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**DECOMPOSITION THEOREMS FOR 3-CONVEX SUBSETS OF  
THE PLANE**

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## DECOMPOSITION THEOREMS FOR 3-CONVEX SUBSETS OF THE PLANE

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Let  $S$  be a 3-convex subset of the plane. If  $(\text{cl } S \sim S) \subseteq \text{int } (\text{cl } S)$  or if  $(\text{cl } S \sim S) \subseteq \text{bdry } (\text{cl } S)$ , then  $S$  is expressible as a union of four or fewer convex sets. Otherwise,  $S$  is a union of six or fewer convex sets. In each case, the bound is best possible.

1. Introduction. Let  $S$  be a subset of  $R^d$ . Then  $S$  is said to be 3-convex iff for every three distinct points in  $S$ , at least one of the segments determined by these points lies in  $S$ . Valentine [2] has proved that for  $S$  a closed, 3-convex subset of the plane,  $S$  is expressible as a union of three or fewer closed convex sets. We are interested in obtaining a similar decomposition without requiring the set  $S$  to be closed. The following definitions and results obtained by Valentine will be useful.

For  $S \subseteq R^d$ , a point  $x$  in  $S$  is a *point of local convexity* of  $S$  iff there is some neighborhood  $U$  of  $x$  such that, if  $y, z \in S \cap U$ , then  $[y, z] \subseteq S$ . If  $S$  fails to be locally convex at some point  $q$  in  $S$ , then  $q$  is called a *point of local nonconvexity* (lnc point) of  $S$ .

Let  $S$  be a closed, connected, 3-convex subset of the plane, and let  $Q$  denote the closure of the set of isolated lnc points of  $S$ . Valentine has proved that for  $S$  not convex, then  $\text{card } Q \geq 1$ ,  $Q$  lies in the convex kernel of  $S$ , and  $Q \subseteq \text{bdry } (\text{conv } Q)$ . An *edge* of  $\text{bdry } (\text{conv } Q)$  is a closed segment (or ray) in  $\text{bdry } (\text{conv } Q)$  whose endpoints are in  $Q$ . We define a *leaf* of  $S$  in the following manner: In case  $\text{card } Q \geq 3$ , let  $L$  be the line determined by an edge of  $\text{bdry } (\text{conv } Q)$ ,  $L_1, L_2$  the corresponding open halfspaces. Then  $L$  supports  $\text{conv } Q$ , and we may assume  $\text{conv } Q \subseteq \text{cl } (L_1)$ . We define  $W = \text{cl } (L_2 \cap S)$  to be a *leaf* of  $S$ . For  $2 \geq \text{card } Q \geq 1$ , constructions used by Valentine may be employed to decompose  $S$  into two closed convex sets, and we define each of these convex sets to be a *leaf* of  $S$ .

By Valentine's results, every point of  $S$  is either in  $\text{conv } Q$  or in some leaf  $W$  of  $S$  (or both), and every leaf  $W$  is convex. Moreover, Valentine obtains his decomposition of  $S$  by showing that for any collection  $\{s_i\}$  of disjoint edges of  $\text{bdry } (\text{conv } Q)$ , with  $\{W_i\}$  the corresponding collection of leaves,  $\text{conv } Q \cup (\bigcup W_i)$  is closed and convex.

Finally, we will use the following familiar definitions: For  $x, y$  in  $S$ , we say  $x$  *see*  $y$  *via*  $S$  iff the corresponding segment  $[x, y]$  lies in  $S$ . A subset  $T$  of  $S$  is *visually independent via*  $S$  iff for every

$x, y$  in  $T$ ,  $x$  does not see  $y$  via  $S$ .

Throughout the paper,  $S$  will denote a 3-convex subset of the plane,  $Q$  the closure of the set of isolated lnc points of  $\text{cl } S$ .

2. Preliminary lemmas. The following sequence of lemmas will be useful in obtaining the desired representation theorems. We begin with an easy result.

LEMMA 1.  $\text{Cl } S$  is 3-convex.

*Proof.* Let  $x, y, z$  be distinct points in  $\text{cl } S$  and select disjoint sequences  $(x_i), (y_i), (z_i)$  in  $S$  converging to  $x, y, z$  respectively. For each  $i$ , one of the corresponding segments is in  $S$ , and for one pair, say  $x$  and  $y$ , infinitely many of the segments  $[x_i, y_i]$  lie in  $S$ . Since these segments converge to  $[x, y]$ ,  $[x, y]$  lies in  $\text{cl } S$ .

The remaining lemmas are technical in nature. Lemmas 2, 3, and 4 reveal various pleasant features of  $\text{int}(\text{cl } S) \sim S$ , while 5 and 6 are concerned with lnc points of  $\text{cl } S$ .

LEMMA 2. If  $p \in \text{int}(\text{cl } S) \sim \ker(\text{cl } S) \neq \emptyset$ , then  $p \in S$ .

*Proof.* Since  $p \notin \ker(\text{cl } S)$ , there is some point  $x$  in  $\text{cl } S$  for which  $[x, p] \not\subseteq \text{cl } S$ . Moreover,  $x$  may be chosen in  $S$  (for if  $p$  saw every member of  $S$  via  $\text{cl } S$ , then  $p$  would see every member of  $\text{cl } S$  via  $\text{cl } S$  and  $p$  would lie in  $\ker(\text{cl } S)$ ).

There is a convex neighborhood  $N$  of  $p$ , no point of which sees  $x$  via  $\text{cl } S$ , with  $N \subseteq \text{int}(\text{cl } S)$ . For any  $s, t$  distinct points in  $N \cap S$ , necessarily  $[s, t] \subseteq S$  by the 3-convexity of  $S$ , so  $N \cap S$  is convex. Since  $N \subseteq \text{int}(\text{cl } S)$ ,  $p$  is interior to some triangle  $\text{conv}\{w, y, z\}$  with vertices belonging to  $N \cap S$ . Then since  $N \cap S$  is convex,  $\text{conv}\{w, y, z\} \subseteq S$ , and  $p \in S$ . In fact,  $p \in \text{int } S$ .

COROLLARY. If  $p \in \text{cl } S \sim S$ , then either  $p \in \text{bdry}(\text{cl } S)$  or  $p \in \ker(\text{cl } S)$  (or both).

LEMMA 3. Let  $T \neq \emptyset$  be the set of points  $p$  of  $\text{cl } S \sim S$  for which  $p \in \text{bdry}(\text{cl } S)$ . Then every connected component of  $T$  is either an isolated point of  $\text{cl } S \sim S$  or an interval. Moreover, there can be at most one isolated point, and all components of  $T$  lie on a common line.

