Decorating Regular Tiles with Arcs

David A. Reimann
Department of Mathematics and Computer Science
Albion College
Albion, Michigan, 49224, USA
E-mail: dreimann@albion.edu

Abstract

This paper introduces a notation for describing a regular polygon tile decorated with arcs. Given a tile decorated with a number of arcs having endpoints uniformly spaced around the polygon, the number of possible decorated tiles is given, including the special case where no arcs intersect. The tiles decorated in this manner provide an enormous number of patterns.

1 Introduction

Tiles comprised of regular polygons (regular tiles) form the basis of many two-dimensional patterns, such as the Archimedean and more general k-uniform tilings. Tiles can also be decorated with a simple motif to produce more intricate patterns. For example, Truchet [1] explored the patterns obtainable from a single square tile that was bisected along a diagonal between opposite vertices. Browne [2] investigated patterns on regular polygons using arcs connecting midpoints of polygon sides. Reimann [3] investigated motifs on regular polygons where each side was subdivided into an equal number of segments and connected using Bézier curves. Making the divisions equal in length allows arcs from adjacent tiles in a tessellation to form continuous segments.

This paper introduces a notation for describing a regular tile decorated with arcs as in [3]. An expression for the number of tiles is developed for polygons with n sides and d divisions per side. In a minority of cases, a decoration will consists of arcs that do not cross. A separate expression is given for computing the number of patterns where there are no arc crossings.

2 Methods and Results

Given a regular polygon with n sides and d divisions per side. The product nd must be even to allow the uniqueness condition on the endpoints so that each arc connects two distinct endpoints. Each vertex can be assigned a unique integer $0,1,\ldots n-1$ in a clockwise fashion starting at the top of the triangle. Likewise, each endpoint can be assigned a unique integer $0,1,\ldots nd-1$ starting with the first side clockwise from the first vertex. Figure 1 shows an example of this with a square ($n=4$) that contains $d=1$ divisions per side.

A specific decoration can be fully described using the following notation:

$$(\alpha,\beta|\gamma,\delta|\varepsilon,\zeta|\cdots)$$

where $\alpha,\beta,\gamma,\ldots$ represent the endpoint number. There are exactly $m=nd/2$ pairs of endpoints representing a single arc separated using the '|' character all enclosed by parentheses. For a triangle ($n=3$) with two endpoints per side ($d=2$), the possibilities including rotationally equivalent decorations are shown along with the corresponding decoration in Figure 2. Note there are 15 possible decorations for this configuration with 10 containing crossings and 5 with no crossings.

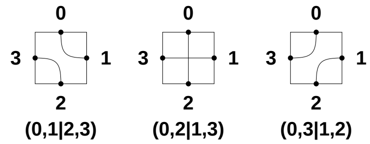

Figure 1: *The collection of all 3 possible decorations of a square with 1 division per side. The notation for each decorated square is described in the text. Note there are two unique geometric tiles.*

Since the arc endpoints lie along the perimeter of the polygon, these points and corresponding arcs can be mapped onto a circle, resulting in a chord diagram associated with the decoration. Riordan [4] states the number of distinct chord diagrams with m arcs is given by

$$f(m) = \frac{(2m)!}{2^m m!}$$

which is the Sloane sequence A001147 [5].

This result can be obtained using the following reasoning, which is the basis for generating all possible decorations. For each pair λ, μ representing an arc, list the points in increasing order so that $\lambda < \mu$. Furthermore, list the m pairs so that the first components of all pairs are increasing in order. The first number of the first pair must be 0, with $nd-1$ choices for the second number. In the second pair, the first number is the smallest remaining value, leaving $nd-3$ choices for the second number. The values in last pairing are fixed because of ordering. For $N = 2m = nd$ endpoints, this results in

$$f(N) = (N-1)(N-3)(N-5)\cdots(1) = \frac{N!}{N(N-2)(N-4)\cdots(2)} = \frac{N!}{(N/2)!2^{N/2}} = \frac{(2m)!}{2^m m!}.$$

The number of chord diagrams where there are no crossings is given by Errara [6] as

$$g(m) = \frac{(2m)!}{m!(m+1)!}$$

which is just the Catalan numbers. Algorithmically, one can verify a decoration is crossing free by considering the relationship between endpoints of the arcs, namely pairs of endpoints should nest. A recursive procedure can be used to construct a crossing free decoration using the fact that an arc will partition the set of endpoints into two distinct subsets. If each subset contains an even number of points, the subset can then be recursively split until the subset contains only two endpoints. Values for the functions f and g are given in Table 1. Similar logic can be used to see that all tiles decorated in this manner are all 2-colorable.

