

Decorrelation of a class of Gibbs particle processes and asymptotic properties of U -statistics

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September 8, 2020

Abstract

We study a stationary Gibbs particle process with deterministically bounded particles on Euclidean space defined in terms of an activity parameter and non-negative interaction potentials of finite range. Using disagreement percolation we prove exponential decay of the correlation functions, provided a dominating Boolean model is subcritical. We also prove this property for the weighted moments of a U -statistic of the process. Under the assumption of a suitable lower bound on the variance, this implies a central limit theorem for such U -statistics of the Gibbs particle process. A byproduct of our approach is a new uniqueness result for Gibbs particle processes.

Keywords: Gibbs process, particle process, pair potential, disagreement percolation, correlation functions, central limit theorem, U -statistics

AMS MSC 2010: 60G55, 60F05, 60D05

1 Introduction

Starting with the seminal paper [26], the limit theory for functionals of Gibbs point processes on Euclidean space has recently attracted a lot of attention [1, 27, 30]. In the present paper we derive asymptotic moment properties of certain Gibbs processes of geometrical objects. There are different background frames to deal with this problem in the literature. The first one is to extend asymptotic results to Gibbs marked point processes [19]. In applications marks describe the geometric properties of particles or they can be particles themselves. The generalization of results from point processes to marked point processes is sometimes claimed as obvious in the literature (cf. [6, Remark 3.7]). However, depending on the circumstances the details of such an extension require additional effort. Another approach is to parametrize some particle attributes and to deal with the point processes on the parameter space. See [28] for an application of the method of moments to a specific Gibbs model of this type. In

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the present paper we have chosen a third approach dealing directly with particle processes, defined as point processes on the space of compact sets equipped with the Hausdorff distance as in [25].

We study a stationary Gibbs particle process Ξ on \mathbb{R}^d defined in terms of a family of higher-order potentials with finite interaction range and an activity parameter, assuming that the size of the particles is deterministically bounded. Some first limit results for Gibbs particle processes with pair potentials have been derived in [9], using Stein's method as in [27]. Let W_n denote a centered cube of volume $n \in \mathbb{N}$. We are interested in the asymptotic behaviour of U -statistics of the form

$$F_n := \frac{1}{k!} \int h(K_1, \dots, K_k) \Xi_n^{(k)}(d(K_1, \dots, K_k)), \quad n \in \mathbb{N},$$

where h is a symmetric and measurable function of $k \in \mathbb{N}$ particles and $\Xi_n^{(k)}$ is the restriction of the k^{th} factorial measure of Ξ to $(W_n)^k$. For small activity parameter (and under some additional technical assumptions) we prove exponential decay of correlations for weighted moments of U -statistics. Under an additional assumption on the asymptotic variance this implies a central limit theorem (CLT) for the standardized sequence $(F_n)_{n \in \mathbb{N}}$. Our main technical tools are some methods from [1] and a disagreement coupling [13] of two Gibbs processes with a dominating Poisson particle process. The exponential decay of the pair correlation function via disagreement percolation property has been proved previously in [13] in a comparable setting.

The paper is organized as follows. In Section 2, we define our Gibbs model and provide some of its basic properties. Lemma 2.5 provides the Papangelou intensity of the Palm distribution with respect to the conditional Gibbs distribution for Gibbs processes with arbitrary Papangelou intensity and is new in this generality. Likewise, Lemma 2.6 on the stochastic domination of reduced Palm distribution by a Poisson process might also be of some independent interest.

In Section 3, first the existence of disagreement percolation in our setting is proved in Theorem 3.2. Then we prove the fast decay of correlations provided that the activity is below the percolation threshold of the associated Boolean model in Theorem 3.4. This result not only strengthens and generalizes the results in [26], but also holds for a wider range of the activity parameter. As a byproduct we obtain with Corollary 3.3 a new uniqueness result.

In Section 4 we study a U -statistics of order k of the Gibbs particle process in the above subcritical regime. We prove exponential decay of correlations for weighted moments (Theorem 4.9) and derive mean and variance asymptotics (Theorem 4.10). Under an additional assumption on the variance asymptotics this implies a central limit theorem (Theorem 4.11).

2 Preliminaries

2.1 Particle processes

Let \mathbb{R}^d be Euclidean d -dimensional space with Borel σ -field \mathcal{B}^d , and let \mathcal{B}_b^d denote the system of bounded Borel sets. Let \mathcal{C}^d be the space of compact subsets (particles) of \mathbb{R}^d . Let $\mathcal{C}^{(d)} := \mathcal{C}^d \setminus \{\emptyset\}$ be equipped with the Hausdorff metric d_H (see [18, 25]) and the associated Borel σ -field $\mathcal{B}(\mathcal{C}^{(d)})$. As usual for metric spaces, we define for non-empty sets $\Psi, \Gamma \subset \mathcal{C}^{(d)}$

$$d(\Psi, \Gamma) := \inf_{A \in \Psi, B \in \Gamma} d_H(A, B). \quad (2.1)$$

To avoid confusion, our notation $d(\Psi, \Gamma)$ does not reflect the underlying metric d_H . Let \mathbf{N} denote the space of all measures ξ on $\mathcal{C}^{(d)}$ with values in $\mathbb{N}_0 \cup \{\infty\}$ such that $\xi(B(K, r)) < \infty$, for each $K \in \mathcal{C}^{(d)}$ and each $r \geq 0$, where $B(K, r) := \{L \in \mathcal{C}^{(d)} \mid d_H(K, L) \leq r\}$ is the ball with radius r centred at K . As usual (see e.g. [18]), we equip this space with the smallest σ -field \mathcal{N} such that the mappings $\xi \mapsto \xi(\Psi)$ are measurable for each $\Psi \in \mathcal{B}(\mathcal{C}^{(d)})$.

Definition 2.1. A *particle process* Ξ in \mathbb{R}^d is a random element of \mathbf{N} , defined over some fixed probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Such a particle process is said to be *stationary* if $\theta_x \Xi \stackrel{d}{=} \Xi$, for each $x \in \mathbb{R}^d$, where, for each measure ξ on $\mathcal{C}^{(d)}$, we set $\theta_x \xi := \int \mathbf{1}\{K + x \in \cdot\} \xi(dK)$ with $K + x := \{y + x \mid y \in K\}$.

Let $z(K)$ denote the centre of the circumscribed ball of $K \in \mathcal{C}^{(d)}$ and note that $z(K + x) = z(K) + x$ for all $(K, x) \in \mathcal{C}^{(d)} \times \mathbb{R}^d$. We say that a particle process is *simple*, if $\Xi(z^{-1}(x)) \in \{0, 1\}$, $x \in \mathbb{R}^d$, holds almost surely. In the following, we consider only simple stationary particle processes. We also assume that $\mathbb{P}(\Xi(\mathcal{C}^{(d)}) \neq 0) = 1$. The *intensity* γ of such a particle process Ξ is defined by

$$\gamma := \mathbb{E} \left[\int \mathbf{1}\{z(K) \in [0, 1]^d\} \Xi(dK) \right].$$

The *intensity measure* $\mathbb{E}[\Xi]$ of Ξ is the measure $A \mapsto \mathbb{E}[\Xi(A)]$, $A \in \mathcal{B}(\mathcal{C}^{(d)})$.

An important example of a particle process is a *Poisson process* Π_μ on $\mathcal{C}^{(d)}$, whose intensity measure μ is defined by

$$\mu := \iint \mathbf{1}\{K + x \in \cdot\} \mathbb{Q}(dK) dx, \quad (2.2)$$

where dx refers to integration with respect to the Lebesgue measure \mathcal{L}_d on \mathbb{R}^d and \mathbb{Q} is some fixed probability measure on $\mathcal{C}^{(d)}$. We refer to [18, Chapter 3] for the definition and fundamental properties of general Poisson processes. More generally, we consider the Poisson processes $\Pi_{\lambda\mu}$ with intensity measure $\lambda\mu$, where $\lambda > 0$. Under some integrability assumptions on \mathbb{Q} , the Poisson process $\Pi_{\lambda\mu}$ exists as a stationary particle process [25]. The number λ is the intensity of $\Pi_{\lambda\mu}$ while \mathbb{Q} is called the *particle distribution* of $\Pi_{\lambda\mu}$. It is no restriction of generality to assume that

$$\mathbb{Q}(\mathcal{C}_0^{(d)}) = 1, \quad (2.3)$$

where $\mathcal{C}_0^{(d)} := \{K \in \mathcal{C}^{(d)} \mid z(K) = \mathbf{0}\}$, and $\mathbf{0}$ denotes the origin in \mathbb{R}^d . However, we make the crucial assumption that there exists $R > 0$ such that

$$\mathbb{Q}(\{K \in \mathcal{C}^{(d)} \mid K \subseteq B(\mathbf{0}, R)\}) = 1, \quad (2.4)$$

where $B(x, R)$ is the closed Euclidean ball with radius R centered at $x \in \mathbb{R}^d$. This is the deterministic bound on the particle size.

Given $m \in \mathbb{N}$ and $\xi \in \mathbf{N}$, the m^{th} *factorial measure* $\xi^{(m)}$ of ξ is the measure on $(\mathcal{C}^{(d)})^m$ defined by

$$\xi^{(m)}(\cdot) := \int \mathbf{1}\{(K_1, \dots, K_m) \in \cdot\} \mathbf{1}\{K_i \neq K_j \text{ for } i \neq j\} \xi^m(d(K_1, \dots, K_m)).$$

For us this is only of relevance if $\xi(\{K\}) \leq 1$, for each $K \in \mathcal{C}^{(d)}$. Then ξ is called *simple*. In this case, $\xi^{(m)}$ coincides with the standard definition of the factorial measure [18, Chapter 4]. The m^{th} factorial moment measure $\alpha^{(m)}$ of a simple particle process Ξ is defined by $\alpha^{(m)} := \mathbb{E}[\Xi^{(m)}]$.

Definition 2.2. Let $p \in \mathbb{N}$. The p^{th} Palm distributions of a particle process Ξ is a family $\mathbb{P}_{K_1, \dots, K_p}$, $K_1, \dots, K_p \in \mathcal{C}^{(d)}$, of probability measures on \mathbf{N} satisfying

$$\begin{aligned} & \mathbb{E} \left[\int f(K_1, \dots, K_p, \Xi) \Xi^{(p)}(d(K_1, \dots, K_p)) \right] \\ &= \iint f(K_1, \dots, K_p, \xi) \mathbb{P}_{K_1, \dots, K_p}(d\xi) \alpha^{(p)}(d(K_1, \dots, K_p)), \end{aligned} \quad (2.5)$$

for each non-negative measurable function f on $(\mathcal{C}^{(d)})^p \times \mathbf{N}$.

Palm distributions are well-defined whenever the p^{th} factorial moment measure $\alpha^{(p)}$ of Ξ is σ -finite. They can be chosen such that $(K_1, \dots, K_p) \mapsto \mathbb{P}_{K_1, \dots, K_p}(A)$ is a measurable function on $(\mathcal{C}^{(d)})^p$, for each $A \in \mathcal{N}$. The reduced Palm distribution $\mathbb{P}_{K_1, \dots, K_p}^!$ of Ξ is defined by means of the equality

$$\int f(K_1, \dots, K_p, \xi) \mathbb{P}_{K_1, \dots, K_p}^!(d\xi) = \int f(K_1, \dots, K_p, \xi - \delta_{K_1} - \dots - \delta_{K_p}) \mathbb{P}_{K_1, \dots, K_p}(d\xi), \quad (2.6)$$

valid for every non-negative measurable function f on $(\mathcal{C}^{(d)})^p \times \mathbf{N}$. We abuse our notation by writing, for each measurable $g: \mathbf{N} \rightarrow \mathbb{R}$,

$$\mathbb{E}_{K_1, \dots, K_p}[g(\Xi)] := \int g(\xi) \mathbb{P}_{K_1, \dots, K_p}(d\xi) \quad \text{and} \quad \mathbb{E}_{K_1, \dots, K_p}^![g(\Xi)] := \int g(\xi) \mathbb{P}_{K_1, \dots, K_p}^!(d\xi).$$

Given $\Psi \in \mathcal{B}(\mathcal{C}^{(d)})$, we define $\mathbf{N}_\Psi := \{\xi \in \mathbf{N} \mid \xi(\Psi^c) = 0\}$ and let \mathcal{N}_Ψ denote the σ -field on this set of measures. Given $\xi \in \mathbf{N}$, $B \in \mathcal{B}^d$ and $\Psi \in \mathcal{B}(\mathcal{C}^{(d)})$, we denote by ξ_B and ξ_Ψ the restrictions of ξ to $z^{-1}(B)$ and Ψ , respectively. Finally, we set $\mathcal{B}_b(\mathcal{C}^{(d)}) := \{z^{-1}(B) \mid B \in \mathcal{B}_b^d\}$.

