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Decompositions of $\tau_{\mathcal{G}}$-Continuity and Continuity

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Abstract. In this paper, we introduce and investigate the notion of weakly $\mathcal{G}$-locally closed sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.

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1 Introduction

The idea of grills on a topological space was first introduced by Choquet [6]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [5], [4], [18] for details). In [17], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [9] defined new classes of sets and obtained a new decomposition of continuity in terms of grills. In this paper, we introduce and investigate the notion of weakly $\mathcal{G}$-locally closed sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.

2 Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. The power set of X will be denoted by $\mathcal{P}(X)$. A subcollection $\mathcal{G}$ (not containing the empty set) of $\mathcal{P}(X)$ is called a grill [6] on X if $\mathcal{G}$ satisfies the following conditions:

1. $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,
2. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any point x of a topological space (X, τ), $\tau(x)$ denotes the collection of all open neighborhoods of x.

Definition 2.1. [17] *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \tau(x)\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ.*

Proposition 2.1. [17] *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$:*

1. *$A \subseteq B$ implies that $\Phi(A) \subseteq \Phi(B)$,*
2. *$\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,*
3. *$\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$.*

Let G be a grill on a space X. Then in [17] a map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ satisfies a Kuratowski closure axiom. Thus a subset A of X is $\tau_{\mathcal{G}}$-closed if $\Psi(A) = A$ or equivalently $\Phi(A) \subseteq A$. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}$-$Cl(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subseteq \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

Corollary 2.2. [17] *Let $(X, \tau, \mathcal{G})$ be a grill topological space and suppose $A, B \subseteq X$ with $B \notin \mathcal{G}$. Then $\Phi(A \cup B) = \Phi(A) = \Phi(A - B)$.*

Proposition 2.3. [17] *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$ with $A \subseteq \Phi(A)$. Then $Cl(A) = \Psi(A) = Cl(\Phi(A)) = \Phi(A)$.*

Lemma 2.4. [17] *Let $(X, \tau, \mathcal{G})$ be a grill topological space with $\tau - \phi \subseteq \mathcal{G}$. Then for all $U \in \tau$, $U \subseteq \Phi(U)$.*

Definition 2.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. A subset A in X is said to be*

1. *Φ-open* [9] *if $A \subseteq Int(\Phi(A))$,*

2. *$\mathcal{G}$-preopen* [9] *if $A \subseteq Int(\Psi(A))$.*

3 weakly $\mathcal{G}$-locally closed sets

A subset A of a topological space (X, τ) is said to be locally closed [3] if A is the intersection of an open set and a closed set. Locally closed sets are further investigated by Ganster and Reilly in [7]. It is easy to see that all open sets as well as all closed sets are locally closed. Recently Mandal and Mukherjee [13] introduced the notion of $\mathcal{G}$-locally closed sets as a new type of locally closed sets.

Definition 3.1. [13] *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-locally closed if $A = U \cap \Phi(A)$ for some $U \in \tau$.*

We now introduce a new type of locally closed sets called weakly $\mathcal{G}$-locally closed as follows:

Definition 3.2. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be weakly $\mathcal{G}$-locally closed (briefly weakly- $\mathcal{G}$-LC) if $A = U \cap V$, where U is open and V is $\tau_{\mathcal{G}}$-closed.*

Remark 3.1.

1. [13] Every $\mathcal{G}$-locally closed set in a grill topological space $(X, \tau, \mathcal{G})$ is locally closed. But the converse is false.

2. Every locally closed set in a grill topological space $(X, \tau, \mathcal{G})$ is weakly $\mathcal{G}$-locally closed. But the converse is false as is shown below.

Example 3.1. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $\mathcal{G} = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then $A = \{a, b\}$ is weakly $\mathcal{G}$-locally closed but it is not locally closed.

Proposition 3.1. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. Then the following properties hold:*

1. *If A is open, then A is weakly-$\mathcal{G}$-LC.*

2. *If A is $\tau_{\mathcal{G}}$-closed, then A is weakly-$\mathcal{G}$-LC.*

Proof. The proof is obvious. □

The converses of the statements in Proposition 3.1 need not be true as shown in the following example.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then

1. $A = \{b\}$ is a weakly-$\mathcal{G}$-LC set but it is not open.