*Proof.* If  $T$  is a singleton point, the result is immediate, so assume that  $T$  contains at least two distinct points  $x, y$ . Let  $L(x, y)$  denote

the line determined by these points. It is clear that not both  $x$  and  $y$  can be isolated in  $\text{cl } S \sim S$ , for otherwise, since  $x, y \in \text{int}(\text{cl } S)$ , it would be easy to select three points of  $S$  on  $L(x, y)$  visually independent via  $S$ .

Again using the 3-convexity of  $S$ ,  $L(x, y) \cap S$  has at most two components, and  $L(x, y) \cap T \subseteq \ker(\text{cl } S)$  has at most three components. By an earlier argument, at most one component of  $L(x, y) \cap T$  is an isolated point, and clearly each component is either an isolated point or an interval.

To complete the proof, it suffices to show that  $T \subseteq L(x, y)$ . Let  $t \in \text{int}(\text{cl } S) \sim L(x, y)$  to show  $t \notin T$ . Since  $L(x, y) \cap T$  contains at most one isolated point,  $L(x, y) \cap T$  contains at least one interval  $(r, s) \subseteq \text{int}(\text{cl } S)$ , and we may choose some point  $u$  in  $S$  for which  $(u, t)$  cuts  $(r, s)$ . Then select a convex neighborhood  $N$  of  $t$ ,  $N \subseteq \text{int}(\text{cl } S)$ , so that for every  $q$  in  $N$ ,  $(u, q)$  cuts  $(r, s)$ . By techniques similar to those used in the proof of Lemma 2,  $N \cap S$  is convex and  $t \in S$ . Hence  $t \notin T$  and  $T \subseteq L(x, y)$ .

LEMMA 4. *If  $\text{cl } S \sim S$  contains an interval  $(r, s)$  disjoint from  $\text{bdry}(\text{cl } S)$ , then every lnc point of  $\text{cl } S$  lies on  $L(r, s)$ .*

*Proof.* Assume that for some lnc point  $t$  of  $\text{cl } S$ ,  $t \notin L(r, s)$ . As in the proof of Lemma 3, choose a point  $u$  and a neighborhood  $N$  of  $t$  so that  $u$  sees no point of  $N \cap S$  via  $S$ . Since  $t$  is an lnc point of  $\text{cl } S$ ,  $N$  contains points  $v, w$  in  $S$  which are visually independent via  $S$ . Hence  $u, v, w$  are visually independent via  $S$ , a contradiction, and  $t$  must lie on  $L(r, s)$ .

LEMMA 5. *If  $p$  is in  $\ker(\text{cl } S)$  and  $q, r$  are in  $Q$ , then  $q \notin (p, r)$  (where  $p, q, r$  are distinct points).*

*Proof.* Assume, on the contrary, that the points are collinear, with  $p < q < r$ . Let  $L$  denote the line containing  $p, q, r$ ,  $L_1, L_2$  the corresponding open halfspaces. Since  $p \in \ker(\text{cl } S)$  and  $\text{cl } S$  is not convex, there must be some point  $x$  of  $\text{cl } S$  not on  $L$ , say in  $L_1$ . Our hypothesis implies that  $\text{cl } S$  is connected, so by [2], Corollary 1,  $r \in \ker(\text{cl } S)$ , and the triangle  $\text{conv}\{p, x, r\}$  has its boundary in  $\text{cl } S$ . It is easy to see that the closed, 3-convex set  $\text{cl } S$  is simply connected, so  $\text{conv}\{p, x, r\} \subseteq \text{cl } S$ . Thus since  $q$  is an lnc point for  $\text{cl } S$ , there must be some point  $y$  of  $\text{cl } S$  in  $L_2$ ,  $\text{conv}\{p, y, r\} \subseteq \text{cl } S$ , and  $q$  cannot be an lnc point for  $\text{cl } S$ , clearly impossible. Our assumption is false, and  $q \notin (p, r)$ .

COROLLARY. *No three members of  $Q$  are collinear.*

LEMMA 6. *If  $p \in \text{conv } Q$ ,  $q \in Q$ ,  $q \neq p$ , and  $W_1, W_2$  are leaves of  $\text{cl } S$  containing  $q$ , then  $W_1, W_2$  are in opposite closed halfspaces determined by  $L(p, q)$ .*

*Proof.* Clearly the hypothesis implies that  $\text{cl } S$  is connected and that  $\text{card } Q \geq 2$ . If  $\text{card } Q = 2$ , the result is an immediate consequence of an argument used by Valentine (Case 2, Theorem 3 of [2]), so we may assume that  $\text{card } Q \geq 3$ . Let  $r$  lie on the edge of  $\text{bdry}(\text{conv } Q)$  which defines  $W_1$ ,  $r \neq q$ . If  $r \in L(p, q) \equiv L$ , then by the definition of  $W_1$ , it is obvious that  $W_1$  is in one of the closed halfspaces determined by  $L$ , say  $\text{cl } L_1$ . Otherwise, without loss of generality, assume that  $r$  is in the open halfspace  $L_1$ . Clearly  $p$  and  $W_1$  are separated by  $L(r, q)$ . Now if any point  $x$  of  $W_1$  lay in  $L_2$ , then  $q$  would lie interior to the triangle  $\text{conv } \{p, x, r\} \subseteq \text{cl } S$ , and  $q$  could not be an lnc point for  $\text{cl } S$ , a contradiction. Hence  $W_1$  lies in  $\text{cl } L_1$  in either case.

Since  $W_1 \cup \text{conv } Q$  is convex (by Valentine's results) and  $q$  is an lnc point for  $\text{cl } S$ ,  $W_2$  necessarily contains points in  $L_2$ , and  $W_2 \subseteq \text{cl } L_2$ , finishing the proof.

3. **Decomposition theorems.** With the preliminary lemmas behind us, we begin to investigate conditions under which  $S$  may be represented as a union of four or fewer convex sets, dealing primarily with the case for  $(\text{cl } S \sim S) \subseteq \text{int}(\text{cl } S)$ .

The first theorem, allowing us to restrict attention to the case for  $\text{cl } S = \text{cl}(\text{int } S)$ , will be helpful later.

THEOREM 1. *If  $\text{cl } S \neq \text{cl}(\text{int } S)$ , then  $S$  is a union of two or fewer convex sets.*

*Proof.* Without loss of generality, assume  $S$  is connected, for otherwise the result is trivial. Let  $x \in S \sim \text{cl}(\text{int } S) \neq \emptyset$ , and let  $N$  be a convex neighborhood of  $x$  disjoint from  $\text{int } S$ . Since  $S$  is connected,  $x$  is not an isolated point of  $S$ , and it is clear that  $N \cap S$  contains at least one segment.

