann algebra $\mathfrak{M}$ is a factor. Then every $*$-derivation δ in $\mathfrak{M}$ is either σ-weakly closable or else σ-singular.*

REMARK 2.7. There exists a non-trivial σ-singular $*$-derivation in a von Neumann algebra. In fact, it is shown in [1] that there exists a non-norm-closable $*$-derivation δ_0 in a UHF-algebra $\mathfrak{A}$. Take a factor representation π of $\mathfrak{A}$ on a Hilbert space. Since π is faithful by the simplicity of $\mathfrak{A}$, we can define a $*$-derivation δ_π in the weak closure (von Neumann algebra) $\overline{\pi(\mathfrak{A})}^{w}$ of $\pi(\mathfrak{A})$ such that $\delta_\pi(\pi(a)) = \pi(\delta_0(a))$ for all $a \in$

$\mathscr{D}(\delta_0)$. Then δ_π is also non-norm-closable and so δ_π is not σ-weakly closable. It follows from Corollary 2.6 that δ_π is σ-singular.

PROPOSITION 2.8. *Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space and let δ be a σ-weakly closable $*$-derivation in $\mathfrak{M}$. If there exist a bounded $*$-derivation δ_1 in $\mathfrak{M}$ (i.e., $\mathscr{D}(\delta_1) = \mathfrak{M}$) and a σ-singular $*$-derivation δ_2 in $\mathfrak{M}$ such that $\delta = \delta_1 + \delta_2$, then $\delta_2 = 0$; that is, δ is bounded.*

PROOF. Take an element a in $\overline{\mathscr{R}(\delta_2)}^{\sigma-w}$. Then there exists a net $\{a_i\}$ in $\mathscr{D}(\delta_2)$ such that $a_i \to 0$ $(\sigma - w)$ and $\delta_2(a_i) \to a$ $(\sigma - w)$. Since δ_1 is σ-weakly continuous, we have

$$\delta_1(a_i) \to 0 \quad (\sigma - w) .$$

Hence we have $\delta(a_i) \to a$ $(\sigma - w)$. Since δ is σ-weakly closable we have $a = 0$. We complete the proof.

Now in connection with a $*$-derivation we introduce operator algebras on an indefinite inner product space, which will be used in the remainder of this paper.

Let $\mathscr{K}$ be a Hilbert space with the usual inner product $(\cdot, \cdot)$. We denote by $\mathscr{B}(\mathscr{K})$ the algebra of all bounded linear operators on $\mathscr{K}$. Let J_0 be a self-adjoint unitary operator on $\mathscr{K}$. We introduce an *indefinite inner product* induced by J_0 as follows:

$$[x, y]_{J_0} = (J_0 x, y)$$

for all $x, y \in \mathscr{K}$. With this indefinite metric, $\mathscr{K}$ is called a *J_0-space* denoted by $\{\mathscr{K}, J_0\}$. For $A \in \mathscr{B}(\mathscr{K})$, we denote by A^{J_0} the adjoint operation of A with respect to $[\cdot,\cdot]_{J_0}$. With $A \in \mathscr{B}(\mathscr{K})$, we write A^* for the usual adjoint operation of A with respect to $(\cdot,\cdot)$. It then is easily seen that

$$A^{J_0} = J_0 A^* J_0 .$$

Let $\mathfrak{B}$ be a subalgebra of $\mathscr{B}(\mathscr{K})$. If $\mathfrak{B}$ is an involutive algebra with this adjoint operation $A \to A^{J_0}$ as involution, then $\mathfrak{B}$ is said to be a *J_0-algebra* on $\{\mathscr{K}, J_0\}$. It is clear that $\mathscr{B}(\mathscr{K})$ is such an algebra. If $\mathfrak{C}$ is an involutive subalgebra of a J_0-algebra $\mathfrak{B}$ then $\mathfrak{C}$ is said to be a J_0-subalgebra of $\mathfrak{B}$. This involution $A \to A^{J_0}$ is called a J_0-involution.

Suppose that $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathscr{H}$ and δ is a $*$-derivation in $\mathfrak{A}$.