2. $A = \{a\}$ is a weakly-$\mathcal{G}$-LC set but it is not $\tau_{\mathcal{G}}$-closed.

Theorem 3.2. *For a subset A of a grill topological space $(X, \tau, \mathcal{G})$, the following are equivalent:*

1. *A is open.*

2. *A is weakly-$\mathcal{G}$-LC and $\mathcal{G}$-preopen.*

Proof. (1) ⇒ (2): It is obvious since X is $\tau_{\mathcal{G}}$-closed.
(2) ⇒ (1): Let A be a weakly-$\mathcal{G}$-LC set and $\mathcal{G}$-preopen. Then, we have $A \subseteq Int(\Psi(A))$ and $A = U \cap V$, where $U \in \tau$ and V is $\tau_{\mathcal{G}}$-closed, respectively. Therefore, we have

$$\begin{aligned} A \subseteq & Int(\Psi(A)) \\ = & Int(\Psi(U \cap V)) \\ \subseteq & Int(\Psi(U) \cap \Psi(V)) \\ = & Int(\Psi(U)) \cap Int(\Psi(V)) \\ = & Int(\Psi(U)) \cap Int(V). \end{aligned}$$

Since $A = U \cap V$ and $A \subseteq U$, we have

$$\begin{aligned} A = & A \cap U \\ \subseteq & [Int(\Psi(U)) \cap Int(V)] \cap U \\ = & [Int(\Psi(U)) \cap U] \cap Int(V) \\ = & Int\,[\Psi(U) \cap U] \cap Int(V) \\ = & Int[U \cap V] = Int(A). \end{aligned}$$

Hence A is an open set. □

The notions of weakly-$\mathcal{G}$-LC sets and $\mathcal{G}$-preopen sets are independent as shown in the following examples.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then $A = \{b\}$ is a weakly-$\mathcal{G}$-LC set but it is not $\mathcal{G}$-preopen.

Example 3.4. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $A = \{a, b\}$ is $\mathcal{G}$-preopen but it is not weakly-$\mathcal{G}$-LC.

Theorem 3.3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A be a weakly-$\mathcal{G}$-LC subset of X. Then the following properties hold:*

1. *If B is a $\tau_{\mathcal{G}}$-closed set, then $A \cap B$ is a weakly-$\mathcal{G}$-LC set.*
2. *If B is an open set, then $A \cap B$ is a weakly-$\mathcal{G}$-LC set.*
3. *If B is a weakly-$\mathcal{G}$-LC set, then $A \cap B$ is a weakly-$\mathcal{G}$-LC set.*

Proof. (1) Let B be $\tau_{\mathcal{G}}$-closed, then $A \cap B = (U \cap V) \cap B = U \cap (V \cap B)$, where $V \cap B$ is $\tau_{\mathcal{G}}$-closed and U is open. Hence $A \cap B$ is weakly-$\mathcal{G}$-LC.
(2) Let B be open, then $A \cap B = (U \cap V) \cap B = (U \cap B) \cap V$, where $U \cap B$ is open and V is $\tau_{\mathcal{G}}$-closed. Hence $A \cap B$ is weakly-$\mathcal{G}$-LC.
(3) Let B be weakly-$\mathcal{G}$-LC, then $A \cap B = (U \cap V) \cap (F \cap G) = (U \cap F) \cap (V \cap G)$, where $U \cap F$ is open and $V \cap G$ is $\tau_{\mathcal{G}}$-closed. Hence $A \cap B$ is weakly-$\mathcal{G}$-LC. □

Definition 3.3. [12] *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then a subset A of X is said to be $\mathcal{G}$-g-closed if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*

Theorem 3.4. *A subset of a grill topological space $(X, \tau, \mathcal{G})$ is $\tau_{\mathcal{G}}$-closed if and only if it is weakly-$\mathcal{G}$-LC and $\mathcal{G}$-g-closed.*