Note the number of tiles is the same for any combination of m arcs. For example triangles with 4 divisions, squares with 3 divisions, hexagons with two divisions, and dodecagons with 1 division will have the same number of tile possibilities because $m = 6$ for each of these situations. However, the number of geometrically unique tiles is different because the underlying symmetries are different as shown for $m = 3$ in Figures 2 and 3 and for $m = 4$ in Figures 4 and 5.

			all possible tiles								geometrically unique tiles							
			$f(m)$	$g(m)$	crossings							crossings						
n	d	m	total	0	1	2	3	4	5	6	number	0	1	2	3	4	5	6
4	1	2	3	2	1						2	1	1					
3	2	3	15	5	6	3	1				7	3	2	1	1			
6	1	3	15	5	6	3	1				5	2	1	1	1			
4	2	4	105	14	28	28	21	9	4	1	30	6	7	6	6	2	2	1
8	1	4	105	14	28	28	21	9	4	1	17	3	4	3	3	2	1	1

Table 1: *Number of decorated tiles with m total arcs.*

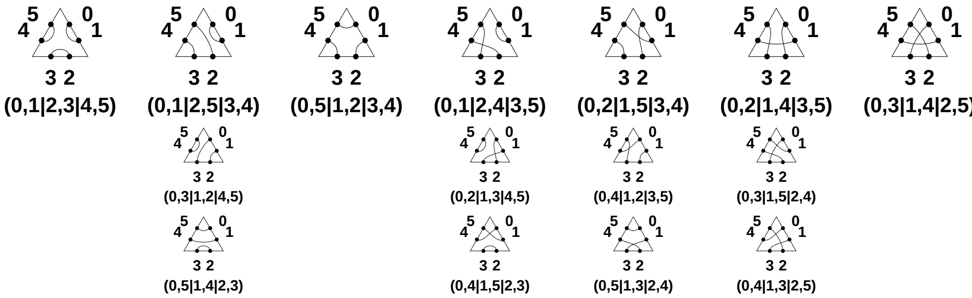

Figure 2: *The collection of all 15 decorations of a triangle with 2 divisions per side. The top row contains all geometrically unique tiles arranged from left to right by increasing number of arc crossings. Columns contain geometric equivalence classes. Note there are 5 total decorations (3 unique) without crossing arcs and 10 decorations containing crossings (4 unique).*

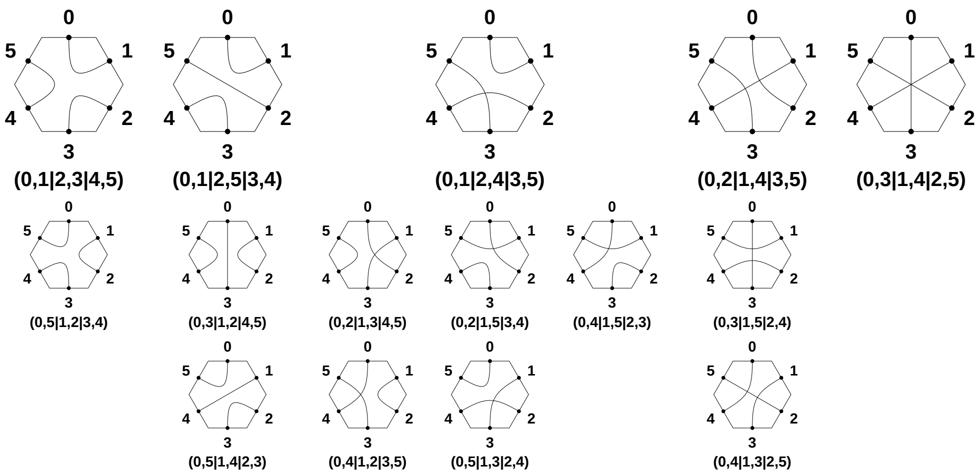

Figure 3: *The collection of all 15 decorations of a hexagon with 1 division per side. The top row contains all geometrically unique tiles arranged from left to right by increasing number of arc crossings. Columns contain the 5 geometric equivalence classes. Compare with patterns in Figure 2.*