DEFINITION 2.9. We define the mapping of the domain $\mathscr{D}(\delta)$ into $\mathscr{B}(\tilde{\mathscr{H}})$ as follows:

$$\pi_\delta(a) = \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}$$

for $a \in \mathscr{D}(\delta)$, where $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}$ denotes the Hilbert direct sum of $\mathscr{H}$.

We write $\pi_\delta(\mathscr{D}(\delta))$ for the image of π_δ; that is, $\pi_\delta(\mathscr{D}(\delta)) = \{\pi_\delta(a): a \in \mathscr{D}(\delta)\}$.

LEMMA 2.10 [6, Proposition 2.1]. *Keep the same notations as in Definition 2.9. Let J be a self-adjoint unitary operator on $\tilde{\mathscr{H}}$ defined by $J(\xi \oplus \eta) = \eta \oplus \xi$ for $\xi, \eta \in \mathscr{H}$. Then $\pi_\delta(\mathscr{D}(\delta))$ is a J-algebra on $\{\tilde{\mathscr{H}}, J\}$.*

We remark that if δ is norm-closed then $\pi_\delta(\mathscr{D}(\delta))$ is a semi-simple Banach algebra with the operator-norm topology and the Hermitian involution mentioned above, and that δ is bounded if and only if $\pi_\delta(\mathscr{D}(\delta))$ is C^*-equivalent as a Banach involutive algebra ([6]).

3. Decomposition of unbouded derivations. In this section we shall discuss a decomposition of $*$-derivations in a von Neumann algebra into a σ-singular part and a normal part and show the following main theorem.

THEOREM 3.1. *Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space. Then every $*$-derivation δ in $\mathfrak{M}$ is decomposed into the sum*

$$\delta = \delta_1 + \delta_2$$

where δ_1 is a σ-weakly closable $*$-derivation in $\mathfrak{M}$ and δ_2 is a σ-singular $*$-derivation in $\mathfrak{M}$.

Before giving the proof of the theorem, we shall present several lemmas, which are concerned with J-algebras induced by $*$-derivations.

Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Let δ be a $*$-derivation in $\mathfrak{M}$. As we mentioned in Definition 2.9 and Lemma 2.10, we introduce the J-algebra $\pi_\delta(\mathscr{D}(\delta))$, which is induced by δ, on a J-space $\{\tilde{\mathscr{H}}, J\}$ where $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We note that a mapping π_δ is an involution-preserving isomorphism of $\mathscr{D}(\delta)$ into $\mathscr{B}(\tilde{\mathscr{H}})$; that is, $\pi_\delta(a^*) = \pi_\delta(a)^J$ for $a \in \mathscr{D}(\delta)$. In fact, we have

$$\pi_\delta(a)^J = J\begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}^* J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^* & 0 \\ \delta(a^*) & a^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^* & \delta(a^*) \\ 0 & a^* \end{pmatrix} = \pi_\delta(a^*) ,$$

for all $a \in \mathscr{D}(\delta)$. For simplicity we write $\mathfrak{A}_\delta \equiv \overline{\pi_\delta(\mathscr{D}(\delta))}^{\sigma-w}$ for the

σ-weak (operator) closure of the J-algebra $\pi_\delta(\mathscr{D}(\delta))$ in $\mathscr{B}(\tilde{\mathscr{H}})$. Then we have the following, which is more or less obvious.

LEMMA 3.2. *$\mathfrak{A}_\delta$ is a σ-weakly closed J-algebra on $\{\tilde{\mathscr{H}}, J\}$.*

Let a be an element in $\mathfrak{A}_\delta$. Suppose that a is expressed in matrix form as

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where each $a_{ij} \in \mathscr{B}(\mathscr{H})$ $(i, j = 1, 2)$. Then it is easy to show that $a_{11} = a_{22}$ and $a_{21} = 0$.

Keeping the same notations as above, we define a mapping ψ_δ of $\mathfrak{A}_\delta$ into $\mathscr{B}(\mathscr{H})$ by

$$\psi_\delta(a) = a_{11}$$

for $a \in \mathfrak{A}_\delta$, and we also define a mapping Δ of $\mathfrak{A}_\delta$ into $\mathscr{B}(\mathscr{H})$ by

$$\Delta(a) = a_{12}$$

for $a \in \mathfrak{A}_\delta$. Then, with these notations, each element a in $\mathfrak{A}_\delta$ is expressed in matrix form as

$$a = \begin{pmatrix} \psi_\delta(a) & \Delta(a) \\ 0 & \psi_\delta(a) \end{pmatrix}.$$

Then we have the following by simple computations.