2.2 Gibbs particle processes

In this subsection we present some fundamental facts on Gibbs processes in a general setting. We base our definition of a Gibbs process on the following GNZ-equation, referring e.g. to [14] for a discussion of the literature.

Definition 2.3. Let $\kappa: \mathcal{C}^{(d)} \times \mathbf{N} \rightarrow [0, \infty)$ be a measurable function and $\lambda > 0$. A particle process Ξ is called a *Gibbs process* with *Papangelou conditional intensity* κ and *activity parameter* $\lambda > 0$, if

$$\mathbb{E} \left[\int f(K, \Xi - \delta_K) \Xi(dK) \right] = \lambda \mathbb{E} \left[\int f(K, \Xi) \kappa(K, \Xi) \mu(dK) \right] \quad (2.7)$$

holds for all measurable $f: \mathcal{C}^{(d)} \times \mathbf{N} \rightarrow [0, \infty)$, where δ_K is the *Dirac measure* located at K and μ is given by (2.2).

In the following, we fix a Gibbs process with Papangelou intensity κ and activity λ as in Definition 2.3. For $p \in \mathbb{N}$, define a measurable function $\kappa_p: (\mathcal{C}^{(d)})^p \times \mathbf{N} \rightarrow [0, \infty)$ by

$$\kappa_p(K_1, \dots, K_p, \xi) := \kappa(K_1, \xi) \kappa(K_2, \xi + \delta_{K_1}) \cdots \kappa(K_p, \xi + \delta_{K_1} + \cdots + \delta_{K_{p-1}}). \quad (2.8)$$

Equation (2.7) can be iterated so as to yield

$$\begin{aligned} & \mathbb{E} \left[\int f(K_1, \dots, K_p, \Xi - \delta_{K_1} - \dots - \delta_{K_p}) \Xi^{(p)}(d(K_1, \dots, K_p)) \right] \\ &= \lambda^p \mathbb{E} \left[\int f(K_1, \dots, K_p, \Xi) \kappa_p(K_1, \dots, K_p, \Xi) \mu^p(d(K_1, \dots, K_p)) \right], \end{aligned} \quad (2.9)$$

for each measurable $f: (\mathcal{C}^{(d)})^p \times \mathbf{N} \rightarrow [0, \infty)$. Therefore, κ_p is called the conditional intensity of p^{th} order. By [20, Satz 1.5], κ_p is a symmetric function of the p particles.

Definition 2.4. Let $p \in \mathbf{N}$. The p^{th} correlation function of a Gibbs process Ξ with Papan-gelou intensity κ and activity λ is the function $\rho_p: (\mathcal{C}^{(d)})^p \rightarrow [0, \infty]$ defined by

$$\rho_p(K_1, \dots, K_p) := \lambda^p \mathbb{E}[\kappa_p(K_1, \dots, K_p, \Xi)]. \quad (2.10)$$

Putting

$$f(K_1, \dots, K_p, \xi) = \mathbf{1}\{K_1 \in B_1, \dots, K_p \in B_p\}, \quad B_1, \dots, B_p \in \mathcal{B}(\mathcal{C}^{(d)}),$$

in (2.9), we obtain that the p^{th} factorial moment measure of Ξ is given by

$$\alpha^{(p)}(\cdot) = \int \mathbf{1}\{(K_1, \dots, K_p) \in \cdot\} \rho_p(K_1, \dots, K_p) \mu^p(d(K_1, \dots, K_p)), \quad (2.11)$$

justifying our terminology.

We define a measurable function $H: \mathbf{N} \times \mathbf{N} \rightarrow (-\infty, \infty]$, the Hamiltonian, by

$$H(\xi, \chi) := \begin{cases} 0, & \text{if } \xi(\mathcal{C}^{(d)}) = 0, \\ -\log \kappa_m(K_1, \dots, K_m, \chi), & \text{if } \xi = \delta_{K_1} + \dots + \delta_{K_m} \text{ with } K_1, \dots, K_m \in \mathcal{C}^{(d)}, \\ \infty, & \text{if } \xi(\mathcal{C}^{(d)}) = \infty. \end{cases}$$

For $\Psi \in \mathcal{B}(\mathcal{C}^{(d)})$, denote by $\Pi_{\Psi, \lambda\mu} := (\Pi_{\lambda\mu})_{\Psi}$ the restriction of the Poisson process $\Pi_{\lambda\mu}$ to Ψ . The partition function $Z_{\Psi}: \mathbf{N} \rightarrow (0, \infty]$ of Ξ (on Ψ) is defined by

$$Z_{\Psi}(\chi) := \mathbb{E} \left[e^{-H(\Pi_{\Psi, \lambda\mu}, \chi)} \right], \quad (2.12)$$

For $\Psi \in \mathcal{B}_b(\mathcal{C}^{(d)})$ we have that $\mu(\Psi) < \infty$ and hence $Z_{\Psi} > 0$. It was shown in [20] that $Z_{\Psi}(\Xi_{\Psi^c}) < \infty$ \mathbb{P} -a.s. and that the following DLR-equations [24, 16, 19] hold:

$$\mathbb{E}[f(\Xi_{\Psi}) \mid \Xi_{\Psi^c} = \chi] = Z_{\Psi}(\chi)^{-1} \mathbb{E} \left[f(\Pi_{\Psi, \lambda\mu}) e^{-H(\Pi_{\Psi, \lambda\mu}, \chi)} \right], \quad \Psi \in \mathcal{B}_b(\mathcal{C}^{(d)}), \quad (2.13)$$

for $\mathbb{P}(\Xi_{\Psi^c} \in \cdot)$ -a.s. $\chi \in \mathbf{N}_{\Psi^c}$ and each measurable $f: \mathbf{N} \rightarrow [0, \infty)$.

Given $p \in \mathbf{N}$, $\Psi \in \mathcal{B}(\mathcal{C}^{(d)})$ and $\chi \in \mathbf{N}_{\Psi^c}$, we write $\mathbb{P}_{\Psi, \chi, K_1, \dots, K_p}^!$, $K_1, \dots, K_p \in \Psi$, for the reduced Palm distribution of Ξ_{Ψ} with boundary condition $\chi \in \mathbf{N}_{\Psi^c}$. Formally, this is the Palm distribution of the conditional distribution $\mathbb{P}(\Xi_{\Psi} \in \cdot \mid \Xi_{\Psi^c} = \chi)$. The corresponding Papan-gelou intensity is denoted as $\kappa_{\Psi, \chi, K_1, \dots, K_p}$.

Lemma 2.5. Let $p \in \mathbf{N}$, $\Psi \in \mathcal{B}_b(\mathcal{C}^{(d)})$ and $\chi \in \mathbf{N}_{\Psi^c}$. A version of $\kappa_{\Psi, \chi, K_1, \dots, K_p}$ is given by

$$\kappa_{\Psi, \chi, K_1, \dots, K_p}(K, \xi) = \kappa(K, \xi + \chi + \delta_{K_1} + \dots + \delta_{K_p}), \quad K_1, \dots, K_p \in \Psi, \xi \in \mathbf{N}.$$

Proof. The proof is straightforward and given here for completeness [4, page 17]. Without restricting generality we assume that $\lambda = 1$.

Let $g: \mathcal{C}^{(d)} \times \mathbf{N} \rightarrow [0, \infty)$ be measurable. By the DLR-equation and the Mecke equation for $\Pi := \Pi_\mu$ (see [18, Theorem 4.1]),

$$\begin{aligned} & \iint g(K, \xi) \xi(dK) \mathbb{P}(\Xi_\Psi \in d\xi \mid \Xi_{\Psi^c} = \chi) \\ &= Z_\Psi(\chi)^{-1} \mathbb{E} \left[\int_\Psi g(K, \Pi_\Psi + \delta_K) e^{-H(\Pi_\Psi + \delta_K, \chi)} \mu(dK) \right]. \end{aligned}$$

Since $H(\Pi_\Psi + \delta_K, \chi) = H(\Pi_\Psi, \chi) - \log \kappa(K, \Pi_\Psi + \chi)$ the above right-hand side equals (again by the DLR-equation)

$$\int \int_\Psi g(K, \xi + \delta_K) \kappa(K, \xi + \chi) \mu(dK) \mathbb{P}(\Xi_\Psi \in d\xi \mid \Xi_{\Psi^c} = \chi).$$

Hence $\mathbb{P}(\Xi_\Psi \in \cdot \mid \Xi_{\Psi^c} = \chi)$ is a Gibbs distribution whose Papangelou kernel $\kappa_{\Psi, \chi}$ is given by $\kappa_{\Psi, \chi}(K, \xi) = \mathbf{1}\{K \in \Psi\} \kappa(K, \xi + \chi)$.

By the first step of the proof it suffices to determine the reduced Palm distributions of a Gibbs process Ξ . It is convenient to write $\mathbf{K}_p := (K_1, \dots, K_p)$ and $\delta_{\mathbf{K}_p} := \delta_{K_1} + \dots + \delta_{K_p}$, for $K_1, \dots, K_p \in \mathcal{C}^{(d)}$. We proceed by induction over p . Let $g: (\mathcal{C}^{(d)})^p \rightarrow [0, \infty)$ and $f: \mathcal{C}^{(d)} \times \mathbf{N} \rightarrow [0, \infty)$ be measurable. By (2.6) and (2.5),

$$\begin{aligned} & \iiint g(\mathbf{K}_p) f(K, \xi - \delta_K) \xi(dK) \mathbb{P}_{\mathbf{K}_p}^! (d\xi) \rho_p(\mathbf{K}_p) \mu^p(d\mathbf{K}_p) \\ &= \mathbb{E} \left[\iint g(\mathbf{K}_p) f(K_{p+1}, \Xi - \delta_{\mathbf{K}_{p+1}}) (\Xi - \delta_{\mathbf{K}_{p+1}})(dK_{p+1}) \Xi^{(p)}(d\mathbf{K}_p) \right]. \end{aligned}$$

Since $(\Xi - \delta_{\mathbf{K}_{p+1}})(dK_{p+1}) \Xi^{(p)}(d\mathbf{K}_p) = \Xi^{(p+1)}(d\mathbf{K}_{p+1})$, we obtain from (2.9) that the above right-hand side equals

$$\begin{aligned} & \mathbb{E} \left[\int g(\mathbf{K}_p) f(K_{p+1}, \Xi) \kappa_{p+1}(\mathbf{K}_{p+1}, \Xi) \mu^{p+1}(d\mathbf{K}_{p+1}) \right] \\ &= \mathbb{E} \left[\iint g(\mathbf{K}_p) f(K_{p+1}, \Xi) \kappa(K_{p+1}, \Xi + \delta_{\mathbf{K}_p}) \kappa_p(\mathbf{K}_p, \Xi) \mu^p(d\mathbf{K}_p) \mu(dK_{p+1}) \right], \quad (2.14) \end{aligned}$$

where the identity comes from the definition of κ_{p+1} . It follows directly from (2.5) and (2.9) that $\mathbb{P}_{\mathbf{K}_p}^!$ is for $\alpha^{(p)}$ -a.e. \mathbf{K}_p absolutely continuous with respect to the distribution of Ξ with density $\kappa_p(\mathbf{K}_p, \cdot) / \rho_p(\mathbf{K}_p)$, where $a/0 := 0$ for all $a \geq 0$. Therefore, expression (2.14) equals

$$\iiint g(\mathbf{K}_p) f(K_{p+1}, \xi) \kappa(K_{p+1}, \xi + \delta_{\mathbf{K}_p}) \mathbb{P}_{\mathbf{K}_p}^! (d\xi) \mu(dK_{p+1}) \rho_p(\mathbf{K}_p) \mu^p(d\mathbf{K}_p).$$

This shows that the Papangelou intensity of $\mathbb{P}_{\mathbf{K}_p}^!$ is for $\alpha^{(p)}$ -a.e. \mathbf{K}_p given by the function $(K, \xi) \mapsto \kappa(K, \xi + \delta_{\mathbf{K}_p})$, as required. \square

2.3 Stochastic domination

For $\xi, \xi' \in \mathbf{N}$, we write $\xi \leq \xi'$ if $\xi(\Psi) \leq \xi'(\Psi)$ for all $\Psi \in \mathcal{B}(\mathcal{C}^{(d)})$. An event $E \in \mathcal{N}$ is increasing, if, for all $\xi, \xi' \in \mathbf{N}_\Psi$, $E \ni \xi \leq \xi'$ implies that $\xi' \in E$. Another viewpoint is that E is closed under the addition of point measures.