Proof. Necessity is trivial. We prove only sufficiency. Let A be weakly-$\mathcal{G}$-LC and $\mathcal{G}$-g-closed. Since A is weakly-$\mathcal{G}$-LC, $A = U \cap V$, where U is open and V is $\tau_{\mathcal{G}}$-closed. So, we have $A = U \cap V \subseteq U$. Since A is $\mathcal{G}$-g-closed, $\Phi(A) \subseteq U$. Also $A = U \cap V \subseteq V$ and V is $\tau_{\mathcal{G}}$-closed, then $\Phi(A) \subseteq V$. Consequently, we have $\Phi(A) \subseteq U \cap V = A$ and hence A is $\tau_{\mathcal{G}}$-closed. □

The notions of weakly-$\mathcal{G}$-LC sets and $\mathcal{G}$-g-closed sets are independent.

Example 3.5. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{a, b\}$ is $\mathcal{G}$-g-closed but it is not weakly-$\mathcal{G}$-LC.

2. $A = \{c, d\}$ is a weakly-$\mathcal{G}$-LC set but it is not $\mathcal{G}$-g-closed .

Theorem 3.5. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. Then the following properties are equivalent:*

1. *A is weakly-$\mathcal{G}$-LC;*
2. *$A = U \cap \Psi(A)$ for some open set U;*
3. *$\Psi(A) - A = \Phi(A) - A$ is closed;*
4. *$A \cup [X - \Phi(A)] = A \cup [X - \Psi(A)]$ is open;*
5. *$A \subseteq Int[A \cup (X - \Phi(A))]$.*

Proof. (1) $\Rightarrow$ (2): If A is weakly-$\mathcal{G}$-LC, then there exist an open set U and a $\tau_{\mathcal{G}}$-closed set F such that $A = U \cap F$. Clearly, $A \subseteq U \cap \Psi(A)$. Since F is $\tau_{\mathcal{G}}$-closed, $\Psi(A) \subseteq \Psi(F) = F$ and so $U \cap \Psi(A) \subseteq U \cap F = A$. Therefore, $A = U \cap \Psi(A)$.
(2) $\Rightarrow$ (3): Now $\Phi(A) - A = \Phi(A) \cap (X - A) = \Phi(A) \cap [X - (U \cap \Psi(A))] = \Phi(A) \cap (X - U)$. Therefore, $\Psi(A) - A = \Phi(A) - A$ is closed.
(3) $\Rightarrow$ (4): Since $X - (\Phi(A) - A) = (X - \Phi(A)) \cup A$, then $[X - \Phi(A)] \cup A$ is open. Clearly, $A \cup [X - \Phi(A)] = A \cup [X - \Psi(A)]$.
(4) $\Rightarrow$ (5): It is clear.
(5) $\Rightarrow$ (1): $X - \Phi(A) = Int(X - \Phi(A)) \subseteq Int[A \cup (X - \Phi(A))]$ which implies that $A \cup [X - \Phi(A)] \subseteq Int[A \cup (X - \Phi(A))]$ and so $A \cup [X - \Phi(A)]$ is open. Since $A = [A \cup [X - \Phi(A)]] \cap \Psi(A)$, A is weakly-$\mathcal{G}$-LC. □

Remark 3.2. In a grill topological space $(X, \tau, \mathcal{G})$, if $A \subseteq \Phi(A)$ for every subset A of X, then every weakly-$\mathcal{G}$-LC set is $\mathcal{G}$-locally closed.

4 Strongly $\mathcal{G}$-locally closed sets

Definition 4.1. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be strongly $\mathcal{G}$-locally closed (briefly strongly-$\mathcal{G}$-LC) (resp. strongly-LC* [10]*) if $A = U \cap V$, where U is regular open and V is $\tau_{\mathcal{G}}$-closed (resp. closed).*

Proposition 4.1. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. Then the following properties hold:*

1. *If A is regular open, then A is strongly-$\mathcal{G}$-LC.*
2. *If A is $\tau_{\mathcal{G}}$-closed, then A is strongly-$\mathcal{G}$-LC.*

3. *If A is strongly-$\mathcal{G}$-LC, then A is weakly-$\mathcal{G}$-LC.*

The converses of the statements in Proposition 4.1 need not be true as shown in the following example.