LEMMA 3.3. *The following statements hold.*

1. *The mapping ψ_δ is a σ-weakly continuous homomorphism of $\mathfrak{A}_\delta$ into $\mathfrak{M}$ with $\psi_\delta(a^J) = \psi_\delta(a)^*$ for each $a \in \mathfrak{A}_\delta$.*

2. *The mapping Δ is a σ-weakly continuous linear mapping of $\mathfrak{A}_\delta$ into $\mathfrak{M}$ such that*

$$\Delta(ab) = \psi_\delta(a)\Delta(b) + \Delta(a)\psi_\delta(b)$$

and

$$\Delta(a^J) = \Delta(a)^*$$

for each $a, b \in \mathfrak{A}_\delta$.

We remark that the mapping $\psi_\delta \circ \pi_\delta$ is the identity mapping on $\mathscr{D}(\delta)$ and the mapping $\Delta \circ \pi_\delta$ is equal to δ.

We put $K = \{\Delta(a)\colon\ a \in \mathfrak{A}_\delta$ and $\psi_\delta(a) = 0\}$. Then we have $K = \mathscr{S}(\delta; \sigma - w)$. In fact, for each b in $\mathscr{S}(\delta; \sigma - w)$, there exists a net $\{b_\alpha\}$ in $\mathscr{D}(\delta)$ such that $b_\alpha \to 0$ $(\sigma - w)$ and $\delta(b_\alpha) \to b$ $(\sigma - w)$, and so we have

$$\pi_\delta(b_\alpha) \to \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \quad (\sigma - w)\ .$$

Thus $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \mathfrak{A}_\delta$. This shows that $b \in K$, hence $\mathscr{S}(\delta;\ \sigma - w) \subset K$. The reverse inclusion is also shown analogously. Hence it follows from Lemma 2.4 that K is a σ-weakly closed two sided ideal of $\mathfrak{M}$, which implies that there exists a central projection p of $\mathfrak{M}$ with $K = p\mathfrak{M}$. We put $q = 1 - p$.

Let a be an arbitrary element in $\mathfrak{A}_\delta$. Then a is decomposed into the sum $a = a_q + a_p$, where

$$(*) \qquad a_q = \begin{pmatrix} \psi_\delta(a) & q\Delta(a) \\ 0 & \psi_\delta(a) \end{pmatrix}$$

and

$$a_p = \begin{pmatrix} 0 & p\Delta(a) \\ 0 & 0 \end{pmatrix}.$$

We write $\mathfrak{A}_\delta(q)$ for the set of all a_q $(a \in \mathfrak{A}_\delta)$. Then we have the following:

LEMMA 3.4. *In the above decomposition, both a_q and a_p belong to $\mathfrak{A}_\delta$. Furthermore $\mathfrak{A}_\delta(q)$ is a σ-weakly closed J-subalgebra of $\mathfrak{A}_\delta$.*

PROOF. Take any a in $\mathfrak{A}_\delta$. By the definition of K, we have $a_p \in \mathfrak{A}_\delta$, so that $a_q = a - a_p \in \mathfrak{A}_\delta$. We next show that $\mathfrak{A}_\delta(q) = \{a \in \mathfrak{A}_\delta;\ p\Delta(a) = 0\}$. Suppose that a_q is an element in $\mathfrak{A}_\delta(q)$ and is represented as $(*)$ with $a \in \mathfrak{A}_\delta$. Since $\mathfrak{A}_\delta(q) \subset \mathfrak{A}_\delta$, we have

$$a_q = \begin{pmatrix} \psi_\delta(a_q) & \Delta(a_q) \\ 0 & \psi_\delta(a_q) \end{pmatrix}.$$

Hence $p\Delta(a_q) = p \cdot q\Delta(a) = 0$. The reverse inclusion is clear.

Since $a \in \mathfrak{A}_\delta \to p\Delta(a)$ is a σ-weakly continuous linear mapping by Lemma 3.3, $\mathfrak{A}_\delta(q)$ is σ-weakly closed.