A particle process Ξ is stochastically dominated by another particle process Ξ' if $\mathbb{P}(\Xi \in E) \leq \mathbb{P}(\Xi' \in E)$, for each increasing $E \in \mathcal{N}$. In this case we write $\Xi \stackrel{d}{\leq} \Xi'$ and also $\mathbb{P}(\Xi \in \cdot) \stackrel{d}{\leq} \mathbb{P}(\Xi' \in \cdot)$. By the famous Strassen theorem this implies the existence of a coupling $(\tilde{\Xi}, \tilde{\Xi}')$ of (Ξ, Ξ') such that $\tilde{\Xi} \leq \tilde{\Xi}'$ almost surely. In this context we call $\tilde{\Xi}$ a thinning of $\tilde{\Xi}'$. It then follows that $\mathbb{E}f(\Xi) \leq \mathbb{E}f(\Xi')$ for all increasing measurable $f: \mathbf{N} \rightarrow [0, \infty]$, where f is called increasing if $f(\xi) \leq f(\xi')$ whenever $\xi \leq \xi'$.

A classical example is the stochastic domination of $\Pi_{\alpha\mu}$ by $\Pi_{\beta\mu}$ for $\alpha \leq \beta$. Later we use the following deeper fact.

Lemma 2.6. *Suppose that Ξ is a Gibbs particle process with Papangelou intensity $\kappa \leq 1$ and activity λ . Then $\mathbb{P}(\Xi \in \cdot) \stackrel{d}{\leq} \mathbb{P}(\Pi_{\lambda\mu} \in \cdot)$. Furthermore we have for each $p \in \mathbb{N}$ that $\mathbb{P}_{K_1, \dots, K_p}^! \stackrel{d}{\leq} \mathbb{P}(\Pi_{\lambda\mu} \in \cdot)$, for $\alpha^{(p)}$ -a.e. (K_1, \dots, K_p) , where $\alpha^{(p)}$ is the p^{th} factorial moment measure of Ξ .*

Proof. We only prove the second assertion. The proof of the first assertion is simpler (and in fact a special case). Let $p \in \mathbb{N}$. We use the notation of the proof of Lemma 2.5. By this lemma and [10, Theorem 1.1] used for finite Gibbs processes, we have that $\mathbb{P}_{\Psi, \chi, \mathbf{K}_p}^! \stackrel{d}{\leq} \mathbb{P}(\Pi_{\Psi, \lambda\mu} \in \cdot)$, for each $\Psi \in \mathcal{B}_b(\mathcal{C}^d)$ for $\mathbb{P}(\Xi_{\Psi^c} \in \cdot)$ -a.e. χ and $\alpha^{(p)}$ -a.e. $\mathbf{K}_p \in \Psi^p$. Hence, the definition of Palm distributions implies for each measurable increasing $f: \mathbf{N} \rightarrow [0, \infty)$ and each measurable $g: \mathcal{C}^d \times \mathbf{N} \rightarrow [0, \infty)$ \mathbb{P} -a.s., that

$$\begin{aligned} & \iint f(\xi - \delta_{\mathbf{K}_p})g(\mathbf{K}_p) \xi^{(p)}(d\mathbf{K}_p) \mathbb{P}(\Xi_{\Psi} \in d\xi \mid \Xi_{\Psi^c}) \\ & \leq \iint f(\xi)g(\mathbf{K}_p) \mathbb{P}(\Pi_{\Psi, \lambda\mu} \in d\xi) \mathbb{E}[(\Xi_{\Psi})^{(p)} \in d\mathbf{K}_p \mid \Xi_{\Psi^c}]. \end{aligned}$$

Taking expectations yields

$$\mathbb{E} \left[\int_{\Psi^p} f((\Xi - \delta_{\mathbf{K}_p})_{\Psi})g(\mathbf{K}_p) \Xi^{(p)}(d\mathbf{K}_p) \right] \leq \int_{\Psi^p} \mathbb{E}[f(\Pi_{\Psi, \lambda\mu})]g(\mathbf{K}_p) \alpha^{(p)}(d\mathbf{K}_p),$$

so that $\mathbb{P}_{\mathbf{K}_p}^!(\{\xi \in \mathbf{N} \mid \xi_{\Psi} \in \cdot\}) \stackrel{d}{\leq} \mathbb{P}(\Pi_{\Psi, \lambda\mu} \in \cdot)$ for $\alpha^{(p)}$ -a.e. $\mathbf{K}_p \in \Psi^p$, for each measurable $\Psi' \subset \Psi$. Exactly as in the proof of [11, Corollary 3.4], we let $\Psi \uparrow \mathcal{C}^d$ to obtain that $\mathbb{P}_{\mathbf{K}_p}^! \stackrel{d}{\leq} \mathbb{P}(\Pi_{\lambda\mu} \in \cdot)$ for $\alpha^{(p)}$ -a.e. $\mathbf{K}_p \in \Psi^p$ and hence the assertion. \square

Remark 2.7. The definitions and results of Subsections 2.1 and 2.2 apply to Gibbs processes on a general complete separable metric space equipped with a locally finite measure μ .

2.4 Admissible Gibbs particle processes

A family $\varphi := (\varphi_n)_{n \geq 2}$ of higher-order interaction potentials consists of measurable, symmetric and translation-invariant functions $\varphi_n: (\mathcal{C}^d)^n \rightarrow (-\infty, \infty]$. The potentials have finite interaction range R_φ , if $\varphi_n(K_1, \dots, K_n) = 0$, for every $n \geq 2$ and all $K_1, \dots, K_n \in \mathcal{C}^d$ with $\max\{d_H(K_i, K_j) \mid 1 \leq i < j \leq n\} > R_\varphi$.

Define the Papangelou intensity $\kappa: \mathcal{C}^d \times \mathbf{N} \rightarrow [0, \infty)$ by $\kappa(K, \xi) := 0$, if $K \in \text{supp } \xi$, and otherwise

$$\kappa(K, \xi) := \exp \left[- \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int \varphi_n(K, L_1, \dots, L_{n-1}) \xi^{(n-1)}(d(L_1, \dots, L_{n-1})) \right]. \quad (2.15)$$

The function κ is measurable and translation-invariant. Here and later we make the following convention regarding the series in the exponent of (2.15). If the sum over the negative terms diverges, then the whole series is set to zero. We assume that this is not the case for all $\xi \in \mathbf{N}$ and μ -a.e. K . We also assume that $\kappa \leq 1$. While individual potentials might be attractive (i.e., negative), their cumulative effect must be repulsive (i.e., non-negative).

Proving the existence of a Gibbs process with a given Papangelou intensity is a non-trivial task. The literature contains many existence results under varying assumptions of generality [6, 9, 19, 24, 26], none of which seems to cover our current setting. For our main findings (for instance Theorems 3.4 and 4.11) we need to restrict the range of the activity parameter to a finite interval, the subcritical percolation regime of the associated Poisson particle process, c.f. Section 3.1. In that case, the Gibbs distribution is not only uniquely determined (see Corollary 3.3) but can be expected to exist. In the case of a non-negative pair potential we have the following result.

Remark 2.8. Assume that the Papangelou intensity κ is given by a non-negative pair potential, i.e., assume that κ is given by (2.15) with $n = 2$. Assume also that φ_2 has a finite interaction range, or, more generally that $\int (1 - e^{-\varphi_2(K,L)}) \mu(dK) < \infty$ for μ -a.e. L . (By assumption (2.4) we have that $\mu(\Psi) < \infty$ for any ball $\Psi \subset \mathcal{C}^{(d)}$.) Under these assumptions it has been shown in [14] that a Gibbs process exists.

We do not further address the existence problem in this paper and proceed under the assumption that the Gibbs process exists.

For all $\xi, \chi \in \mathbf{N}$ with disjoint supports, the Hamiltonian H takes the form

$$H(\xi, \chi) := \sum_{n=2}^{\infty} \sum_{k=1}^n \frac{1}{k!(n-k)!} \iint \varphi_n(K_1, \dots, K_k, L_1, \dots, L_{n-k}) \\ \times \xi^{(k)}(d(K_1, \dots, K_k)) \chi^{(n-k)}(d(L_1, \dots, L_{n-k})),$$

provided that ξ is finite. The assumption $\kappa \leq 1$ implies that $H \geq 0$. If assumptions (2.3) and (2.4) hold, then (2.13) shows that the Gibbs process Ξ has bounded particles, that is

$$\int \mathbf{1}\{K \not\subseteq B(z(K), R)\} \Xi(dK) = 0, \quad \mathbb{P}\text{-a.s.}$$

For clarity and to avoid lengthy formulations we make the following definition.

Definition 2.9. Assume that φ is a family of higher-order potentials with finite interaction range R_φ . Define κ by (2.15) and assume that $\kappa \leq 1$. Assume also that \mathbb{Q} is a probability measure on $\mathcal{C}^{(d)}$ satisfying (2.3) and (2.4). Let $\lambda > 0$ be given. Assume that Ξ is a Gibbs particle process as in Definition 2.3, where μ is defined by (2.2). Then, we call Ξ an *admissible Gibbs process*.

For an admissible Gibbs particle process it follows from (2.15), (2.8) and (2.9) that

$$\rho_p(K_1, \dots, K_p) \leq \lambda^p, \quad K_1, \dots, K_p \in \mathcal{C}^{(d)}.$$

A classic setup of a repulsive intersection-based pair potential arises from a measurable translation invariant function $U: \mathcal{C}^d \rightarrow [0, \infty]$ with $U(\emptyset) = 0$ and setting $\varphi_2(K, L) := U(K \cap L)$ and $\varphi_n := 0$, for $n \geq 3$. Assumption (2.4) implies an interaction range of at most $4R$.