Example 4.1. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then

1. $A = \{b\}$ is a strongly-$\mathcal{G}$-LC set but it is not regular open.
2. $A = \{a\}$ is a strongly-$\mathcal{G}$-LC set but it is not $\tau_{\mathcal{G}}$-closed.
3. $A = \{a, c\}$ is a weakly-$\mathcal{G}$-LC set but it is not strongly-$\mathcal{G}$-LC.

Theorem 4.2. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A be a strongly-$\mathcal{G}$-LC subset of X. Then the following properties hold:*

1. *If B is a $\tau_{\mathcal{G}}$-closed set, then $A \cap B$ is a strongly-$\mathcal{G}$-LC set.*
2. *If B is a regular open set, then $A \cap B$ is a strongly-$\mathcal{G}$-LC set.*
3. *If B is a strongly-$\mathcal{G}$-LC set, then $A \cap B$ is a strongly- $\mathcal{G}$-LC set.*

Definition 4.2. *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then a subset A of X is said to be $\mathcal{G}$-gr-closed if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*

Lemma 4.3. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. If A is $\mathcal{G}$-g-closed, then A is $\mathcal{G}$-gr-closed.*

Theorem 4.4. *For a subset A of a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:*

1. *A is $\tau_{\mathcal{G}}$-closed;*
2. *A is strongly-$\mathcal{G}$-LC and $\mathcal{G}$-g-closed;*
3. *A is strongly-$\mathcal{G}$-LC and $\mathcal{G}$-gr-closed.*

Proof. (1) $\Rightarrow$ (2): Obvious.
(2) $\Rightarrow$ (3): The proof follows from Lemma 4.3.
(3) $\Rightarrow$ (1): Let A be strongly-$\mathcal{G}$-LC and $\mathcal{G}$-*gr*-closed. Since A is strongly-$\mathcal{G}$-LC, $A = U \cap V$, where U is regular open and V is $\tau_{\mathcal{G}}$-closed. Since $A \subseteq U$ and A is $\mathcal{G}$-*gr*-closed, $\Phi(A) \subseteq U$. Since $A \subseteq V$ and V is $\tau_{\mathcal{G}}$-closed, $\Phi(A) \subseteq V$. Thus $\Phi(A) \subseteq U \cap V = A$. Hence A is $\tau_{\mathcal{G}}$-closed. $\square$

Remark 4.1.

1. The notions of strongly-$\mathcal{G}$-LC sets and $\mathcal{G}$-g-closed sets are independent.
2. The notions of strongly-$\mathcal{G}$-LC sets and $\mathcal{G}$-gr-closed sets are independent.

Example 4.2. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{b, c, d\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{b\}$ is $\mathcal{G}$-gr-closed but it is not strongly-$\mathcal{G}$-LC.
2. $A = \{a, b, c\}$ is $\mathcal{G}$-g-closed but it is not strongly-$\mathcal{G}$-LC.

Example 4.3. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \{\{b\}, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then

1. $A = \{a\}$ is strongly-$\mathcal{G}$-LC but it is not $\mathcal{G}$-g-closed.
2. $A = \{b\}$ is strongly-$\mathcal{G}$-LC but it is not $\mathcal{G}$-gr-closed.

5 Decompositions of $\tau_{\mathcal{G}}$-continuity and continuity

Definition 5.1. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\tau_{\mathcal{G}}$-continuous (resp. $\mathcal{G}$-g-continuous* [12]*, $\mathcal{G}$-gr-continuous, weakly $\mathcal{G}$-LC-continuous, strongly $\mathcal{G}$-LC-continuous) if $f^{-1}(A)$ is a $\tau_{\mathcal{G}}$-closed (resp. $\mathcal{G}$-g-closed, $\mathcal{G}$-gr-closed, weakly-$\mathcal{G}$-LC, strongly-$\mathcal{G}$-LC) set in $(X, \tau, \mathcal{G})$ for every closed set A of (Y, σ).*