Finally we show that $\mathfrak{A}_\delta(q)$ is a J-subalgebra of $\mathfrak{A}_\delta$. For $a, b \in \mathfrak{A}_\delta(q)$, we have, by Lemma 3.3, $p\Delta(ab) = p\Delta(a)\psi_\delta(b) + \psi_\delta(a)p\Delta(b) = 0$, and $p\Delta(a^J) = p\Delta(a)^* = (p\Delta(a))^* = 0$. Hence $ab \in \mathfrak{A}_\delta(q)$ and $a^J \in \mathfrak{A}_\delta(q)$. This shows that $\mathfrak{A}_\delta(q)$ is a J-subalgebra of $\mathfrak{A}_\delta$, and so completes the proof of the lemma.

PROOF OF THEOREM 3.1. Let δ be a $*$-derivation in a von Neumann algebra $\mathfrak{M}$ acting on a Hilbert space $\mathscr{H}$. Keep the same notations as in the previous three lemmas. Take an element a in $\mathfrak{A}_\delta$ with $\psi_\delta(a) = 0$. Then $\Delta(a) \in K$, so that $q\Delta(a) = 0$. Therefore we can define the linear mapping Δ_q of $\psi_\delta(\mathfrak{A}_\delta)$, which is the set of all $\psi_\delta(a)$ with $a \in \mathfrak{A}_\delta$, into $\mathfrak{M}$ as

$$\Delta_q(\psi_\delta(a)) = q\Delta(a)$$

for $a \in \mathfrak{A}_\delta$. Then we shall show that Δ_q is a σ-weakly closed $*$-derivation in $\mathfrak{M}$ with the domain $\psi_\delta(\mathfrak{A}_\delta)$. Since $\psi_\delta(\mathfrak{A}_\delta)$ contains the domain $\mathscr{D}(\delta)$ of δ, $\psi_\delta(\mathfrak{A}_\delta)$ is σ-weakly dense in $\mathfrak{M}$. For $a, b \in \mathfrak{A}_\delta$, we have, by Lemma 3.3, $\Delta_q(\psi_\delta(a)\psi_\delta(b)) = \Delta_q(\psi_\delta(ab)) = q\Delta(ab) = q\Delta(a)\psi_\delta(b) + \psi_\delta(a)q\Delta(b) = \Delta_q(\psi_\delta(a))\psi_\delta(b) + \psi_\delta(a)\Delta_q(\psi_\delta(b))$, and $\Delta_q(\psi_\delta(a)^*) = \Delta_q(\psi_\delta(a^J)) = q\Delta(a^J) = (q\Delta(a))^* = (\Delta_q(\psi_\delta(a)))^*$.

We now define a mapping of $\mathfrak{A}_\delta$ into the von Neumann direct sum $\mathfrak{M} \oplus \mathfrak{M}$ by

$$a = \begin{pmatrix} \psi_\delta(a) & \Delta(a) \\ 0 & \psi_\delta(a) \end{pmatrix} \rightarrow \{\psi_\delta(a), \Delta(a)\} \in \mathfrak{M} \oplus \mathfrak{M} .$$

It is easily seen that this mapping is a homeomorphism with respect to the σ-weak operator topology. Since $\mathfrak{A}_\delta(q)$ is σ-weakly closed by Lemma 3.4 and $G(\Delta_q) = \{\{b, \Delta_q(b)\}: b \in \psi_\delta(\mathfrak{A}_\delta)\} = \{\{\psi_\delta(a), q\Delta(a)\}: a \in \mathfrak{A}_\delta\}$, the graph $G(\Delta_q)$ of Δ_q is σ-weakly closed in $\mathfrak{M} \oplus \mathfrak{M}$. This shows that Δ_q is a σ-weakly closed $*$-derivation in $\mathfrak{M}$.