We examine the maximal segments of  $N \cap S$  (i.e., the segments which are not proper subsets of segments in  $N \cap S$ ). It is easy to show that  $N \cap S$  has at most two maximal segments, for otherwise, the 3-convexity of  $S$  together with the simple connectedness of  $\text{cl } S$  would yield an open region in  $\text{cl } S \cap N$ . Since by Lemma 3 the points of  $\text{int}(\text{cl } S) \sim S$  are collinear, this would imply that  $N \cap S$  has interior points, clearly impossible by our choice of  $N$ .

In case  $N \cap S$  has exactly two maximal segments, an argument similar to the one above may be used to show that any point of  $S$

lies on one of the corresponding lines, and  $S$  is a union of two segments (possibly infinite). If  $N \cap S$  has just one segment, let  $K_1$  denote a maximal convex subset of  $S$  containing it, and let  $K_2 \equiv \text{conv}(S \sim K_1)$ . Again using the facts that  $N$  contains no interior points of  $\text{cl } S$  and  $\text{cl } S$  is simply connected, it is not hard to show that  $K_2 \subseteq S$ , and  $S = K_1 \cup K_2$ , completing the proof.

Theorems 2 and 3 show that a decomposition is possible when  $(\text{cl } S \sim S) \subseteq \text{int}(\text{cl } S)$ . There are two cases to consider, depending on the cardinality of  $Q$ .

**THEOREM 2.** *If  $(\text{cl } S \sim S) \cap \text{bdry}(\text{cl } S) = \emptyset$ , and  $\text{card } Q = n$  for  $n$  an odd integer,  $n > 1$ , then  $S$  is expressible as a union of four or fewer convex sets.*

*Proof.* Clearly the hypothesis implies that  $\text{cl } S = \text{cl}(\text{int } S)$ . By the Corollary to Lemma 2,  $\text{cl } S \sim S \subseteq \text{ker}(\text{cl } S)$ , and by Lemma 3, every component of  $\text{cl } S \sim S$  is either an isolated point or an interval. Since  $\text{card } Q \geq 3$  and (by the corollary to Lemma 5) no three members of  $Q$  can be collinear, Lemma 4 implies that  $\text{cl } S \sim S$  cannot contain an interval. Hence  $\text{cl } S \sim S$  consists of exactly one isolated point  $p$  in  $\text{ker}(\text{cl } S)$ .

Select  $q \in Q$  in the following manner: If  $p \in \text{conv } Q$ , choose  $q \in Q$  so that the line  $L(p, q)$  contains no other member of  $Q$ . (Clearly this is possible since  $\text{card } Q$  is odd and no three members of  $Q$  are collinear.) If  $p \notin \text{conv } Q$ , let  $\{e_i: 1 \leq i \leq n\}$  denote the edges of  $\text{conv } Q$ ,  $\{E_i: 1 \leq i \leq n\}$  the corresponding lines, with  $\text{conv } Q$  in the closed halfspace  $\text{cl}(E_{i_1})$  for each  $i$ . Then  $p \in E_{i_2}$  for exactly one  $i$ , for otherwise, if  $p \in E_{i_2} \cap E_{j_2}$ , then  $\text{int } \text{conv}(\{p\} \cup e_1 \cup e_2)$  would contain an lnc point of  $\text{cl } S$ , clearly impossible since  $\{p\} \cup e_1 \cup e_2 \subseteq \text{ker}(\text{cl } S)$  and  $\text{conv}(\{p\} \cup e_1 \cup e_2) \subseteq \text{cl } S$ . Thus we may choose some  $q \in Q$  so that  $p \in \text{cl } E_{i_1}$  for each edge  $e_i$  containing  $q$ . Then  $(p, q)$  contains points of  $\text{conv } Q$ . Since all points of  $L(p, q) \cap \text{conv } Q$  are on the open ray at  $p$  emanating through  $q$ , Lemma 5 implies that  $L(p, q)$  contains no other members of  $Q$  (and in fact  $p$  cannot lie on any line  $E_i$ ).

To review, in either case we have chosen  $q$  in  $Q$  so that  $L(p, q)$  contains no other member of  $Q$  and  $(p, q)$  contains points of  $\text{conv } Q$ . Letting  $L_1, L_2$  denote distinct open halfspaces determined by  $L = L(p, q)$ , define  $A \equiv \text{cl}(S \cap L_1)$ ,  $B \equiv \text{cl}(S \cap L_2)$ . If  $W_1, W_2$  are leaves of  $\text{cl } S$  containing  $q$ , then by Lemma 6,  $W_1$  and  $W_2$  are in opposite closed halfspaces determined by  $L$ , say  $W_1 \subseteq \text{cl } L_1$ ,  $W_2 \subseteq \text{cl } L_2$ .

Let  $R_1, R_2$  denote opposite closed rays at  $p$ ,  $R_1 \cup R_2 = L$ , labeled so that  $q \in R_2$ . Each of  $R_1 \cap S$ ,  $R_2 \cap S$  is an interval by the 3-convexity of  $S$ . Points of  $R_1 \cap S$  necessarily lie in  $A \cap B$ , for otherwise

$R_1$  would contain an lnc point of  $\text{cl } S$ , clearly impossible. If there are any points of  $R_2 \cap S$  not in  $A \cap B$ , without loss of generality we may assume such points lie in  $W_1$  and hence in  $A \sim B$ . Then  $R_2 \cap S \subseteq A$ .

By Case 4 in Theorems 2 and 3 of [2],  $\text{cl}(S \sim W_2)$  is a union of two closed convex sets  $C_1, C_2$ , selected as in Valentine's proof. Since  $A = \text{cl}[\text{cl}(S \sim W_2) \cap L_1]$ ,  $A$  is the union of the two closed convex sets  $A_1, A_2$ , where  $A_i = \text{cl}(C_i \cap L_1)$ ,  $i = 1, 2$ . Moreover,  $(R_1 \cap S) \cup (p, q]$  lies in one of these sets, say  $A_1$ , and  $R_2 \sim (p, q]$  is either in  $A_1$  or in  $A_2$ .

Using an identical argument for  $B$  and  $\text{cl}(S \sim W_1)$ , we may write  $B$  as a union of two closed convex sets  $B_1, B_2$  with  $(R_1 \cap S) \cup (p, q]$  in  $B_1$ , and  $R_2 \sim (p, q]$  disjoint from  $B$ .