Example 2.10. For $d \geq 2$, let \mathcal{G}_d be the space of $(d-1)$ -dimensional linear subspaces of \mathbb{R}^d . Let $R > 0$ and let the measure \mathbb{Q} be concentrated on

$$V := \{A \cap B(\mathbf{0}, R) \mid A \in \mathcal{G}_d\},$$

Then (2.4) holds. The particles are called facets and \mathbb{Q} can be interpreted as the distribution of their normal directions. The space of facets is

$$\tilde{V} := \{B + x \mid B \in V, x \in \mathbb{R}^d\}. \quad (2.16)$$

Let \mathbb{H}^m be the Hausdorff measure of order $m \in \{1, \dots, d\}$ on \mathbb{R}^d . For $j \in \{1, \dots, d\}$ and $K_1, \dots, K_j \in \tilde{V}$ define, setting $0 \cdot \infty = 0$,

$$Q_j(K_1, \dots, K_j) := \mathbb{H}^{d-j} \left(\bigcap_{i=1}^j K_i \right) \mathbf{1} \left\{ \mathbb{H}^{d-j} \left(\bigcap_{i=1}^j K_i \right) < \infty \right\}. \quad (2.17)$$

A family $\varphi := (\varphi_j)_{j \geq 2}$ of higher-order potentials is defined by

$$\varphi_j(K_1, \dots, K_j) := a_j Q_j(K_1, \dots, K_j), \quad K_1, \dots, K_j \in \tilde{V},$$

for $j \in \{2, \dots, d\}$, and $\varphi_j := 0$ otherwise, where $a_2, \dots, a_d \geq 0$ are given parameters. All these potentials have the finite range $R_\varphi = 2R$. The corresponding Gibbs particle process Ξ is called the Gibbs facet process. It is admissible and its existence follows from [6, Remarks 3.7 and 3.1]. For $j \in \{2, \dots, d\}$, the j^{th} submodel is the special case of only the j^{th} potential being active, i.e., only $a_j > 0$, and we denote it by ${}^j\Xi$.

3 Disagreement percolation and moment decorrelation

In the first subsection of this section we discuss some percolation properties of a Poisson process. In the remaining two subsections we fix an admissible Gibbs process and discuss disagreement percolation and prove decorrelation of moments in a subcritical regime.

3.1 Percolation

Define a symmetric relation on $\mathcal{C}^{(d)}$ by setting $K \sim L$, if and only if, $K \cap L \neq \emptyset$. For $\xi \in \mathbf{N}$, this defines a graph $(\text{supp } \xi, \sim)$. For $K, L \in \mathcal{C}^{(d)}$, we say that ξ connects K and L , if there exists a finite path between K and L in the graph on $\xi + \delta_K + \delta_L$. For disjoint $\Psi, \Gamma \in \mathcal{B}(\mathcal{C}^{(d)})$, we say that ξ connects Ψ and Γ , if there exist $K \in \Psi$ and $L \in \Gamma$ such that ξ connects K and L . We write $\Psi \xleftrightarrow{\xi} \Gamma$ for this.

We say that ξ percolates, if its graph contains an infinite connected component. Because connectedness is an increasing event in \mathcal{N} , there is a critical percolation intensity

$$\lambda_c(d) \equiv \lambda_c(d, \mathbb{Q}) \in [0, \infty]$$

for percolation of $\Pi_{\lambda\mu}$; see [21]. The following consequence of a result in [31] is of crucial importance for our main results.

Lemma 3.1. *For $\lambda < \lambda_c(d)$ and $\Psi, \Gamma \in \mathcal{B}_b(\mathcal{C}^{(d)})$ with $\Psi \subseteq \Gamma$, there exist a monotone increasing $C_1 : [0, \infty) \rightarrow [0, \infty)$ and $C_2 \in (0, \infty)$ such that*

$$\mathbb{P} \left(\Psi \xleftrightarrow{\Pi_{\lambda\mu}} \Gamma^c \right) \leq C_1(\text{diam}(\Psi)) \exp(-C_2 d(\Psi, \Gamma^c)). \quad (3.1)$$

Proof. Let $\mathcal{C}_0^{(d,R)} := \{K \in \mathcal{C}_0^{(d)} \mid K \subseteq B(\mathbf{0}, R)\}$. By (2.4) we have that $\mathbb{Q}(\mathcal{C}_0^{(d,R)}) = 1$. A slightly weakened form of the bound [31, Equation (3.7)] in our notation is as follows. For $\lambda < \lambda_c(d)$, there exists a constant $C \in (0, \infty)$, such that, for all $r \geq 0$,

$$\int_{\mathcal{C}_0^{(d)}} \mathbb{P}\left(K \xleftrightarrow{\Pi_{\lambda\mu} + \delta_K} B(\mathbf{0}, r)^c \times \mathcal{C}_0^{(d,R)}\right) \mathbb{Q}(dK) \leq e^{-C(r-2R)}. \quad (3.2)$$

The weakening is a result of a switch from the notion of connecting two sets in \mathbb{R}^d used in [31] to the one connecting two sets in $\mathbb{R}^d \times \mathcal{C}_0^{(d,R)}$ used above. Thus, we correct twice by R in the exponent on the right-hand side of (3.2) to account for the maximum size of K and of particles in $B(0, r)^c \times \mathcal{C}_0^{(d,R)}$ respectively.

If $\Psi = \emptyset$, then the probability is zero anyway. Proceed by assuming that $\Psi \neq \emptyset$.

Let $D := \text{diam}(\Psi)$. Let $s := d(\Psi, \Gamma^c) - D - 10R$. If $s \leq 0$, then choosing $C_1(D) \geq e^{C(D+10R)}$ finishes the proof. From here on, assume that $s > 0$ and that all particles are deterministically bounded by R .

Choose and fix $K_\Psi \in \Psi$ and let $x := z(K_\Psi)$. Let $\Upsilon := A \times \mathcal{C}_0^{(d,R)}$ be the particles with centres of circumscribed balls within the annulus $A := B(x, D + 6R) \setminus B(x, D + 4R)$.

For $K \in B(x, D + 4R) \times \mathcal{C}_0^{(d,R)} =: \Upsilon^-$ and $L \in B(x, D + 6R)^c \times \mathcal{C}_0^{(d,R)} =: \Upsilon^+$, we have $|z(K) - z(L)| \geq 2R$, whence $K \cap L = \emptyset$. For $\xi \in \mathbf{N}$, let $P(\xi)$ be the particles of ξ_Υ connected in $\xi_{\Upsilon \setminus \Upsilon^-}$ to Γ^c . Then $\Psi \xleftrightarrow{\xi} \Gamma^c$ implies that $P(\xi) \neq \emptyset$, because particles in Υ^- do not intersect particles in Υ^+ and ξ needs to contain at least one particle in Υ . Together with a first moment bound, this yields

$$\mathbb{P}\left(\Psi \xleftrightarrow{\Pi_{\lambda\mu}} \Gamma^c\right) \leq \mathbb{P}(P(\Pi_{\lambda\mu}) \neq \emptyset) \leq \mathbb{E}|P(\Pi_{\lambda\mu})|.$$

We apply the Mecke equation to rewrite

$$\mathbb{E}|P(\Pi_{\lambda\mu})| = \mathbb{E} \int \mathbf{1}\{K \in P(\Pi_{\lambda\mu})\} \Pi_{\lambda\mu}(dK) = \int_{\Upsilon} \mathbb{P}\left(K \in P(\Pi_{\lambda\mu} + \delta_K)\right) \lambda\mu(dK).$$

For $K \in \Upsilon$ and $L \in \Gamma^c$ we have that

$$d_H(K, L) \geq d_H(K_\Psi, L) - d_H(K_\Psi, K) \geq d(\Psi, \Gamma^c) - (D + 6R + 2R) = s + 2R.$$

In particular, we have $d_H(K, L) \geq 2R$, so that $K \cap L = \emptyset$. Moreover, for each $\xi \in \mathbf{N}$ satisfying $K \xleftrightarrow{\xi} \Gamma^c$, we obtain that $K \xleftrightarrow{\xi} B(z(K), s)^c \times \mathcal{C}_0^{(d,R)}$. This implies that

$$\mathbb{P}\left(K \in P(\Pi_{\lambda\mu} + \delta_K)\right) \leq \mathbb{P}\left(K \xleftrightarrow{\Pi_{\lambda\mu} + \delta_K} \Gamma^c\right) \leq \mathbb{P}\left(K \xleftrightarrow{\Pi_{\lambda\mu} + \delta_K} B(z(K), s)^c \times \mathcal{C}_0^{(d,R)}\right).$$

Combining these upper bounds and using the definition (2.2) of μ we see that

$$\begin{aligned} \mathbb{P}\left(\Psi \xleftrightarrow{\Pi_{\lambda\mu}} \Gamma^c\right) &\leq \int_{\Upsilon} \mathbb{P}\left(K \xleftrightarrow{\Pi_{\lambda\mu} + \delta_K} B(z(K), s)^c \times \mathcal{C}_0^{(d,R)}\right) \mu(dK) \\ &= \int_{\mathcal{C}_0^{(d)}} \int_A \mathbb{P}\left(K + x \xleftrightarrow{\Pi_{\lambda\mu} + \delta_{K+x}} B(x, s)^c \times \mathcal{C}_0^{(d,R)}\right) dx \mathbb{Q}(dK). \end{aligned}$$

By stationarity of $\Pi_{\lambda\mu}$ this equals

$$\lambda \mathcal{L}_d(A) \int_{\mathcal{C}_0^{(d)}} \mathbb{P}\left(K \xleftrightarrow{\Pi_{\lambda\mu} + \delta_K} B(\mathbf{0}, s)^c \times \mathcal{C}_0^{(d,R)}\right) \mathbb{Q}(dK) \leq \lambda \mathcal{L}_d(A) e^{-C(s-2R)},$$

where the inequality comes from (3.2). Choosing $C_1(D) \geq \lambda \mathcal{L}_d(A) e^{C(D+12R)}$ concludes the argument. \square

Monotonicity in the particle shapes allows to control the percolation threshold. In the special case of $\mathbb{Q} = \delta_{B(\mathbf{0}, R)}$, the measure μ becomes $\mu_R := \int \mathbf{1}\{B(x, R) \in \cdot\} dx$. Assumption (2.4) implies for each $\lambda > 0$ that $\Pi_{\lambda\mu}$ -a.e. ξ fulfils

$$\bigcup_{K \in \xi} K \subseteq \bigcup_{K \in \xi} B(z(K), R).$$

Hence, we can couple $\Pi_{\lambda\mu}$ and $\Pi_{\lambda\mu_R}$ such that

$$\mathbb{P}\left(\bigcup_{K \in \Pi_{\lambda\mu}} K \subseteq \bigcup_{B \in \Pi_{\lambda\mu_R}} B\right) = 1.$$

A well known lower bound (see [22] and [21, Section 3.9]) is

$$\lambda_c(d, \mathbb{Q}) \geq \lambda_c(d, \delta_{B(\mathbf{0}, R)}) \geq \frac{1}{v_d 2^d R^d}, \quad (3.3)$$

where v_d is the volume of the d -dimensional unit ball.

3.2 Disagreement percolation

For $\xi, \xi' \in \mathbf{N}$, we write $\xi \triangle \xi'$ for the absolute difference measure $\max\{\xi, \xi'\} - \min\{\xi, \xi'\}$, equivalent to $|\xi - \xi'|$. In the relevant case of ξ and ξ' both being simple, there is a simpler geometric interpretation of the also simple $\xi \triangle \xi'$. Switching to the support of a simple point measure, we see that $\text{supp}(\xi \triangle \xi') = (\text{supp } \xi) \triangle (\text{supp } \xi')$, which motivates this overloading of the set difference operator \triangle to point measures.

The space $(\mathcal{C}^{(d)}, d_H)$ is a complete and separable metric space. By [7, Theorem 13.1.1], the spaces $(\mathcal{C}^{(d)}, \mathcal{B}(\mathcal{C}^{(d)}))$ and \mathbb{R} equipped with the Borel σ -algebra are Borel isomorphic. That is, there exists a measurable bijection from $\mathcal{C}^{(d)}$ to \mathbb{R} with measurable inverse. We use this bijection to pull back the total order from \mathbb{R} to $\mathcal{C}^{(d)}$ and denote it by \prec . Hence, intervals with respect to \prec are in $\mathcal{B}(\mathcal{C}^{(d)})$.

For the remainder of the section we fix an admissible Gibbs process Ξ as in Definition 2.9.