We write δ_q for the restriction of Δ_q to the domain $\mathscr{D}(\delta)$ of δ, and put $\delta_p = \delta - \delta_q$. Then $\delta_p(a) = p\delta(a)$ and $\delta_q(a) = q\delta(a)$ for each $a \in \mathscr{D}(\delta)$. Indeed, as we remarked in Lemma 3.3, for $a \in \mathscr{D}(\delta)$ we have $\delta_q(a) = \Delta_q(a) = \Delta_q(\psi_\delta(\pi_\delta(a))) = q\Delta(\pi_\delta(a)) = q\delta(a)$, and $\delta_p(a) = \delta(a) - \delta_q(a) = p\delta(a)$.

It is clear that δ_q is a σ-weakly closable $*$-derivation in $\mathfrak{M}$, and δ_p is a $*$-derivation in $\mathfrak{M}$. Finally we shall verify that δ_p is σ-singular. Take an arbitrary element x in $\overline{\mathscr{R}(\delta_p)}^{\sigma-w}$. Since $\overline{\mathscr{R}(\delta_p)}^{\sigma-w} \subset \mathfrak{M}p = K$, there exists a in $\mathfrak{A}_\delta$ such that $x = \Delta(a)$ and $\psi_\delta(a) = 0$. Then there exists a net $\{x_\alpha\}$ in $\mathscr{D}(\delta)$ such that $\pi_\delta(x_\alpha) \rightarrow a$ $(\sigma - w)$; that is, $x_\alpha \rightarrow 0$ $(= \psi_\delta(a))$ $(\sigma - w)$ and $\delta(x_\alpha) \rightarrow \Delta(a) = x$ $(\sigma - w)$. Therefore $\delta_p(x_\alpha) = p\delta(x_\alpha) \rightarrow px = x$ $(\sigma - w)$, which implies that δ_p is σ-singular. This completes the proof of the theorem.

REMARK 3.5. It is easily shown, in the proof of Theorem 3.1, that the minimal σ-weakly closed extension of δ_q as a linear mapping is just equal to Δ_q, and that $\mathscr{S}(\delta; \sigma - w)$ coincides with the σ-weak closure $\overline{\mathscr{R}(\delta_p)}^{\sigma-w}$ of the range of δ_p.

REEMARK 3.6. Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and δ be a $*$-derivation in $\mathfrak{M}$ with $\mathscr{D}(\delta)$ containing the identity 1. Then the radical of $\mathfrak{A}_\delta$ is the set of all a in $\mathfrak{A}_\delta$ with $\psi_\delta(a) = 0$. In fact, if a belongs to the radical of $\mathfrak{A}_\delta$, then we have

$$0 = \lim_{n\to\infty} \|(a^J a)^n\|^{1/n} \geqq \lim_{n\to\infty} \|(\psi_\delta(a)^*\psi_\delta(a))^n\|^{1/n} = \|\psi_\delta(a)\|^2$$

and hence $\psi_\delta(a) = 0$. Suppose, conversely, that $\psi_\delta(a) = 0$. Then for any $b \in \mathfrak{A}_\delta$, we have

$$1_{\tilde{\mathscr{H}}} + ba = \begin{pmatrix} 1 + \psi_\delta(ba) & \Delta(ba) \\ 0 & 1 + \psi_\delta(ba) \end{pmatrix} = \begin{pmatrix} 1 & \psi_\delta(b)\Delta(a) \\ 0 & 1 \end{pmatrix},$$

where $1_{\tilde{\mathscr{H}}}$ denotes the identity operator on $\tilde{\mathscr{H}}$. Hence we have

$$(1_{\tilde{\mathscr{H}}} + ba)^{-1} = \begin{pmatrix} 1 & -\psi_\delta(b)\Delta(a) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \psi_\delta(1_{\tilde{\mathscr{H}}} - ba) & \Delta(1_{\tilde{\mathscr{H}}} - ba) \\ 0 & \psi_\delta(1_{\tilde{\mathscr{H}}} - ba) \end{pmatrix} = 1_{\tilde{\mathscr{H}}} + ba,$$

which implies that a belongs to the radical of $\mathfrak{A}_\delta$. Thus the radical of $\mathfrak{A}_\delta$ is the set of all a in $\mathfrak{A}_\delta$ with $\psi_\delta(a) = 0$.

Furthermore, in the same situation, δ is σ-weakly closable if and only if $\mathfrak{A}_\delta$ is semi-simple. This follows from the equality $K = \mathscr{S}(\delta; \sigma - w)$ immediately after Lemma 3.3.