At last, define sets  $A'_1, A'_2, B'_1, B'_2$  in the following manner: If  $(R_2 \cap S) \sim (p, q] \subseteq A_2$ , let

$$\begin{aligned} A'_1 &\equiv A_1 \sim R_2, & A'_2 &\equiv A_2 \sim R_1, \\ B'_1 &\equiv B_1 \sim R_1, & B'_2 &\equiv B_2 \sim R_2. \end{aligned}$$

And if  $(R_2 \cap S) \sim (p, q] \subseteq A_1$ , let

$$\begin{aligned} A'_1 &\equiv A_1 \sim R_1, & A'_2 &\equiv A_2 \sim R_2, \\ B'_1 &\equiv B_1 \sim R_2, & B'_2 &\equiv B_2 \sim R_1. \end{aligned}$$

We assert that these are convex subsets of  $S$  whose union is  $S$ : Clearly each is a convex subset of  $S$ , and  $S \sim L$  is contained in their union. For  $(R_2 \cap S) \sim (p, q] \subseteq A_2$ ,  $R_2 \cap S \subseteq A'_2 \cup B'_1$ ,  $R_1 \cap S \subseteq A'_1$ . For  $(R_2 \cap S) \sim (p, q] \subseteq A_1$ ,  $R_2 \cap S \subseteq A'_1$ ,  $R_1 \cap S \subseteq B'_1$ . Hence in either case  $S \cap L$  is contained in the union of these sets, and  $S = A'_1 \cup A'_2 \cup B'_1 \cup B'_2$ , completing the proof of the theorem.

**THEOREM 3.** *If  $(\text{cl } S \sim S) \cap \text{bdry}(\text{cl } S) = \emptyset$  and  $\text{card } Q = n \geq 0$ , where  $n$  (possibly infinite) is not an odd integer greater than one, then  $S$  is expressible as a union of four or fewer convex sets.*

*Proof.* If  $S$  is not connected, the result is trivial. Otherwise, by Theorem 3 of Valentine [2],  $\text{cl } S$  may be expressed as a union of two or fewer closed convex sets  $A, B$ . Using Lemma 3, let  $L$  be a line containing  $\text{cl } S \sim S$ ,  $L_1, L_2$  the corresponding open halfspaces. Since  $S$  is 3-convex and  $A$  is convex,  $S \cap A$  is 3-convex, and hence  $(S \cap A) \cap L$  has at most two components, say  $C_1, C_2$ . Let  $R_1, R_2$  denote opposite rays on  $L$  with  $C_1 \subseteq R_1, C_2 \subseteq R_2$ .

Define

$$\begin{aligned} A_1 &\equiv (A \cap S \cap \text{cl } L_1) \sim R_1, \\ A_2 &\equiv (A \cap S \cap \text{cl } L_2) \sim R_2. \end{aligned}$$

Then  $A_1, A_2$  are convex subsets of  $S$  whose union is  $A \cap S$ .

Similarly define convex sets  $B_1, B_2$  whose union is  $B \cap S$ . Clearly  $S = A_1 \cup A_2 \cup B_1 \cup B_2$ , the desired result.

**COROLLARY.** *If  $(\text{cl } S \sim S) \cap \text{bdry}(\text{cl } S) = \emptyset$ , then  $S$  is expressible as a union of four or fewer convex sets. The number four is best possible.*

That the number four in the corollary is best possible is evident from Example 1.

**EXAMPLE 1.** Let  $S$  be the set in Figure 1, with  $p \in S$ . Then  $S$  is not expressible as a union of fewer than four convex sets.

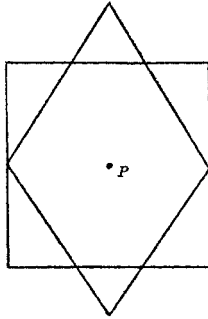


FIGURE 1

The preceding theorems allow us to obtain the following decomposition for open sets.

**THEOREM 4.** *If  $S$  is open, then  $S$  is expressible as a union of four or fewer convex sets. The result is best possible.*

*Proof.* Let  $T \equiv S \cup \text{bdry}(\text{cl } S)$ . Applying arguments identical to those used in the proofs of Theorems 2 and 3,  $T$  is expressible as a union of four or fewer convex sets  $A_i, 1 \leq i \leq 4$ . Define  $B_i \equiv A_i \cap S, 1 \leq i \leq 4$ . We assert that each  $B_i$  is convex. The proof follows:

By Valentine's results,  $\text{cl } S$  is expressible as a union of three or fewer closed convex sets  $C_j, 1 \leq j \leq 3$ , each consisting of an appropriate selection of leaves of  $\text{cl } S$ , together with  $\text{conv } Q$ . Examining the proofs of Theorems 2 and 3, it is clear that each  $A_i$  may be considered as a subset of some  $C_j$  set. Thus we may assume  $B_1 \subseteq C_1 \subseteq C_1$  for an appropriate  $C_1$ .

Let  $x, y \in B_1$ , and let  $p \in (x, y)$  to show  $p \in B_1$ . If  $x$  (or  $y$ ) is interior to some leaf  $W$ , then  $W \subseteq C_1, y$  sees a neighborhood of  $x$  via



$C_1$ , and  $p$  is interior to  $\text{cl } S$ . Since  $p \in A_1$  and  $p \in \text{bdry}(\text{cl } S)$ ,  $p$  is in  $A_1 \cap S = B_1$ . A similar argument holds if  $x$  (or  $y$ ) is interior to  $\text{conv } Q$ . Since neither  $x$  nor  $y$  is in  $\text{bdry}(\text{cl } S)$ , the only other possibility to consider is the case in which  $x, y \in \text{bdry}(\text{conv } Q) \sim Q \subseteq \ker(\text{cl } S)$ . Then  $x \in \text{int}(\text{cl } S)$ ,  $y \in \ker(\text{cl } S)$ ,  $y$  sees some neighborhood of  $x$  via  $\text{cl } S$ , and  $p \in \text{int}(\text{cl } S)$ . Again  $p \in A_1 \cap S = B_1$  and  $B_1$  is indeed convex. Thus  $S$  is the union of the convex sets  $B_i$ ,  $1 \leq i \leq 4$ , and the theorem is proved.

To see that the number four is best possible, let  $S$  denote the set in Example 1 with its boundary deleted. Then  $S$  is an open 3-convex set not expressible as a union of fewer than four convex sets.

4. The general case. It remains to investigate the case for  $S$  an arbitrary 3-convex subset of the plane. A decomposition of  $S$  into six convex sets may be obtained from our previous results, together with Theorems 5 and 6, which deal with the case for  $(\text{cl } S \sim S) \subseteq \text{bdry}(\text{cl } S)$ .

The following result by Lawrence, Hare, and Kenelly [1, Theorem 2] will be useful:

Lawrence, Hare, Kenelly Theorem. Let  $T$  be a subset of a linear space such that each finite subset  $F \subseteq T$  has a  $k$ -partition,  $\{F_1, \dots, F_k\}$ , where  $\text{conv } F_i \subseteq T$ ,  $1 \leq i \leq k$ . Then  $T$  is a union of  $k$  convex sets.