Theorem 3.2. *For all $\Psi \in \mathcal{B}_b(\mathcal{C}^{(d)})$ and $\chi_1, \chi_2 \in \mathbf{N}_{\Psi^c}$, there exists a simultaneous thinning from $\Pi_{\Psi, \lambda\mu}$ to two particle processes Θ^{Ψ, χ_1} and Θ^{Ψ, χ_2} such that Θ^{Ψ, χ_i} has the distribution $\mathbb{P}(\Xi_{\Psi} \in \cdot \mid \Xi_{\Psi^c} = \chi_i)$, for $i \in \{1, 2\}$, and, \mathbb{P} -a.s.,*

$$\forall K \in \text{supp}\left(\Theta^{\Psi, \chi_1} \triangle \Theta^{\Psi, \chi_2}\right): \{K\} \xleftrightarrow{\Theta^{\Psi, \chi_1} \triangle \Theta^{\Psi, \chi_2}} \chi_1 \triangle \chi_2. \quad (3.4)$$

Proof. Using \prec restricted to Ψ in place of the measurable ordering of a bounded Borel subset in [13, Section 4.1], and the DLR-equations as formulated in (2.13), this theorem becomes a literal copy of the construction leading to [13, Theorem 3.3]. \square

The term disagreement percolation comes from the fact that in the subcritical percolation regime of $\Pi_{\lambda\mu}$, there is control of a disagreement cluster by a percolation cluster. The finiteness of the percolation clusters guarantees uniqueness of the Gibbs process.

Corollary 3.3. *If $\lambda < \lambda_c(d)$, then the distribution of Ξ is uniquely determined.*

Proof. The proof generalises in a straightforward way the proof of [13, Theorem 3.2] and Theorem 3.2, with the only change being that the interaction range and particle size are here two separate parameters. Because of the deterministic bound R from (2.4) on the particle size and the finiteness of the interaction range, the arguments remain the same. \square

3.3 Decorrelation of moments

With the following theorem we establish the particle counterpart of fast decay of correlations in [1, Definition 1.1] in the subcritical regime.

Theorem 3.4. *Assume that $\lambda < \lambda_c(d)$. There exist $c_1, c_2 \in (0, \infty)$ such that Gibbs process Ξ satisfies, for all $p, q \in \mathbb{N}$ and $\alpha^{(p+q)}$ -a.e. (K_1, \dots, K_{p+q}) ,*

$$\begin{aligned} & |\rho_{p+q}(K_1, \dots, K_{p+q}) - \rho_p(K_1, \dots, K_p)\rho_q(K_{p+1}, \dots, K_{p+q})| \\ & \leq \lambda^{p+q} \min(p, q) c_1 \exp(-c_2 d(\{K_1, \dots, K_p\}, \{K_{p+1}, \dots, K_{p+q}\})). \end{aligned} \quad (3.5)$$

Combining Theorem 3.4 with the bound (3.3) on the percolation threshold gives the following constraint on the activity as sufficient condition for the exponential decay of correlations (3.5):

$$\lambda < \frac{1}{v_d 2^d R^d}. \quad (3.6)$$

Remark 3.5. To compare our results with Proposition 2.1 in [26] we consider *hard spheres in equilibrium*, that is we assume that $\mathbb{Q} = \delta_{B(\mathbf{0}, R)}$, $\varphi_2(K, L) = \infty \cdot \mathbf{1}\{K \cap L \neq \emptyset\}$ and $\varphi_n \equiv 0$ for $n \geq 3$. Then Ξ can be identified with a point process on \mathbb{R}^d . In the language from [26] we have that $r^\Psi = 2R$ and $m_0^\Psi = 0$. Proposition 2.1 in [26] then shows exponential decay of correlations whenever

$$\lambda < \frac{1}{v_d(1 + 2R)^d}. \quad (3.7)$$

This is comparable with (3.6).

Our Theorem 3.4 shows exponential decay of correlations for a broader range of activities. In fact, it is known from simulations that in low dimensions $\lambda_c(d)$ is considerably larger than the right-hand side of (3.6). (As $d \rightarrow \infty$ we have $\lambda_c(d)v_d 2^d R^d \rightarrow 1$; see [22].) We do not see how the methods from [26] can be used to prove Theorem 3.4.

Remark 3.6. In the recent paper [14] uniqueness of the distribution of a Gibbs process on \mathbb{R}^d has been proved with very different methods, based on a fixed point argument for correlation functions. It is an interesting problem to explore the relationship between this method and disagreement percolation. Some first answers are given in [14].

The proof of Theorem 3.4 is based on the following lemmas. For these lemmas and the proof of Theorem 3.4, fix $\lambda < \lambda_c(d)$ and let C_1 and C_2 as in Lemma 3.1.

Lemma 3.7. *Let $p \in \mathbb{N}$ and let $\Psi_1, \dots, \Psi_p \in \mathcal{B}_b(\mathcal{C}^d)$ be disjoint. Let $\Gamma \in \mathcal{B}_b(\mathcal{C}^d)$ with $\bigcup_{i=1}^p \Psi_i =: \Psi \subseteq \Gamma$. For $\chi \in \mathbf{N}_{\Gamma^c}$ and increasing $E \in \mathcal{N}$,*

$$\begin{aligned} & \left| \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi) - \mathbb{P}(\Xi_\Psi \in E) \right| \\ & \leq \mathbb{P}(\Pi_{\Psi, \lambda\mu} \in E) \sum_{i=1}^p C_1(\text{diam}(\Psi_i)) \exp(-C_2 d(\Psi, \Gamma^c)). \end{aligned} \quad (3.8)$$

Proof. First, we show that, for $\chi_1, \chi_2 \in \mathbf{N}_{\Gamma^c}$,

$$\begin{aligned} & \left| \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi_1) - \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi_2) \right| \\ & \leq \mathbb{P}(\Pi_{\Psi, \lambda\mu} \in E) \sum_{i=1}^p C_1(\text{diam}(\Psi_i)) \exp(-C_2 d(\Psi, \Gamma^c)). \end{aligned} \quad (3.9)$$

By Theorem 3.2,

$$\begin{aligned}
& \left| \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi_1) - \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi_2) \right| \\
&= \left| \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \in E) - \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_2} \in E) \right| \\
&= \left| \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \in E, \Theta_\Psi^{\Gamma, \chi_2} \notin E) - \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \notin E, \Theta_\Psi^{\Gamma, \chi_2} \in E) \right| \\
&\leq \max\{\mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \in E, \Theta_\Psi^{\Gamma, \chi_2} \notin E), \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \notin E, \Theta_\Psi^{\Gamma, \chi_2} \in E)\}.
\end{aligned}$$

By symmetry we only need to bound the first term in the above maximum. It follows from (3.4) that

$$\begin{aligned}
& \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \in E, \Theta_\Psi^{\Gamma, \chi_2} \notin E) \\
&= \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \in E, \Theta_\Psi^{\Gamma, \chi_2} \notin E, \emptyset \neq \Theta_\Psi^{\Gamma, \chi_1} \triangle \Theta_\Psi^{\Gamma, \chi_2}) \\
&= \mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \in E, \Theta_\Psi^{\Gamma, \chi_2} \notin E, \emptyset \neq \Theta_\Psi^{\Gamma, \chi_1} \triangle \Theta_\Psi^{\Gamma, \chi_2} \xrightarrow{\Theta^{\Gamma, \chi_1} \triangle \Theta^{\Gamma, \chi_2}} \chi_1 \triangle \chi_2).
\end{aligned}$$

Since

$$\left\{ \Theta_\Psi^{\Gamma, \chi_1} \triangle \Theta_\Psi^{\Gamma, \chi_2} \xrightarrow{\Theta^{\Gamma, \chi_1} \triangle \Theta^{\Gamma, \chi_2}} \chi_1 \triangle \chi_2 \right\} \subseteq \left\{ \Psi \xrightarrow{\Theta^{\Gamma, \chi_1} \triangle \Theta^{\Gamma, \chi_2}} \chi_1 \triangle \chi_2 \right\} \subseteq \left\{ \Psi \xrightarrow{\Pi_{\lambda^\mu}} \Gamma^c \right\},$$

we obtain that

$$\begin{aligned}
\mathbb{P}(\Theta_\Psi^{\Gamma, \chi_1} \in E, \Theta_\Psi^{\Gamma, \chi_2} \notin E) &\leq \mathbb{P}(\Pi_{\Psi, \lambda^\mu} \in E, \Psi \xrightarrow{\Pi_{\lambda^\mu}} \Gamma^c) \\
&= \mathbb{P}(\Pi_{\Psi, \lambda^\mu} \in E) \mathbb{P}(\Psi \xrightarrow{\Pi_{\lambda^\mu}} \Gamma^c) \\
&\leq \mathbb{P}(\Pi_{\Psi, \lambda^\mu} \in E) \sum_{i=1}^p \mathbb{P}(\Psi_i \xrightarrow{\Pi_{\lambda^\mu}} \Gamma^c) \\
&\leq \mathbb{P}(\Pi_{\Psi, \lambda^\mu} \in E) \sum_{i=1}^p C_1(\text{diam}(\Psi_i)) \exp(-C_2 d(\Psi, \Gamma^c)),
\end{aligned}$$

where the equality results from the complete independence of a Poisson process, the second inequality is a Boolean bound and the final equality uses (3.1) and the fact that $d(\Psi, \Gamma^c) \leq d(\Psi_i, \Gamma^c)$. This proves (3.9).

To prove (3.8), we use the DLR-equation (2.13) to obtain that

$$\begin{aligned}
& \left| \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi) - \mathbb{P}(\Xi_\Psi \in E) \right| \\
&\leq \int \left| \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi) - \mathbb{P}(\Xi_\Psi \in E \mid \Xi_{\Gamma^c} = \chi') \right| \mathbb{P}(\Xi_{\Gamma^c} \in d\chi').
\end{aligned}$$

An application of (3.9) to the integrand shows (3.8). \square

For $\Psi \in \mathcal{B}_b(\mathcal{C}^d)$ and $n \in \mathbb{N}$, let $E_{\Psi, n} := \{\xi \in \mathbf{N} \mid \xi(\Psi) \geq n\}$. The event $E_{\Psi, n}$ is increasing. Fix $p \in \mathbb{N}$ and disjoint $\Psi_1, \dots, \Psi_p \in \mathcal{B}_b(\mathcal{C}^d)$. Let Φ be a particle process. Then,

$$\mathbb{E} \left(\prod_{i=1}^p \Phi(\Psi_i) \right) = \sum_{\vec{n} \in \mathbb{N}^d} \mathbb{P}(\forall 1 \leq i \leq p: \Phi_{\Psi_i} \in E_{\Psi_i, n_i}), \quad (3.10)$$

where we use the notation $\vec{n} =: (n_1, \dots, n_d)$.

Lemma 3.8. For $p \in \mathbb{N}$ and disjoint $\Psi_1, \dots, \Psi_p \in \mathcal{B}_b(\mathcal{C}^d)$,

$$\mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) \leq \lambda^p \prod_{i=1}^p \mu(\Psi_i). \quad (3.11)$$

Proof. Applying (3.10), then stochastic domination from Lemma 2.6, applying (3.10) again and writing out the moment of the Poisson process yields the bound. \square

Lemma 3.9. Let $p \in \mathbb{N}$ and let $\Psi_1, \dots, \Psi_p \in \mathcal{B}_b(\mathcal{C}^d)$ be disjoint. Suppose that $\Gamma \in \mathcal{B}_b(\mathcal{C}^d)$ satisfies $\Gamma \supseteq \Psi := \bigcup_{i=1}^p \Psi_i$ and let $\chi \in \mathbf{N}_{\Gamma^c}$. Then

$$\left| \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) - \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \mid \Xi_{\Gamma^c} = \chi \right) \right| \leq \lambda^p \left(\prod_{i=1}^p \mu(\Psi_i) \right) C_{\Psi} \exp(-C_2 d(\Psi, \Gamma^c)), \quad (3.12)$$

where $C_{\Psi} := \sum_{i=1}^p C_1(\text{diam}(\Psi_i))$.