In the above theorem, we have also shown that a unique central projection p with $\mathscr{S}(\delta; \sigma - w) = \mathfrak{M}p$ gives the decomposition in Theorem 3.1. In what follows, we denote this central projection by p_δ. We note that δ is σ-weakly closable if and only if $p_\delta = 0$. By $P(\delta)$ we denote the set of all central projections in $\mathfrak{M}$, each of which gives the decomposition of δ as mentioned in Theorem 3.1; that is, $\delta = \delta_{1-p} + \delta_p$ where $\delta_{1-p} = (1 - p)\delta$ is a σ-weakly closable $*$-derivation in $\mathfrak{M}$ and $\delta_p = p\delta$ is a σ-singular $*$-derivation in $\mathfrak{M}$. Such a decomposition of δ given by a central projection p in $P(\delta)$ is called a *canonical decomposition* of δ induced by p. It is clear that δ is σ-weakly closable if and only if $P(\delta) \ni 0$. It is also shown, by the definition of $P(\delta)$, that δ is σ-singular if and only if there exists p in $P(\delta)$ with $\delta = \delta_p$. The following lemma characterizes p_δ in $P(\delta)$.

LEMMA 3.7. *Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space and let δ be a $*$-derivation in $\mathfrak{M}$. Then the projection p_δ is a minimal projection in $P(\delta)$.*

PROOF. Take an arbitrary element p in $P(\delta)$. Since $p_\delta \in \mathscr{S}(\delta; \sigma-w)$, there exists a net $\{x_\lambda\}$ in $\mathscr{D}(\delta)$ such that $x_\lambda \to 0$ $(\sigma - w)$ and $\delta(x_\lambda) \to p_\delta$ $(\sigma - w)$. Therefore we have

$$\delta_{1-p}(x_\lambda) = (1 - p)\delta(x_\lambda) \to (1 - p)p_\delta \quad (\sigma - w).$$

Since δ_{1-p} is σ-weakly closable, $(1-p)p_\delta = 0$ which implies that $p_\delta \leqq p$. Thus p_δ is a minimal projection in $P(\delta)$.

In general we may not expect that the sum of two σ-weakly closable linear mappings in a von Neumann algebra is also σ-weakly closable, and hence we may not obtain the uniqueness of such a decomposition in Theorem 3.1.

PROPOSITION 3.8. *Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space and let δ be a $*$-derivation in $\mathfrak{M}$. If the domain $\mathscr{D}(\delta)$ of δ contains the center $\mathfrak{Z}$ of $\mathfrak{M}$, then the canonical decomposition of δ is unique.*

PROOF. Let p be an element in $P(\delta)$. Then we have only to show that $\delta_p(x) = \delta_{p_\delta}(x)$ for every $x \in \mathscr{D}(\delta)$. If $p = 0$, then δ is σ-weakly closable and hence $p_\delta = 0$. Therefore we may assume that p is a nonzero projection. Take y in $\mathscr{S}(\delta_p; \sigma - w)$. Then there exists a net $\{x_\lambda\}$ in $\mathscr{D}(\delta)$ such that $x_\lambda \to 0$ $(\sigma - w)$ and $\delta_p(x_\lambda) \to y$ $(\sigma - w)$. Hence $\delta(px_\lambda) = \delta(p)x_\lambda + p\delta(x_\lambda) = \delta(p)x_\lambda + \delta_p(x_\lambda) \to y$ $(\sigma - w)$. This implies that $y \in \mathscr{S}(\delta; \sigma - w)$. Now let x be an arbitrary element in $\mathscr{D}(\delta)$. Since δ_p is σ-singular, the range of δ_p is contained in $\mathscr{S}(\delta_p; \sigma - w)$, hence in $\mathscr{S}(\delta; \sigma - w)$. Thus $p_\delta\delta_p(x) = \delta_p(x)$. By Lemma 3.6, we have $\delta_{p_\delta}(x) = p_\delta\delta(x) = p_\delta p\delta(x) = \delta_p(x)$ for $x \in \mathscr{D}(\delta)$. This completes the proof.