THEOREM 5. If  $\text{cl } S$  is convex and  $(\text{cl } S \sim S) \subseteq \text{bdry}(\text{cl } S)$ , then  $S$  is a union of three or fewer convex sets. The bound of three is best possible.

*Proof.* Consider the collection of all intervals in  $\text{bdry}(\text{cl } S)$  having endpoints in  $S$  and some relatively interior point not in  $S$ . Each interval determines a line  $L$ , and by the 3-convexity of  $S$ ,  $L \cap S$  has exactly two components. Let  $\mathcal{L}$  denote the collection of all such lines. By the Lawrence, Hare, Kenelly Theorem, without loss of generality we may assume that  $\mathcal{L}$  is finite. Hence the set  $\bigcup \{L \cap S: L \text{ in } \mathcal{L}\}$  has finitely many components, and we may order these components in a clockwise direction along  $\text{bdry}(\text{cl } S)$ . If  $c_i$  denotes the  $i$ th component in our ordering, let

$$\begin{aligned} A' &\equiv \{c_i: i \text{ odd}, i < n\}, \\ B' &\equiv \{c_i: i \text{ even}, i < n\}, \\ C' &\equiv \{c_n\}. \end{aligned}$$

Define

$$\begin{aligned} A &\equiv S \sim (B' \cup C'), \\ B &\equiv S \sim (A' \cup C'), \\ C &\equiv S \sim (A' \cup B'). \end{aligned}$$

We assert that  $A, B, C$  are convex sets whose union is  $S$ . The proof follows:

For  $x, y$  in  $A$ , if  $[x, y]$  contains any point of  $\text{int}(\text{cl } S)$ , then  $(x, y) \subseteq \text{int}(\text{cl } S) \subseteq A$ , and  $[x, y] \subseteq A$ . Otherwise,  $[x, y]$  lies in the boundary of the convex set  $\text{cl } S$ . If the corresponding line  $L(x, y)$  is not in  $\mathcal{L}$ , the result is clear, so suppose  $L(x, y) \in \mathcal{L}$ . Then  $x, y$  must lie in the same  $c_i$  set for some  $i$  odd,  $i < n$ , again giving the desired result. Hence  $A$  is convex. Similarly,  $B, C$  are convex. It is easy to see that  $A \cup B \cup C = S$  and the proof is complete.

The surprising fact that three is best possible is illustrated by Example 2.

EXAMPLE 2. Let  $S$  denote the set in Figure 2, where dotted lines represent segments not in  $S$ . Then  $S$  is not expressible as a union of fewer than three convex sets.

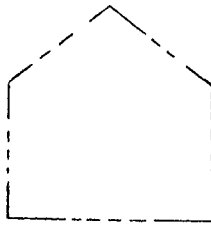


FIGURE 2

THEOREM 6. *If  $(\text{cl } S \sim S) \subseteq \text{bdry}(\text{cl } S)$ , then  $S$  is a union of four or fewer convex sets. The number four is best possible.*

*Proof.* We assume that  $S$  is connected and  $\text{cl } S = \text{cl}(\text{int } S)$ , for otherwise  $S$  is a union of two convex sets. Furthermore, by the Lawrence, Hare, Kenelly Theorem, we may assume that  $\text{cl } S$  has finitely many leaves, and hence  $\text{card } Q = n$  is finite. Notice also that since  $\text{cl } S$  is simply connected and  $(\text{cl } S \sim S) \subseteq \text{bdry}(\text{cl } S)$ ,  $S$  is simply connected.

For the moment, suppose  $3 \leq n$ . Order the points of  $Q$  in a clockwise direction along  $\text{bdry}(\text{conv } Q)$ , letting  $W_i$  denote the leaf of  $\text{cl } S$  determined by  $n$  consecutive points  $q_i, q_{i+1}$  (where  $n + 1 \equiv 1$ ). By Valentine's results in [2], for any pair of disjoint leaves  $W_i, W_j$  of  $\text{cl } S$ , the set  $R \equiv \text{conv } Q \cup W_i \cup W_j$  is a closed convex set. (In case there are no disjoint leaves,  $n = 3$ ,  $W_j = \emptyset$ , and  $R \equiv \text{conv } Q \cup W_i$  is closed and convex.) Consider the collection of intervals in  $\text{bdry } R$  having end-

points  $x, y$  in  $S$  and some relatively interior point  $p$  not in  $S$ . Either such an interval is contained in one leaf, or  $x \in W_i \cup \text{conv } Q$ ,  $y \in W_j \cup \text{conv } Q$ . We examine the latter case. It is clear that for an appropriate labeling,  $j = i + 2$ , so to simplify notation, say  $i = 1$ ,  $j = 3$ , and  $L(x, y)$  supports  $W_2$ . Clearly not both  $x, y$  can lie in  $\text{conv } Q$ , for then  $p \in \text{int } S \subseteq S$ . However, we assert that either  $x$  or  $y$  must lie in  $\text{conv } Q$  and that  $W_2 \cap S$  is convex. The proof follows:

Assume that  $x$  is not an lnc point and that  $x < p \leq q_2 < q_3$ , where  $q_2, q_3$  are the lnc points in  $W_1 \cap W_2$ ,  $W_2 \cap W_3$  respectively. Then  $q_2 \leq y$ . For  $w$  in  $W_2 \cap S$ ,  $w$  cannot see  $x$  via  $S$ , so necessarily  $w$  sees  $y$  via  $S$ , by the 3-convexity of  $S$ . This implies that  $y \leq q_3$  (for otherwise  $q_3$  could not be an lnc point for  $\text{cl } S$ ). Moreover, since no two points of  $W_2 \cap S$  see  $x$  via  $S$ , the 3-convexity of  $S$  together with the convexity of  $W_2$  imply that  $W_2 \cap S$  is convex.

Here we digress briefly for future reference. The set  $L(x, y) \cap S$  has two components, and by the above argument, one must lie in the interval  $[q_2, q_3]$ , the other in  $W_1 \sim Q$  (by our labeling). For general  $W_{i-1}, W_{i+1}$  (disjoint if and only if  $n > 3$ ), we let  $T_i$  denote the connected set of all the somewhat troublesome points  $y$  in  $[q_i, q_{i+1}] \cap S$  having the above property. That is, there exist points  $x$  in exactly one of  $(W_{i-1} \cap S) \sim Q$ ,  $(W_{i+1} \cap S) \sim Q$  for which  $[x, y] \not\subseteq S$  ( $n + 1 \equiv 1$ ).