Definition 5.2. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be g-continuous* [2] *(resp. gr-continuous* [15]*, strongly LC-continuous* [10]*, LC-continuous* [7]*) if $f^{-1}(A)$ is a g-closed, (resp. gr-closed, strongly-LC, locally closed) set in $(X, \tau, \mathcal{G})$ for every closed set A of (Y, σ).*

Definition 5.3. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be contra weakly $\mathcal{G}$-LC-continuous (resp. $\mathcal{G}$-precontinuous* [9]*) if $f^{-1}(A)$ is weakly-$\mathcal{G}$-LC (resp. $\mathcal{G}$-preopen) set in $(X, \tau, \mathcal{G})$ for every open set A of (Y, σ).*

Theorem 5.1. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is $\tau_{\mathcal{G}}$-continuous;*
2. *The inverse image of each open set in Y is $\tau_{\mathcal{G}}$-open;*

3. *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau_{\mathcal{G}}$ containing x such that $f(U) \subseteq V$;*

4. *$f : (X, \tau_{\mathcal{G}}) \to (Y, \sigma)$ is continuous.*

Theorem 5.2. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is continuous if and only if it is contra weakly $\mathcal{G}$-LC-continuous and $\mathcal{G}$-precontinuous.*

Proof. This is an immediate consequence of Theorem 3.2. □

Theorem 5.3. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is $\tau_{\mathcal{G}}$-continuous if and only if it is weakly $\mathcal{G}$-LC-continuous and $\mathcal{G}$-g-continuous.*

Proof. This is an immediate consequence of Theorem 3.4. □

Corollary 5.4. [14] *Let $(X, \tau, \mathcal{G})$ be a grill space and $\mathcal{G} = \mathcal{P}(X) \setminus \{\phi\}$. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is continuous if and only if it is LC-continuous and g-continuous.*

Theorem 5.5. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is $\tau_{\mathcal{G}}$-continuous;*

2. *f is strongly $\mathcal{G}$-LC-continuous and $\mathcal{G}$-g-continuous;*

3. *f is strongly $\mathcal{G}$-LC-continuous and $\mathcal{G}$-gr-continuous.*

Proof. This is an immediate consequence of Theorem 4.4. □

Corollary 5.6. *Let $(X, \tau, \mathcal{G})$ be a grill space and $\mathcal{G} = \mathcal{P}(X) \setminus \{\phi\}$. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is continuous;*

2. *f is strongly LC-continuous and g-continuous;*

3. *f is strongly LC-continuous and gr-continuous.*

6 Additions

The concept of ideals in topological spaces is treated in the classic text by Kuratowski [11] and Vaidyanathaswamy [19]. Janković and Hamlett [8] investigated further properties of ideal spaces. An ideal $\mathcal{I}$ on a topological

space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties: (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An ideal topological space or simply an ideal space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the local function of A with respect to $\mathcal{I}$ and τ [11]. We simply write A^* in case there is no chance for confusion. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$ called the $*$-topology, finer than τ, is defined by $Cl^*(A) = A \cup A^*$ [8].
The following lemma will be useful in the sequel.

Lemma 6.1. [16] *Let (X, τ) be a topological space. Then the following hold.*

1. *$\mathcal{G}$ is a grill on X if and only if $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$ is an ideal on X,*

2. *The operators Cl^* on $(X, \tau, \mathcal{I})$, where $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$, and Ψ on $(X, \tau, \mathcal{G})$ are equal.*

Remark 6.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X.

1. Since $\tau \subseteq \tau_{\mathcal{G}}$, then every strongly-LC set is strongly-$\mathcal{G}$-LC.

2. If $\mathcal{G}= \mathcal{P}(X) \setminus \{\phi\}$, then $\tau = \tau_{\mathcal{G}}$ and hence both the notions of strongly-$\mathcal{G}$-LC and strongly-LC are equal.

3. If $A \subseteq \Phi(A)$, then $Cl(A) = \tau_{\mathcal{G}}$-$Cl(A)$ and hence both the notions of strongly-$\mathcal{G}$-LC and strongly-LC are equal.