Proof. Applying (3.10) and then (3.8) gives

$$\begin{aligned} & \left| \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) - \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \mid \Xi_{\Gamma^c} = \chi \right) \right| \\ &= \sum_{\vec{n} \in \mathbb{N}^d} \left| \mathbb{P}(\forall 1 \leq i \leq p: \Xi_{\Psi_i} \in E_{\Psi_i, n_i}) - \mathbb{P}(\forall 1 \leq i \leq p: \Xi_{\Psi_i} \in E_{\Psi_i, n_i} \mid \Xi_{\Gamma^c} = \chi) \right| \\ &\leq \sum_{\vec{n} \in \mathbb{N}^d} \mathbb{P}(\forall 1 \leq i \leq p: \Pi_{\lambda \mu} \in E_{\Psi_i, n_i}) \sum_{i=1}^p C_1(\text{diam}(\Psi_i)) \exp(-C_2 d(\Psi, \Gamma^c)). \end{aligned}$$

Conclude by another application of (3.10) and writing out the Poisson moment. \square

Lemma 3.10. For $p, q \in \mathbb{N}$ and disjoint $\Psi_1, \dots, \Psi_p, \Upsilon_1, \dots, \Upsilon_q \in \mathcal{B}_b$,

$$\begin{aligned} & \left| \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) \left(\prod_{j=1}^q \Xi(\Upsilon_j) \right) - \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) \mathbb{E} \left(\prod_{i=1}^q \Xi(\Upsilon_i) \right) \right| \\ &\leq \lambda^{p+q} \left(\prod_{i=1}^p \mu(\Psi_i) \right) \left(\prod_{i=1}^q \mu(\Upsilon_i) \right) \min(C_{\Psi}, C_{\Upsilon}) \exp(-C_2 d(\Psi, \Upsilon)), \quad (3.13) \end{aligned}$$

where $\Psi := \bigcup_{i=1}^p \Psi_i$, $C_{\Psi} := \sum_{i=1}^p C_1(\text{diam}(\Psi_i))$, $\Upsilon := \bigcup_{i=1}^q \Upsilon_i$ and $C_{\Upsilon} := \sum_{i=1}^q C_1(\text{diam}(\Upsilon_i))$.

Proof. Let Γ be a large sphere such that $d(\Psi \cup \Upsilon, \Gamma^c) > d(\Psi, \Gamma^c)$. Hence,

$$d(\Psi, \Upsilon \cup \Gamma^c) = d(\Psi, \Upsilon). \quad (3.14)$$

By the DLR-equation (2.13),

$$\begin{aligned} & \left| \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) \left(\prod_{j=1}^q \Xi(\Upsilon_j) \right) - \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) \mathbb{E} \left(\prod_{i=1}^q \Xi(\Upsilon_i) \right) \right| \\ &\leq \int \left| \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \mid \Xi_{\Upsilon \cup \Gamma^c} = \chi \right) - \mathbb{E} \left(\prod_{i=1}^p \Xi(\Psi_i) \right) \right| \left(\prod_{j=1}^q \chi(\Upsilon_j) \right) \mathbb{P}(\Xi_{\Upsilon \cup \Gamma^c} \in d\chi). \end{aligned}$$

Applying (3.12) with Γ replaced by $\Gamma \setminus \Upsilon$ and noting that $(\Gamma \setminus \Upsilon)^c = \Upsilon \cup \Gamma^c$, we bound

$$\begin{aligned}
&\leq \lambda^p \left(\prod_{i=1}^p \mu(\Psi_i) \right) C_\Psi \exp(-C_2 d(\Psi, \Upsilon \cup \Gamma^c)) \int \left(\prod_{j=1}^q \chi(\Upsilon_j) \right) \mathbb{P}(\Xi_{\Upsilon \cup \Gamma^c} \in d\chi) \\
&= \lambda^p \left(\prod_{i=1}^p \mu(\Psi_i) \right) C_\Psi \exp(-C_2 d(\Psi, \Upsilon \cup \Gamma^c)) \mathbb{E} \left(\prod_{j=1}^q \Xi(\Upsilon_j) \right) \\
&\leq \lambda^{p+q} \left(\prod_{i=1}^p \mu(\Psi_i) \right) \left(\prod_{j=1}^q \mu(\Upsilon_j) \right) C_\Psi \exp(-C_2 d(\Psi, \Upsilon)),
\end{aligned}$$

where we use (3.14) and (3.11) to obtain the final inequality. The improved bound $\min(C_\Psi, C_\Upsilon)$ follows from the symmetry in Ψ and Υ . \square

Proof of Theorem 3.4. Let Ψ_{nj} , $n, j \in \mathbb{N}$, be a *dissection system* in \mathbb{N} [16, p. 20]. Let $k \in \mathbb{N}$ and let $\alpha_k(\cdot) := \mathbb{E} \Xi^k(\cdot)$ denote the k^{th} moment measure of Ξ . Let $\alpha_k = \alpha'_k + \alpha''_k$ be the Lebesgue decomposition ([16, Corollary 1.29]) of α_k with respect to μ^k , that is, α'_k is absolutely continuous with respect to μ^k while α''_k and μ^k are mutually singular. Define

$$g_k(K_1, \dots, K_k) := \limsup_{n \rightarrow \infty} \sum_{j_1, \dots, j_k \in \mathbb{N}} \mu^k(\Psi_{n\vec{j}})^{-1} \alpha_k(\Psi_{n\vec{j}}) \mathbf{1}\{(K_1, \dots, K_k) \in \Psi_{n\vec{j}}\},$$

where we write $\vec{j} := (j_1, \dots, j_k)$ and $\Psi_{n\vec{j}} := \Psi_{nj_1} \times \dots \times \Psi_{nj_k}$ and where we set $a/0 := 0$ for all $a \in \mathbb{R}$. Outside the *generalised diagonal*

$$D_k := \{(K_1, \dots, K_k) \in (\mathcal{C}^{(d)})^k \mid \text{there exist } \{i, j\} \subseteq \{1, \dots, k\} \text{ with } K_i = K_j\}$$

the function g_k coincides with

$$g_k^\neq(K_1, \dots, K_k) := \limsup_{n \rightarrow \infty} \sum_{j_1, \dots, j_k \in \mathbb{N}}^\neq \mu^k(\Psi_{n\vec{j}})^{-1} \alpha_k(\Psi_{n\vec{j}}) \mathbf{1}\{(K_1, \dots, K_k) \in \Psi_{n\vec{j}}\},$$

where the superscript \neq indicates summation over k -tuples with distinct entries. Since Ξ is simple, the measure $\alpha^{(k)}$ is the restriction of α_k to the complement of D_k , see [18, Exercise 6.9] and the proof of [18, Theorem 6.13]. Moreover, Lemma 3.8 shows that $\alpha^{(k)}$ is absolutely continuous with respect to μ^k . Therefore we obtain from the proof of [16, Theorem 1.28] and (2.11) that the above superior limits are actually limits for μ^k -a.e. (K_1, \dots, K_k) and, moreover, that

$$\rho_k(K_1, \dots, K_k) = g_k(K_1, \dots, K_k), \quad \alpha^{(k)}\text{-a.e. } (K_1, \dots, K_k) \in (\mathcal{C}^{(d)})^k. \quad (3.15)$$

Let $p, q \in \mathbb{N}$. Using (3.15) for $k \in \{p+q, p, q\}$ and combining this with Lemma 3.10, we obtain for $\alpha^{(p+q)}$ -a.e. $(K_1, \dots, K_{p+q}) \in (\mathcal{C}^{(d)})^{p+q}$ that

$$\begin{aligned}
&|\rho_{p+q}(K_1, \dots, K_{p+q}) - \rho_p(K_1, \dots, K_p) \rho_q(K_{p+1}, \dots, K_{p+q})| \\
&\leq \lambda^{p+q} \limsup_{n \rightarrow \infty} \sum_{j_1, \dots, j_{p+q} \in \mathbb{N}}^\neq \min(p, q) C_1(1) \exp(-C_2 d(\cup_{i=1}^p \Psi_{nj_i}, \cup_{i=p+1}^{p+q} \Psi_{nj_i})) \mathbf{1}\{(K_i)_{i=1}^{p+q} \in \Psi_{n\vec{j}}\} \\
&= \lambda^{p+q} \min(p, q) C_1(1) \exp(-C_2 d(\{K_1, \dots, K_p\}, \{K_{p+1}, \dots, K_{p+q}\})). \quad (3.16)
\end{aligned}$$

Here the inequality can be obtained from Lemma 3.10 as follows. Fix $(K_1, \dots, K_{p+q}) \in (\mathcal{C}^{(d)})^{p+q}$ such that $K_i \neq K_j$ for $i \neq j$. Then, for all sufficiently large $n \in \mathbb{N}$, there exists a unique $\vec{j} \in \mathbb{N}^{p+q}$ with distinct entries such that $(K_1, \dots, K_{p+q}) \in \Psi_{n\vec{j}} =: \Psi_n(K_1, \dots, K_{p+q})$. As $n \rightarrow \infty$ we have $\Psi_n(K_1, \dots, K_{p+q}) \downarrow \{K_1, \dots, K_{p+q}\}$. Moreover the diameter of $\Psi_n(K_1, \dots, K_{p+q})$ (with respect to the product of the Hausdorff metric) tends to 0. These facts do also imply the identity (3.16). Indeed, we just need combine them with definition (2.1) of $d(\cdot, \cdot)$ and the triangle inequality. \square

4 Asymptotic properties of U -statistics

In this section we fix an admissible Gibbs process Ξ as in Definition 2.9.

For $n \in \mathbb{N}$, let $W_n := [-\frac{1}{2}n^{1/d}, \frac{1}{2}n^{1/d}]^d$ be the centred cube of volume n , $\mathcal{C}_n^d := z^{-1}(W_n)$ and $\xi_n := \xi_{W_n}$, for $\xi \in \mathbf{N}$. Let $\Xi_n := \Xi_{W_n}$ and $\Xi_n^c := \Xi_{W_n^c}$ be the restriction of the Gibbs process to \mathcal{C}_n^d and $(\mathcal{C}_n^d)^c$ respectively. For a given mapping $F: \mathbf{N} \rightarrow \mathbb{R}$, we are interested in the asymptotic properties of $F(\Xi_n)$ as $n \rightarrow \infty$. We focus on special mappings F introduced next.

4.1 Admissible U -statistics

A function $h: (\mathcal{C}^{(d)})^k \rightarrow \mathbb{R}$ is called symmetric if $h(K_1, \dots, K_k) = h(K_{\pi(1)}, \dots, K_{\pi(k)})$, for all $K_1, \dots, K_k \in \mathcal{C}^{(d)}$ and every permutation π of k elements. It is translation invariant, if $h(K_1, \dots, K_k) = h(\theta_x K_1, \dots, \theta_x K_k)$, for all $K_1, \dots, K_k \in \mathcal{C}^{(d)}$ and $x \in \mathbb{R}^d$. Given a measurable symmetric and translation invariant function h we can define

$$F_h(\xi) := \frac{1}{k!} \int h(K_1, \dots, K_k) \xi^{(k)}(d(K_1, \dots, K_k)), \quad \xi \in \mathbf{N}. \quad (4.1)$$

In fact, the functions $F_h(\Xi)$ and $F_h(\Xi_n)$ are U -statistics of order k , cf. [23] and [18, Chapter 12]. Define

$$T(K, \xi) := \frac{1}{k!} \int h(K, K_2, \dots, K_k) \xi^{(k-1)}(d(K_2, \dots, K_k)), \quad (K, \xi) \in \mathcal{C}^{(d)} \times \mathbf{N}, \quad (4.2)$$

where the case $k = 1$ has to be read as $T(K) := h(K)$. Then

$$F_h(\xi) := \int T(K, \xi) \xi(dK), \quad \xi \in \mathbf{N}.$$

The authors of [1] call T a score function.