Continuing the argument, delete  $W_2$  and consider the 3-convex set  $(S \sim W_2) \cup (S \cap L(x, y))$ . Renumber the lnc points and leaves for this set so that the old  $W_1$  and  $W_3$  are contained in the new leaf  $U_1$ . Since we are assuming  $\text{card } Q$  is finite, repeating the procedure finitely many times yields a 3-convex set  $S_0$  having the following property: For  $V_i, V_j$  disjoint leaves of  $\text{cl } S_0$ ,  $x$  in  $V_i \cap S_0$ ,  $y$  in  $V_j \cap S_0$ , then  $[x, y] \subseteq S_0$ . In addition, without loss of generality we may assume that for each leaf  $V_i$  of  $\text{cl } S_0$ ,  $V_i \cap S_0$  is not convex, for otherwise,  $V_i$  may be deleted by the above procedure.

To avoid confusion, let  $Q_0$  denote the set of lnc points of  $\text{cl } S_0$ ,  $Q_0 \subseteq Q$ ,  $\text{card } Q_0 = m \leq n$ . For  $3 \leq m$ , let  $V_i$  denote the leaf determined by lnc points  $p_i, p_{i+1}$  in  $Q_0$  (where  $p_{m+1} = p_1$ ). For  $m = 2$ , let  $V_1, V_2$  denote the leaves of  $\text{cl } S_0$  as defined in the introduction to this paper. If  $0 \leq m \leq 1$ , let  $V_1 = V_2 = \text{cl } S_0$ .

For each  $i$ , consider the collection of intervals in  $\text{bdry } V_i$  having endpoints in  $V_i \cap S_0$  and some relatively interior point not in  $S_0$ . Each interval determines a line  $L$ , and for  $m \neq 1$ ,  $L \cap V_i \cap S_0$  has exactly two components, each in  $\text{bdry } V_i$ . In case  $m = 1$ , an obvious adjustment may be made (by deleting any ray of  $L$  which contains interior points of  $\text{cl } S_0$ ) to yield the same result. For each  $i$ , let  $\mathcal{L}_i$  denote the collection of all such lines. Again using the Lawrence, Hare, Kenelly Theorem, we may assume that each  $\mathcal{L}_i$  is finite. The set  $\bigcup \{L \cap V_i \cap S_0 : L \text{ in } \mathcal{L}_i\}$  has finitely many components, and we

may order them in a clockwise direction along bdry  $V_i$ . Let  $c_{ij}$  denote the  $j$ th such component for  $V_i$ , and let  $\mathcal{C}_i$  denote the collection of all the  $c_{ij}$  sets corresponding to  $V_i$ . Clearly each  $c_{ij}$  is either a point, an interval, or the union of two noncollinear intervals. Moreover, for  $m \geq 2$ , no components for  $V_i, V_{i+1}$  may have common points. (Such a point would necessarily be  $p_{i+1}$ , and if  $s_i \in V_i \cap S_0, s_{i+1} \in V_{i+1} \cap S_0$  with some interior point of each of  $[s_i, p_{i+1}], [p_{i+1}, s_{i+1}]$  not in  $S_0$ , then  $s_i, p_{i+1}, s_{i+1}$  would be visually independent via  $S_0$ , clearly impossible.)

For each  $V_i$ , select every  $c_{i2j}$ . That is, select the members of  $\mathcal{C}_i$  having second subscript even. No two components selected correspond to the same line, and for  $m \neq 0$ , we have chosen one component corresponding to each line in  $\mathcal{L}_i$ . If  $m = 0$ , without loss of generality we may assume  $\mathcal{C}_1$  is ordered in a clockwise direction from some point in  $Q \cap \text{cl } S_0 \neq \emptyset$ . In case no component has been chosen for some line  $L$  in  $\mathcal{L}_1$ , then  $L$  must contain points of both the first and last members of  $\mathcal{C}_1$ , and by a previous argument, one of these components must lie in  $\text{conv } Q$ .

For  $m \neq 1$ , since  $V_i$  is convex, it is easy to show that  $\text{conv } \{c_{i2j}: 1 \leq j\}$  is a subset of  $S_0$  (and this is certainly true even if  $\text{cl } S_0$  is convex). We will prove that  $B_0 \equiv \text{conv } \{c_{i2j}: 1 \leq i \leq m, 1 \leq j\}$  is in  $S_0$  and hence in  $S$ . If  $\text{cl } S_0$  is convex (or empty) the result is immediate, so assume  $\text{cl } S_0$  has at least one lnc point. For convenience, in case  $\text{cl } S_0$  has only one lnc point, call it  $p_2$ , and let  $V_1 = V_2$  follow  $p_2$  in our clockwise ordering.

Recall that  $V_i \cap S_0$  is not convex for any  $i$ , so no  $\mathcal{C}_i$  is empty. Let  $c_0$  denote the last member of  $\mathcal{C}_1$  selected,  $x$  the last point of  $\text{cl } c_0$  (relative to our ordering). If  $x \neq p_2$ , let  $L = L(x, p_2)$ . Otherwise, by the 3-convexity of  $S_0$ ,  $c_0 = \{p_2\}$ , and in this case let  $L$  denote the corresponding member of  $\mathcal{L}_1$ . Let  $L_1, L_2$  be the open halfspaces determined by  $L$ , with  $Q_0 \subseteq \text{cl } L_1$ . Since  $p_2$  is an lnc point of  $S_0$  and  $S_0$  is 3-convex, it is clear that at most one member of  $\mathcal{C}_2$ , namely  $c_{21}$ , may contain points in  $L_2$ . We assert that  $c_0$  sees  $c_{22}$  via  $S_0$ . The proof follows:

In case  $L \in \mathcal{L}_1, L \cap V_1 \cap S_0$  has two components, each in bdry  $V_1$ , and one of these must be  $\{p_2\}$ . Then by the 3-convexity of  $S_0, c_{22} \subseteq L_1$  and  $c_0$  sees  $c_{22}$  via  $S_0$ . Otherwise,  $c_0 \sim \{x\} \subseteq L_1$ . If  $x \notin S_0$ , then since  $c_{22} \subseteq \text{cl } L_1$ , it is clear that  $c_0$  sees  $c_{22}$  via  $S_0$ . If  $x \in S_0$  and  $p_2 \in S_0$ , then again the result is clear. If  $x \in S_0$  and  $p_2 \notin S_0$ , then  $c_{22} \subseteq L_1$  and  $c_0$  sees  $c_{22}$  via  $S_0$ , finishing the argument.

In case  $V_1, V_2$  are the only leaves for  $\text{cl } S_0, V_1 \neq V_2$ , then repeating the argument for the last member of  $\mathcal{C}_2$  and  $c_{12}$  and using the fact that  $S_0$  is simply connected, we have  $B_0 \subseteq S_0 \subseteq S$ . (If  $V_1 = V_2$ , the result is immediate.) Otherwise,  $3 \leq m$  and an inductive argument may be used to show that  $B_0$  is in  $S$ .