4. If $\mathcal{G}= \{X\}$, then $\Phi(A) = \phi$ for any subset A of X and $\Psi(A) = A$. Then any subset A of X is strongly-$\mathcal{G}$-LC.

5. For any subset A of a space X and any grill $\mathcal{G}$ on X, $\Phi(A)$ is $\tau_{\mathcal{G}}$-closed. Then if every open set is regular open, then every $\mathcal{G}$-locally closed set is strongly-$\mathcal{G}$-LC.

6. Let τ be suitable for $\mathcal{G}$, that is, $A - \Phi(A) \notin \mathcal{G}$ for all $A \subseteq X$ [[17], Definition 3.1] and $\tau - \{\phi\} \subseteq \mathcal{G}$. Then if $(X, \tau_{\mathcal{G}})$ is regular then $\tau = \tau_{\mathcal{G}}$ by Theorem 3.8 of [17] and hence both notions of strongly-$\mathcal{G}$-LC and strongly-LC are equal.

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References

[1] **A. Al-Omari and T. Noiri**, Decompositions of continuity via grills, *Jordan J. Math. Stat.*, **4 (1)**, (2011), 33–46

[2] **K. Balachandran and P. Sundaram and H. Maki**, On generalized continuous maps in topological spaces, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.*, **12**, (1991), 5–13

[3] **N. Bourbaki**, *General Topology, Part I*, Addition wesley, Reading Mass, 1966

[4] **K. C. Chattopadhyay and W. J. Thron**, Extensions of closure spaces, *Can. J. Math.*, **29 (6)**, (1977), 1277–1286

[5] **K. C. Chattopadhyay and O. Njåstad and W. J. Thron**, Merotopic spaces and extensions of closure spaces, *Can. J. Math.*, **35 (4)**, (1983), 613–629

[6] **G. Choqet**, Sur les notions de filter et grill, *Comptes Rendus Acad. Sci. Paris*, **224**, (1947), 171–173

[7] **M. Ganster and I. L. Reilly**, Locally closed sets and *LC*-continuous functions, *Internat. J. Math. Math. Sci.*, **12**, (1989), 417–424

[8] **D. Janković and T. R. Hamlet**, New topologies from old via ideals, *Amer. Math. Monthly*, **97**, (1990), 295–310

[9] **E. Hatir and S. Jafari**, On some new calsses of sets and a new decomposition of continuity via grills, *J. Adv. Math. Stud.*, **3 (1)**, (2010), 33–40

[10] **S. Krishnaprakash and M. Rajamani V. Inthumathi**, Strongly-$\mathcal{I}$-localy closed sets and decomposition of $*$-continuity, *Acta Math. Hungar.*, **130 (4)**, (2011), 358–362

[11] **K. Kuratowski**, *Topology I*, Academic Press, 1966

[12] **D. Mandal and M. N. Mukherjee**, On a type of generalized closed sets, *Bol. Soc. Paran. Mat.*, **30 (1)**, (2012), 67–76

[13] **D. Mandal and M. N. Mukherjee**, On some generalized types of sets and functions: certain decompositions, *J. Adv. Res. Pure Math.*, **4(1)**, (2012), 1-8.

[14] **T. Noiri and V. Popa**, A general decomposition of continuity, *Anal. Univ. Sci. Budapest*, **50**, (2007), 41–50

[15] **N. Palaniappan and K. C. Rao**, Regular generalized closed sets, *Kyungpook Math. J.*, **33**, (1993), 211–219

[16] **V. Renukadevi**, Relation between ideals and grills, *J. Adv. Res. Pure Math.*, **2(4)**, (2010), 9–14

[17] **B. Roy and M. N. Mukherjee**, On a typical topology induced by a grill, *Soochow J. Math.*, **33**, (2007), 771–786

[18] **W. J. Thron**, Proximity structure and grills, *Math. Ann.*, **206**, (1973), 35–62

[19] **R. Vaidyanathaswamy**, *Set Topology*, Chelsea Publishing Company, New York, 1960

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