To finish this section, we remark that every (unbounded) derivation in von Neumann algebras (or C^*-algebras) is extended to a norm-continuous module derivation of a Banach algebra $\mathfrak{A}$ to a Banach $\mathfrak{A}$-module. To see this, we recall some definitions [4]. Let $\mathfrak{A}$ be a Banach algebra and $\mathscr{X}$ be a Banach space. Then $\mathscr{X}$ is said to be a Banach $\mathfrak{A}$-module if it is a two sided module and both bilinear mappings $(a, x) \in \mathfrak{A} \times \mathscr{X} \to a \cdot x$, $x \cdot a \in \mathscr{X}$ are bounded. Furthermore a Banach $\mathfrak{A}$-module $\mathscr{X}$ is said to be a dual $\mathfrak{A}$-module if $\mathscr{X}$ is isometrically isomorphic to the dual space of a Banach space $\mathscr{X}_*$, and, for each $a \in \mathfrak{A}$, both mappings $x \in \mathscr{X} \to a \cdot x$, $x \cdot a \in \mathscr{X}$ are weak $*$-continuous.

Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and δ be a $*$-derivation in $\mathfrak{M}$. Keeping the same notations as in Lemmas 3.2 and 3.3, we define bilinear mappings of $\mathfrak{A}_\delta \times \mathfrak{M}$ to $\mathfrak{M}$ by

$$(a, m) \to a \cdot m = \psi_\delta(a)m \quad \text{and} \quad m \cdot a = m\psi_\delta(a) .$$

Then $\mathfrak{M}$ is a Banach $\mathfrak{A}_\delta$-module since, for all $a \in \mathfrak{A}_\delta$ and $m \in \mathfrak{M}$, we have $\|a \cdot m\| \leqq \|\psi_\delta(a)\| \, \|m\| \leqq \|a\| \, \|m\|$ and $\|m \cdot a\| \leqq \|a\| \, \|m\|$. We have by Lemma 3.3 the following:

PROPOSITION 3.9. *Let $\mathfrak{M}$ be a von Neumann algebra and δ be a*

$*$-derivation in $\mathfrak{M}$. Then δ is extended to a norm-continuous module derivation Δ of $\mathfrak{A}_\delta$ to a dual $\mathfrak{A}_\delta$-module $\mathfrak{M}$.

REFERENCES

[1] O. BRATTELI AND D. W. ROBINSON, Unbounded derivations of C^*-algebras, Comm. Math. Phys. 42 (1975), 253-268.

[2] O. BRATTELI AND D. W. ROBINSON, Unbounded derivations of von Neumann algebras, Ann. Inst. H. Poincare, Section A. 25 (1976), 139-164.

[3] J. DIXMIER, Les algèbres d'opérateurs dans l'espace Hilbertian, 2e edition, Gauthier-Villars, Paris, 1969.

[4] B. E. JOHNSON, Cohomology in Banach algebras, Mem. Amer. Math. Soc. 127 (1972).

[5] R. V. KADISON, Derivations of operator algebras, Ann. Math. 83 (1966), 280-293.

[6] S. ÔTA, Certain operator algbras induced by $*$-derivations in C^*-algebras on an indefinite inner product space, J. Functional Anal, 30 (1978), 238-244.

[7] S. ÔTA, Locally closedness of unbounded derivations in C^*-algebras, Proc. Amer. Math. Soc. 71 (1978), 222-226.

[8] J. R. RINGROSE, Cohomology of operator algebras, Lecture notes in Math. 247 (1972), 359-422.

[9] S. SAKAI, Derivations of W^*-algebras, Ann. Math. 83 (1966), 273-279.

[10] S. SAKAI, Recent developments in the theory of unbounded derivations in C^*-algebras, Lecture Notes, The U.S.-Japan Seminor on operator algebras, 1977, Los Angeles (Lecture Notes in Math. 650, Springer Verlag, Berlin, 1978).

[11] S. SAKAI, The theory of unbounded derivations in C^*-algebras, Lecture notes, Copenhagen and Newcastle, 1977.

[12] S. SAKAI, C^*-algebras and W^*-algebras, Springer-Verlag, Berlin/Heidelberg/New York, 1970.

DEPARTMENT OF MATHEMATICS
KYUSHU UNIVERSITY
FUKUOKA, 812
JAPAN