3 Discussion

As seen in Table 1, the number of different decorated tiles increases exponentially with the number of divisions and polygon sides. This provides an enormous number of patterns from the same family that can be used to provide a statistically uniform, yet varying tessellated pattern. Given *m* arcs, the total number of tiles with *k* crossings remains constant, however the number of unique tiles increases as *n* decreases due to the interplay between symmetries in the crossing patterns and the underlying polygons. While the large number might be daunting to produce, the number of unique individual arcs is actually very limited, especially when one considers reflection and rotation. The importance of identifying patterns where there are no crossings is in creating patterns where regions between arcs are filled with a color or other distinctive pattern. Future work includes understanding which patterns form rotation and reflection classes.

4 Acknowledgments

This work was supported by a grant from the Hewlett-Mellon Fund for Faculty Development at Albion College, Albion, MI. The author is grateful Douglas Dunham for helpful discussions on this matter and to the anonymous reviewers for their comments and suggestions.

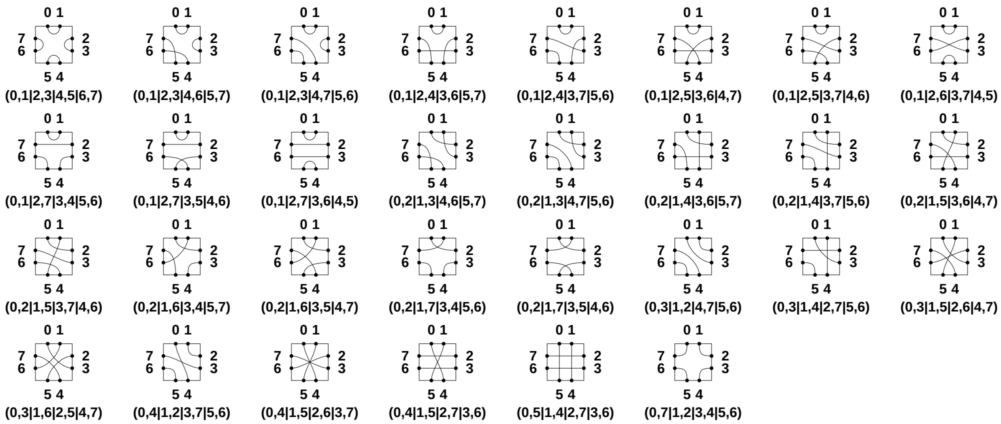

Figure 4: *The collection of the 30 unique geometric decorations of a square with 2 divisions per side. There are 105 total possible decorated tiles.*

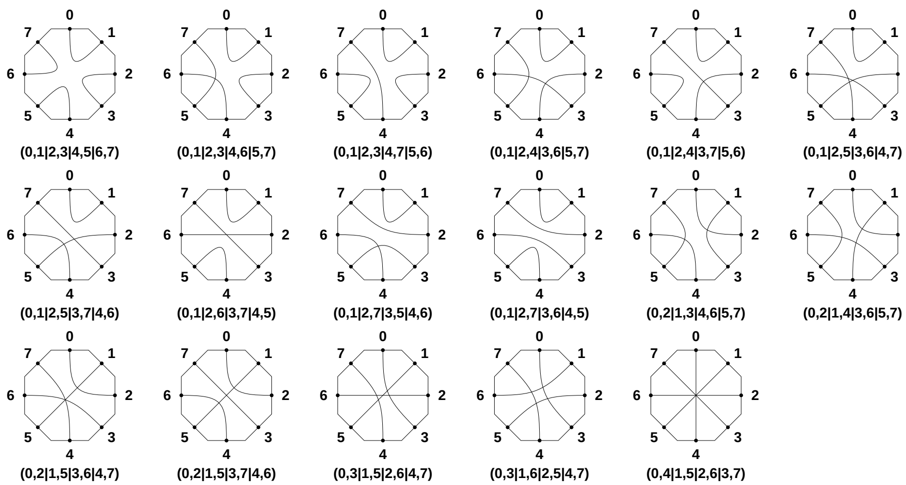

Figure 5: *The collection of the 17 unique geometric decorations of an octagon with 1 division per side. There are 105 total possible decorated tiles. Compare with patterns in Figure 4.*

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