Using Valentine's results, write  $\text{cl } S$  as a union of three or fewer convex sets  $A_j$ ,  $j = 1, 2, 3$ , where for  $n$  odd

$$\begin{aligned} A_1 &\equiv \bigcup \{W_i: i \text{ odd}, i < n\} \cup \text{conv } Q, \\ A_2 &\equiv \bigcup \{W_i: i \text{ even}, i < n\} \cup \text{conv } Q, \\ A_3 &\equiv W_n \cup \text{conv } Q, \end{aligned}$$

and for  $n$  even

$$\begin{aligned} A_1 &\equiv \bigcup \{W_i: i \text{ odd}, i \leq n\} \cup \text{conv } Q, \\ A_2 &\equiv \bigcup \{W_i: i \text{ even}, i \leq n\} \cup \text{conv } Q, \\ A_3 &= \emptyset. \end{aligned}$$

Define  $B_j \equiv S \cap [A_j \sim ((\text{bdry } S) \cap B_0)]$ ,  $j = 1, 2, 3$ .

Recall the  $T_i$  sets defined previously,  $T_i \subseteq [q_i, q_{i+1}] \subseteq W_i$ ,  $1 \leq i \leq n$ . To simplify notation, let  $L_i = L(q_i, q_{i+1})$ , and define sets  $F_i, G_i$  in the following manner: For  $i$  even, let  $F_i = T_i$  if points from both components of  $L_i \cap S$  are in  $B_1$ ,  $F_i = \emptyset$  otherwise. Similarly for  $i$  odd, let  $F_i = T_i$  if points from both components of  $L_i \cap S$  are in  $B_2$ ,  $F_i = \emptyset$  otherwise. For  $i = 1, i = n - 1$ , let  $G_i = T_i$  if points from both components of  $L_i \cap S$  are in  $B_3$ ,  $G_i = \emptyset$  otherwise. By previous remarks, at least one of  $G_1, F_1$  is empty, and at least one of  $G_{n-1}, F_{n-1}$  is empty.

Define

$$\begin{aligned} D_1 &\equiv B_1 \sim \bigcup \{F_i: i \text{ even}\}, \\ D_2 &\equiv B_2 \sim \bigcup \{F_i: i \text{ odd}\}, \\ D_3 &\equiv B_3 \sim \bigcup \{G_i, G_{n-1}\}. \end{aligned}$$

Finally, letting  $P = \{F_i \cap F_j: 1 \leq i < j \leq n\} \cup \{G_i \cap F_j: i = 1, n - 1, 1 \leq j \leq n\}$ , define  $D_0 \equiv \text{conv}(B_0 \cup P)$ . We assert that the sets  $D_j$ ,  $0 \leq j \leq 3$ , are convex sets whose union is  $S$ . The proof follows:

Suppose that one of the sets  $D_1, D_2, D_3$ , say  $D_1$ , is not convex to obtain a contradiction. Choose  $x, y$  in  $D_1$  for which  $[x, y] \not\subseteq D_1$ . It is clear that  $[x, y] \subseteq \text{bdry}(\text{cl } D_1) = \text{bdry } A_1$ . Furthermore,  $x, y$  cannot both belong to  $W \sim Q$  for any leaf  $W$  of  $\text{cl } S$ , for otherwise they would belong to the same leaf of  $\text{cl } S_0$ , and one of  $x, y$  would lie in  $(\text{bdry } S) \cap B_0$  and hence not in  $D_1$ , a contradiction. Employing a previous argument, the set  $L(x, y) \cap S$  has two components, each having points in  $B_1$ , and one of these components is the set  $[q_i, q_{i+1}] \cap S = T_i$  for some  $i$  even ( $n + 1 \equiv 1$ ). Let  $R_i$  denote the other component of  $L(x, y) \cap S$ . If  $R_i \cap B_0 \neq \emptyset$ , then  $R_i, T_i$  would lie on the boundary of a leaf of  $\text{cl } S_0$ ,  $R_i \subseteq B_0$ ,  $T_i \subseteq B_1$ , and  $[x, y] \subseteq T_i \subseteq D_1$ , a contradiction. Thus  $R_i \cap B_0 = \emptyset$  and  $R_i \subseteq D_1$ . However, this implies that one of  $x, y$  must lie in  $F_i$  and not in  $D_1$ , again a contradiction. Our assumption is false and  $D_1$  is convex. Similarly  $D_2, D_3$  are convex,

and clearly each is a subset of  $S$ .

It remains to show that the convex set  $D_0$  lies in  $S$ . Examining the set  $P$ , if  $F_i \cap F_j \neq \emptyset$  for some  $i \neq j$  (or if  $G_i \cap F_j \neq \emptyset$ ), then  $F_i = T_i$ ,  $F_j = T_j$ , for an appropriate labeling  $j = i + 1$ , and  $F_i \cap F_{i+1} = \{q_{i+1}\} \subseteq S$ . We will show that for each  $z$  in  $B_0$ ,  $[q_{i+1}, z] \subseteq S$ . The proof follows:

We have seen that  $W_i \cap S$ ,  $W_{i+1} \cap S$  are both convex, so for every  $z$  in one of these sets,  $[q_{i+1}, z] \subseteq S$ . Moreover, we assert that the components of  $L(q_i, q_{i+1}) \cap S$ ,  $L(q_{i+1}, q_{i+2}) \cap S$  not in  $\text{conv } Q$ , call them  $R_i, R_{i+1}$ , are disjoint from  $B_0$ : If  $R_i \cap B_0 \neq \emptyset$ , then by an earlier argument,  $R_i \subseteq B_0$ ,  $T_i \cap B_0 = \emptyset$ ,  $T_i \subseteq D_1 \cap D_2 \cap D_3$ , and  $F_i = \emptyset$ , a contradiction. Hence for  $z$  in  $B_0 \sim (W_i \cup W_{i+1})$ ,  $(q_{i+1}, z) \subseteq \text{int } S$ , and  $[q_{i+1}, z] \subseteq S$  whenever  $z \in B_0$ , the desired result.

Certainly for  $q_i, q_j, q_k$  in  $P \subseteq S$ ,  $\text{conv } \{q_i, q_j, q_k\} \subseteq S$ .

By Carathéodory's theorem in the plane, to prove that  $D_0 \subseteq \text{conv}(B_0 \cup P)$  is in  $S$ , it is sufficient to show that the convex hull of any three points of  $B_0 \cup P$  is in  $S$ , and from the remarks above, clearly we need only show  $\text{conv } \{q_i, q_j, z\} \subseteq S$  for  $q_i, q_j$  in  $P$ ,  $z$  in  $B_0$ . However, since  $S$  is simply connected and  $\text{bdry}(\text{conv } \{q_i, q_j, z\}) \subseteq S$ ,  $\text{conv } \{q_i, q_j, z\} \subseteq S$  and  $D_0 \subseteq S$ , the desired result.