Definition 4.1. Let $k \in \mathbb{N}$ and let $h: (\mathcal{C}^{(d)})^k \rightarrow \mathbb{R}$ be measurable, symmetric and translation invariant. Then F_h in (4.1) is called an *admissible* function (of order k) if $h(K_1, \dots, K_k) = 0$, whenever either

$$\max_{2 \leq i \leq k} d_H(K_i, K_1) > r, \quad (4.3)$$

for some given $r > 0$, or when $K_i = K_1$, for some $i \in \{2, \dots, k\}$. If, moreover,

$$\|h\|_\infty := \sup_{K_1, \dots, K_k \in \mathcal{C}^{(d)}} |h(K_1, \dots, K_k)| < \infty, \quad (4.4)$$

then $F_h(\Xi_n)$, $n \in \mathbb{N}$, is called an admissible U -statistic of order k (of the Gibbs process Ξ_n).

Example 4.2. Consider Example 2.10 and recall that $\mathbf{N}_{\tilde{V}} := \{\xi \in \mathbf{N} \mid \xi(\tilde{V}^c) = 0\}$. For $\xi \in \mathbf{N}_{\tilde{V}}$ and $j \in \{1, \dots, d\}$, define using Q_j from (2.17)

$$G_j(\xi) := \frac{1}{j!} \int_{\tilde{V}_j} Q_j(K_1, \dots, K_j) \xi^{(j)}(d(K_1, \dots, K_j)). \quad (4.5)$$

Then G_j is an admissible function with $r = 2R$, cf. (2.4). In the special case of $d = 2$, facets are segments in the plane and $G_2(\xi)$ is the total number of intersections between segments in $\text{supp } \xi$ having different orientation. For $d = 3$, facets are thin circular plates and $G_2(\xi)$ is the total length of intersections of pairs of facets in $\text{supp } \xi$.

Admissible functions satisfy a moment condition introduced in [1, Definition 1.8].

Proposition 4.3. *If F_h is an admissible function of order k and $p \in \mathbb{N}$, then*

$$\sup_{n \in \mathbb{N}} \sup_{1 \leq q \leq p} \sup_{K_1, \dots, K_q \in \mathcal{C}_n^d} \mathbb{E}_{K_1, \dots, K_q} [\max\{|T(K_1, \Xi_n)|, 1\}^p] < \infty, \quad (4.6)$$

where T is from (4.2) and the inner supremum is an essential supremum with respect to the q^{th} factorial moment measure of Ξ .

Proof. Let $q \in \{1, \dots, p\}$. Since h is admissible, property (4.4) implies that

$$\max\{|T(K, \xi)|, 1\}^p \leq g_2(K, \xi), \quad (K, \xi) \in \mathcal{C}^{(d)} \times \mathbf{N},$$

where

$$g_2(K, \xi) := c \max\left\{1, \left(\xi(B(K, r))^{k-1}\right)^p\right\}$$

with $c := \max\{1, \|h\|_\infty/k!\}$. In the following we argue for μ^q -a.e. $(K_1, \dots, K_q) \in (\mathcal{C}^{(d)})^q$. Since $\kappa \leq 1$, Lemma 2.6 shows that $\mathbb{P}_{K_1, \dots, K_q}^!$ is stochastically dominated by the distribution of $\Pi_{\lambda\mu}$. Therefore,

$$\begin{aligned} \mathbb{E}_{K_1, \dots, K_q} [\max\{|T(K_1, \Xi_n)|, 1\}^p] &\leq \mathbb{E}[g_2(K_1, \Pi_{\lambda\mu} + \delta_{K_1} + \dots + \delta_{K_q})] \\ &\leq c \mathbb{E}[(q + \Pi_{\lambda\mu}(B(K_1, r)))^{p(k-1)}]. \end{aligned}$$

Using the inequality $(a + b)^{p(k-1)} \leq 2^{p(k-1)-1}(a^{p(k-1)} + b^{p(k-1)})$, for $a, b > 0$, we obtain that

$$\mathbb{E}_{K_1, \dots, K_q} [\max\{|T(K_1, \Xi_n)|, 1\}^p] \leq c 2^{p(k-1)-1} (q^{p(k-1)-1} + \mathbb{E}[\Pi_{\lambda\mu}(B(K_1, r))^{p(k-1)}]).$$

The random variable $\Pi_{\lambda\mu}(B(K_1, r))$ has a Poisson distribution with parameter

$$\begin{aligned} \mathbb{E}[\Pi_{\lambda\mu}(B(K_1, r))] &= \lambda \int \mathbf{1}\{d_H(K, K_1) \leq r\} \mu(dK) \\ &= \lambda \iint \mathbf{1}\{d_H(K + x, K_1) \leq r\} \mathbb{Q}(dK) dx. \end{aligned}$$

By (2.3) and (2.4) we may assume that $K_1 \subset B(z(K_1), R)$. Assume that $K \in \mathcal{C}^{(d)}$ satisfies $K \subset B(\mathbf{0}, R)$ and that $\|x - z(K_1)\| > R$. It follows from the definition of the Hausdorff distance that $d_H(K + x, K_1) \geq \|x - z(K_1)\| - R$. Hence, uniformly in K_1 under our assumptions, we obtain the finite bound

$$\begin{aligned} \mathbb{E}[\Pi_{\lambda\mu}(B(K_1, r))] &\leq \lambda \int \mathbf{1}\{\|x - z(K_1)\| \leq R\} dx + \lambda \int \mathbf{1}\{R < \|x - z(K_1)\| \leq r + R\} dx \\ &= \lambda \int \mathbf{1}\{\|x\| \leq r + R\} dx. \end{aligned}$$

Thus, the assertion follows from the moment properties of a Poisson random variable. \square

4.2 Factorization of weighted mixed moments

In this subsection we study an admissible pair (Ξ, T) defined as follows.

Definition 4.4. We call (Ξ, T) an admissible pair if Ξ is an admissible Gibbs process with $\lambda < \lambda_c(d)$ and the score function T corresponds to an admissible function F_h , cf. Definition 2.9 and Definition 4.1.

Example 4.5. The Gibbs facet process from Example 2.10 together with the admissible function G_j from Example 4.2 forms an admissible pair, for each $j \in \{1, \dots, d\}$.

Given $n, p, k_1, \dots, k_p \in \mathbb{N}$ and $K_1, \dots, K_p \in \mathcal{C}^{(d)}$, we define the weighted mixed moment

$$m^{(k_1, \dots, k_p)}(K_1, \dots, K_p; n) := \rho_p(K_1, \dots, K_p) \int_{\mathbb{N}} \left(T(K_1, \xi_n)^{k_1} \cdots T(K_p, \xi_n)^{k_p} \right) \mathbb{P}_{K_1, \dots, K_p}(d\xi). \quad (4.7)$$

In the following all equations and inequalities involving Palm distributions and correlation functions are to be understood in the a.e.-sense with respect to the appropriate factorial moment measures of Ξ .

Definition 4.6. We adapt the terminology of [1] and say that weighted mixed moments have fast decay of correlations, if there exist constants $a_l, b_l > 0$, $l \in \mathbb{N}$, such that

$$\begin{aligned} & \left| m^{(k_1, \dots, k_{p+q})}(K_1, \dots, K_{p+q}; n) \right. \\ & \left. - m^{(k_1, \dots, k_p)}(K_1, \dots, K_p; n) m^{(k_{p+1}, \dots, k_{p+q})}(K_{p+1}, \dots, K_{p+q}; n) \right| \\ & \leq b_l \exp(-a_l d(\{K_1, \dots, K_p\}, \{K_{p+1}, \dots, K_{p+q}\})), \end{aligned} \quad (4.8)$$

for all $n, p, q, k_1, \dots, k_{p+q} \in \mathbb{N}$ and for all $K_1, \dots, K_{p+q} \in z^{-1}(W_n)$, where $t := \sum_{i=1}^{p+q} k_i$.

We intend to use Theorem 3.4 to show that (4.8) holds in our context. The method from [1] is used and transformed step by step from point processes on \mathbb{R}^d to particle processes. Recall that \prec is the total order of $\mathcal{C}^{(d)}$ introduced in Section 3.2 and that intervals with respect to \prec are in $\mathcal{B}(\mathcal{C}^{(d)})$. In particular, for $K \in \mathcal{C}^{(d)}$, $(-\infty, K) = \{L \in \mathcal{C}^{(d)} \mid L \prec K\}$.

Let o be the zero-measure, i.e., $o(\Psi) = 0$, for all $\Psi \in \mathcal{B}(\mathcal{C}^{(d)})$. Abbreviate $[l] := \{1, \dots, l\}$, for $l \in \mathbb{N}$. We define a difference operator for a measurable function $\psi : \mathbb{N} \rightarrow \mathbb{R}$, $l \in \mathbb{N} \cup \{0\}$ and $K_1, \dots, K_l \in \mathcal{C}^{(d)}$ by

$$D_{K_1, \dots, K_l}^l \psi(\xi) := \begin{cases} \sum_{J \subseteq [l]} (-1)^{l-|J|} \psi(\xi_{(-\infty, K_*)} + \sum_{j \in J} \delta_{K_j}) & \text{if } l > 0, \\ \psi(o) & \text{if } l = 0, \end{cases} \quad (4.9)$$

where $K_* := \min\{K_1, \dots, K_l\}$ with respect to \prec . We say that ψ is \prec -continuous at ∞ , if $\lim_{K \uparrow \mathbb{R}^d} \psi(\xi_{(-\infty, K)}) = \psi(\xi)$, for all $\xi \in \mathbb{N}$.

We use the following factorial moment expansion (FME) proved in [2, Theorem 3.1] on a general Polish space. For stronger results in the special case of a Poisson process we refer to [17] and [18, Chapter 19].

Theorem 4.7. Let $\psi: \mathbf{N} \rightarrow \mathbb{R}$ be \prec -continuous at ∞ . Assume that, for all $l \in \mathbf{N}$,

$$\int_{(\mathcal{C}^{(d)})^l} \mathbb{E}_{K_1, \dots, K_l}^! [|D_{K_1, \dots, K_l}^l \psi(\Xi)|] \rho_l(K_1, \dots, K_l) \mu^l(d(K_1, \dots, K_l)) < \infty \quad (4.10)$$

and

$$\lim_{l \rightarrow \infty} \frac{1}{l!} \int_{(\mathcal{C}^{(d)})^l} \mathbb{E}_{K_1, \dots, K_l}^! [D_{K_1, \dots, K_l}^l \psi(\Xi)] \rho_l(K_1, \dots, K_l) \mu^l(d(K_1, \dots, K_l)) = 0. \quad (4.11)$$

Then, $\mathbb{E}[\psi(\Xi)]$ has the FME

$$\mathbb{E}[\psi(\Xi)] = \psi(o) + \sum_{l=1}^{\infty} \frac{1}{l!} \int_{(\mathcal{C}^{(d)})^l} D_{K_1, \dots, K_l}^l \psi(o) \rho_l(K_1, \dots, K_l) \mu^l(d(K_1, \dots, K_l)). \quad (4.12)$$

For an admissible pair (T, Ξ) , $K_1, \dots, K_p \in \mathcal{C}^{(d)}$ and $\xi \in \mathbf{N}$, set

$$\psi_{k_1, \dots, k_p}(K_1, \dots, K_p; \xi) := \prod_{i=1}^p T(K_i, \xi)^{k_i}, \quad (4.13)$$

$$\psi_{k_1, \dots, k_p}^! (K_1, \dots, K_p; \xi) := \prod_{i=1}^p T\left(K_i, \xi + \sum_{j=1}^p \delta_{K_j}\right)^{k_i}, \quad (4.14)$$

with $k_1, \dots, k_p \geq 1$. It holds that $\mathbb{E}_{K_1, \dots, K_p} [\psi(\Xi_n)] = \mathbb{E}_{K_1, \dots, K_p}^! [\psi^!(\Xi_n)]$. Given $p \in \mathbf{N}$ and $K_1, \dots, K_p \in \mathcal{C}^{(d)}$ we denote by $\rho_l^{(K_1, \dots, K_p)}$ the l th correlation function of $\mathbb{P}_{K_1, \dots, K_p}^!$. Further we let $(\mathbb{P}_{K_1, \dots, K_p}^!)_{L_1, \dots, L_l}^!$, $L_1, \dots, L_l \in \mathcal{C}^{(d)}$ denote the reduced Palm distributions of $\mathbb{P}_{K_1, \dots, K_p}^!$. It is easy to show that

$$\rho_p(K_1, \dots, K_p) \rho_l^{(K_1, \dots, K_p)}(L_1, \dots, L_l) = \rho_{p+l}(K_1, \dots, K_p, L_1, \dots, L_l), \quad (4.15)$$

and

$$(\mathbb{P}_{K_1, \dots, K_p}^!)_{L_1, \dots, L_l}^! = \mathbb{P}_{K_1, \dots, K_p, L_1, \dots, L_l}^!. \quad (4.16)$$