Finally, by inspection, each  $F_i \neq \emptyset$  fails to belong to at most one of the sets  $D_1, D_2, D_3$ . Points in intersecting  $F_i$  sets are in  $D_0$ , so  $\bigcup \{D_j; 0 \leq j \leq 3\} = S$  and the argument for  $3 \leq \text{card } Q$  is complete.

To finish the proof, we must examine the cases for  $0 \leq \text{card } Q \leq 2$ . If  $\text{card } Q = 2$  or if  $\text{card } Q = 1$  and  $S \sim Q$  is connected, then let  $W_1, W_2$  denote the corresponding leaves of  $\text{cl } S$ , and use a simplified version of the previous proof to define  $B_0, B_1, B_2$ . If one of  $B_1, B_2$ , say  $B_1$ , is not convex, then letting  $T = W_1 \cap W_2 \cap S$ ,  $W_2 \cap S = B_2$  is convex,  $T \subseteq B_2$ , and  $B_0, B_1 \sim T, B_2$  are the desired convex sets.

In case  $\text{card } Q = 1$  and  $S \sim Q$  is not connected, then for  $W_1, W_2$  the corresponding leaves of  $\text{cl } S$ , each of  $W_1 \cap S, W_2 \cap S$  is convex. For  $\text{card } Q = 0$ , the result follows from Theorem 5, and the proof of Theorem 6 is complete.

The number four in Theorem 6 is best possible, as the following example illustrates.

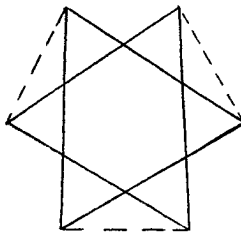


FIGURE 3

EXAMPLE 3. Let  $S$  denote the set in Figure 3, where dotted segments are in  $\text{bdry}(\text{cl } S) \sim S$ . Then  $S$  is a union of no fewer than four convex sets.

At last, using Theorem 6, we have a decomposition theorem for  $S$  an arbitrary 3-convex subset of the plane.

THEOREM 7. *The set  $S$  is a union of six or fewer convex sets. The result is best possible.*

*Proof.* By earlier comments, we may assume that  $S$  is connected,  $\text{cl } S = \text{cl}(\text{int } S)$ , and  $Q$  is finite. Furthermore, we assume  $\text{int}(\text{cl } S) \sim S \neq \emptyset$ , for otherwise the result is an immediate consequence of Theorem 6. Let  $T \equiv S \cup \text{bdry}(\text{cl } S)$ , and let  $L$  be the line containing  $\text{cl } T \sim T$  described in Theorem 2 or Theorem 3 (whichever is appropriate). Clearly  $L$  may be chosen to contain an lnc point  $q$  of  $\text{cl } S$ . If  $L_1, L_2$  are the corresponding open halfspaces, then each of  $T_1 \equiv \text{cl}(T \cap L_1) = \text{cl}(S \cap L_1)$ ,  $T_2 \equiv \text{cl}(T \cap L_2) = \text{cl}(S \cap L_2)$  is 3-convex.

Define  $S_i \equiv T_i \cap S$ ,  $i = 1, 2, \dots$ . We assert that each  $S_i$  is 3-convex: For  $x, y, z$  in  $S_1 = T_1 \cap S$ , assume  $[x, y]$  lies in the 3-convex set  $S$  to show  $[x, y] \subseteq S_1$ . If  $x$  or  $y$  is in  $L_1$ , then certainly  $(x, y) \subseteq L_1 \cap S \subseteq T_1$ , and  $[x, y] \subseteq S_1$ . If  $x, y$  are on  $L$ , then since no lnc points of the closed set  $T_1$  are on  $L$ ,  $x, y$  lie in the same leaf of  $T_1$ , and  $[x, y] \subseteq T_1 \cap S = S_1$ . Thus  $S_1$  is 3-convex. Similarly  $S_2$  is 3-convex. Moreover,  $(\text{cl } S_i \sim S_i) \subseteq \text{bdry}(\text{cl } S_i)$ ,  $i = 1, 2$ .

Using Theorem 6, we will show that each  $S_i$  is a union of three convex sets: By the proofs of Theorems 2 and 3,  $\text{cl } S_1 = T_1$  is a union of two convex sets  $A_1, A_2$ , and each  $A_i$  may be considered a subset of an appropriate  $C_j$  set,  $1 \leq j \leq 3$ , where the  $C_j$  sets are those described in Valentine's paper with  $\text{cl } S = C_1 \cup C_2 \cup C_3$ . In case  $T_1$  has one leaf or an even number of leaves, then clearly the proof of Theorem 6 may be used to write  $S_1$  as a union of three convex sets. If  $T_1$  has  $n$  leaves for  $n$  odd,  $n > 1$ , let  $V$  be the leaf of  $T_1$  bounded by  $L$ ,  $q \in Q \cap L \subseteq A_1 \cap A_2$ . Order the lnc points of  $T_1$  in a clockwise direction so that  $V$  is determined by  $q_n, q_1$ , and let  $U_n, U_{n+1}$  denote the closed subsets of  $V$  bounded by  $L(q_n, q)$ ,  $L(q, q_1)$  respectively. Treating  $U_1, \dots, U_n, U_{n+1}$  as leaves of  $T_1$ ,  $U_i$  determined by lnc points  $q_i, q_{i+1}$ ,  $1 \leq i < n$ , the proof of Theorem 6 may be applied to write  $S_1$  as a union of three convex sets. (Of course, in defining  $B_0$ , points of  $V$  in  $S_0$  belong to the same leaf of  $S_0$ .)

By a parallel argument  $S_2$  is a union of three convex sets, and  $S = S_1 \cup S_2$  is a union of six or fewer convex sets, finishing the proof of the theorem.

Our final example shows that the bound of six in Theorem 7 is

best possible.

EXAMPLE 4. Let  $S$  be the set in Figure 4, with dotted segments in  $\text{bdry}(\text{cl } S) \sim S$  and  $p \in \text{int}(\text{cl } S) \sim S$ . Then  $S$  cannot be expressed as a union of fewer than six convex sets.

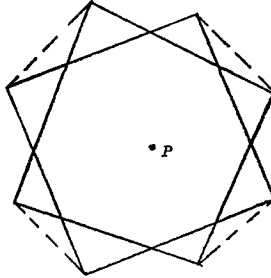


FIGURE 4

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