Lemma 4.8. For distinct $K_1, \dots, K_p \in \mathcal{C}^{(d)}$, $k_1, \dots, k_p, n \in \mathbf{N}$, $t_p := \sum_{i=1}^p k_i$ and k the order of the U -statistic, the functional $\psi^!$ admits the FME

$$\begin{aligned} \mathbb{E}_{K_1, \dots, K_p}^! [\psi_{k_1, \dots, k_p}^! (K_1, \dots, K_p; \Xi_n)] &= \psi_{k_1, \dots, k_p}^! (K_1, \dots, K_p; o) \\ &+ \sum_{l=1}^{t_p(k-1)} \frac{1}{l!} \int_{(\mathcal{C}^{(d)})^l} D_{L_1, \dots, L_l}^l \psi_{k_1, \dots, k_p}^! (K_1, \dots, K_p; o) \rho_l^{(K_1, \dots, K_p)}(L_1, \dots, L_l) \mu^l(d(L_1, \dots, L_l)). \end{aligned}$$

Proof. We abbreviate $\psi_{k_1, \dots, k_p}(K_1, \dots, K_p; \Xi)$ by $\psi(K_1, \dots, K_p; \Xi)$. The radius bound r from (4.3) for the function h implies that $\psi^!$ is \prec -continuous at ∞ . In [1, Lemma 5.1] it is shown that $\psi^!$ is the sum of U -statistics of orders not larger than $t_p(k-1)$. Thus, for $l \in (t_p(k-1), \infty)$ and all $L_1, \dots, L_l \in \mathcal{C}^{(d)}$, we have

$$D_{L_1, \dots, L_l}^l \psi^! (K_1, \dots, K_p; \xi) = 0. \quad (4.17)$$

This implies that (4.10), for $l \in (t_p(k-1), \infty)$, and (4.11) are satisfied for ψ^l from (4.14). We need to verify (4.10), for $l \in [1, t_p(k-1)]$. For $L_1, \dots, L_l \in \mathcal{C}^{(d)}$, $\xi \in \mathbf{N}$ and $J \subseteq [l]$, set

$$\xi_J := \xi_{(-\infty, L_*)} + \sum_{j \in J} \delta_{L_j},$$

where $L_* := \min\{L_1, \dots, L_l\}$ and $(-\infty, L_*)$ are with respect to the order \prec .

The difference operator D_{L_1, \dots, L_l}^l vanishes like in (4.17) as soon as $L_m \notin \bigcup_{i=1}^p B(K_i, 2r)$, for some $m \in [l]$. To prove this, expand (4.9) to obtain

$$\begin{aligned} D_{L_1, \dots, L_l}^l \psi^l(K_1, \dots, K_p; \xi) &= \sum_{J \subseteq [l], m \notin J} (-1)^{l-|J|} \psi^l(K_1, \dots, K_p; \xi_J) \\ &\quad + \sum_{J \subseteq [l], m \notin J} (-1)^{l-|J|-1} \psi^l(K_1, \dots, K_p; \xi_{J \cup \{m\}}) = 0, \end{aligned}$$

since, for fixed $J \subseteq [l]$ and $m \notin J$, $\psi^l(K_1, \dots, K_p; \xi_J) = \psi^l(K_1, \dots, K_p; \xi_{J \cup \{m\}})$. Then, we have

$$\begin{aligned} \psi^l(K_1, \dots, K_p; \xi_J) &\leq \prod_{i=1}^p \|h\|_\infty^{k_i} \left(\xi \left(\bigcup_{i=1}^p B(K_i, 2r) \right) + |J| + p \right)^{k_i(k-1)} \\ &\leq \|h\|_\infty^{t_p} \left(\xi \left(\bigcup_{i=1}^p B(K_i, 2r) \right) + |J| + p \right)^{t_p(k-1)}. \end{aligned}$$

Using this for difference operator we get

$$\begin{aligned} |D_{L_1, \dots, L_l}^l \psi^l(K_1, \dots, K_p; \xi)| &\leq \|h\|_\infty^{t_p} \sum_{J \subseteq [l]} \left(\xi \left(\bigcup_{i=1}^p B(K_i, 2r) \right) + |J| + p \right)^{t_p(k-1)} \\ &\leq \|h\|_\infty^{t_p} 2^l \left(\xi \left(\bigcup_{i=1}^p B(K_i, 2r) \right) + l + p \right)^{t_p(k-1)}. \end{aligned} \quad (4.18)$$

Using (4.16), the defining equation (2.5) and (4.18) results in

$$\begin{aligned} &\frac{1}{l!} \int_{(\mathcal{C}^{(d)})^l} (\mathbb{E}_{K_1, \dots, K_p}^l)_{L_1, \dots, L_l} [|D_{L_1, \dots, L_l}^l \psi^l(K_1, \dots, K_p; \Xi_n)|] \\ &\quad \times \rho_l^{(K_1, \dots, K_p)}(L_1, \dots, L_l) \mu^l(\mathrm{d}(L_1, \dots, L_l)) \\ &= \frac{1}{l!} \int_{(\mathcal{C}^{(d)})^l} \mathbb{E}_{K_1, \dots, K_p, L_1, \dots, L_l}^l [|D_{L_1, \dots, L_l}^l \psi^l(K_1, \dots, K_p; \Xi_n)|] \\ &\quad \times \rho_l^{(K_1, \dots, K_p)}(L_1, \dots, L_l) \mu^l(\mathrm{d}(L_1, \dots, L_l)) \\ &\leq \|h\|_\infty^{t_p} 2^l \mathbb{E}_{K_1, \dots, K_p} \left[\Xi \left(\bigcup_{i=1}^p B(K_i, r) \right)^l \left(\Xi \left(\bigcup_{i=1}^p B(K_i, r) \right) + l + p \right)^{t_p(k-1)} \right]. \end{aligned}$$

Since Ξ has all moments under the Palm distribution the finiteness of the last term and hence the validity of the condition for $l \in [1, t_p(k-1)]$ follows. This justifies the FME expansion. \square

Theorem 4.9. *Let (T, Ξ) be an admissible pair. Then the weighted moments have fast decay of correlations.*

Proof. Let $p, q, k_1, \dots, k_{p+q} \in \mathbb{N}$ be fixed. Let $u := \max(4R + R_\varphi, r)$, with R_φ being the finite interaction range of the admissible particle process as outlined in Definition 2.9 and taking into account the particle size from (2.4), as well as (4.3). Given $n \in \mathbb{N}$, $K_1, \dots, K_{p+q} \in z^{-1}(W_n)$, we set

$$s := d(\{K_1, \dots, K_p\}, \{K_{p+1}, \dots, K_{p+q}\}).$$

Without loss of generality we assume that $s \in (8u, \infty)$. Put t as in Definition 4.6 and t_p as in Lemma 4.8 respectively and let $t_q := \sum_{i=p+1}^{p+q} k_i$.

Then, using Lemma 4.8, (4.17) and (4.15), we obtain

$$\begin{aligned} m^{(k_1, \dots, k_{p+q})}(K_1, \dots, K_{p+q}; n) &= \mathbb{E}_{K_1, \dots, K_{p+q}}^1[\psi^!(K_1, \dots, K_{p+q}; \Xi_n)] \rho_{p+q}(K_1, \dots, K_{p+q}) \\ &= \sum_{l=0}^{t(k-1)} \frac{1}{l!} \int_{(\mathcal{C}_n^d)^l} D_{L_1, \dots, L_l}^l \psi^!(o) \rho_{l+p+q}(K_1, \dots, K_{p+q}, L_1, \dots, L_l) \mu^l(d(L_1, \dots, L_l)) \\ &= \sum_{l=0}^{t(k-1)} \frac{1}{l!} \int_{(\bigcup_{i=1}^{p+q} B(K_i, 2u))^l} D_{L_1, \dots, L_l}^l \psi^!(o) \rho_{l+p+q}(K_1, \dots, K_{p+q}, L_1, \dots, L_l) \mu^l(d(L_1, \dots, L_l)). \end{aligned}$$

Let $\Psi_{j,l} := \left(\bigcup_{i=1}^p B(K_i, 2u)\right)^j \times \left(\bigcup_{i=1}^q B(K_{p+i}, 2u)\right)^l$. Then, using the FME from Lemma 4.8,

$$\begin{aligned} m^{(k_1, \dots, k_{p+q})}(K_1, \dots, K_{p+q}; n) &= \sum_{l=0}^{t(k-1)} \frac{1}{l!} \sum_{j=0}^l \frac{l!}{j!(l-j)!} \int_{\Psi_{j,l-j}} D_{L_1, \dots, L_l}^l \psi^!(K_1, \dots, K_{p+q}; o) \\ &\quad \times \rho_{l+p+q}(K_1, \dots, K_{p+q}, L_1, \dots, L_l) \mu^l(d(L_1, \dots, L_l)) \\ &= \sum_{l=0}^{t(k-1)} \sum_{j=0}^l \frac{1}{j!(l-j)!} \int_{\Psi_{j,l-j}} \sum_{J \subseteq [l]} (-1)^{l-|J|} \psi^!\left(K_1, \dots, K_{p+q}; \sum_{j \in J} \delta_{L_j}\right) \\ &\quad \times \rho_{l+p+q}(K_1, \dots, K_{p+q}, L_1, \dots, L_l) \mu^l(d(L_1, \dots, L_l)). \end{aligned}$$

To compare the $(p+q)^{\text{th}}$ mixed moment with the product of the p -th and q^{th} mixed moments we use a factorization that holds for $L_1, \dots, L_j \in \bigcup_{i=1}^p B(K_i, 2u)$ and $L_{j+1}, \dots, L_l \in \bigcup_{i=1}^q B(K_{p+i}, 2u)$. If

$$K \in \bigcup_{i=1}^p B(K_i, 2u), \quad L \in \bigcup_{i=1}^q B(K_{p+i}, 2u),$$

then $K \cap L = \emptyset$. Hence,

$$\psi^!(K_1, \dots, K_{p+q}; \sum_{i=1}^l \delta_{L_i}) = \psi^!\left(K_1, \dots, K_p; \sum_{i=1}^j \delta_{L_i}\right) \psi^!\left(K_{p+1}, \dots, K_{p+q}; \sum_{i=j+1}^l \delta_{L_i}\right). \quad (4.19)$$

Using (4.19) and similar steps as in the case of the $(p+q)^{\text{th}}$ mixed moment we work with the product of p^{th} and q^{th} mixed moments.

$$\begin{aligned}
& m^{(k_1, \dots, k_p)}(K_1, \dots, K_p; n) m^{(k_{p+1}, \dots, k_{p+q})}(K_{p+1}, \dots, K_{p+q}; n) \\
&= \mathbb{E}_{K_1, \dots, K_p}^! [\psi^!(K_1, \dots, K_p, \Xi_n)] \mathbb{E}_{K_{p+1}, \dots, K_{p+q}}^! [\psi^!(K_{p+1}, \dots, K_{p+q}, \Xi_n)] \\
&\quad \times \rho_p(K_1, \dots, K_p) \rho_q(K_{p+1}, \dots, K_{p+q}) \\
&= \sum_{l_1, l_2=0}^{\infty} \int_{\Psi_{l_1, l_2}} D_{L_1, \dots, L_{l_1}}^{l_1} \psi^!(K_1, \dots, K_p; o) D_{N_1, \dots, N_{l_2}}^{l_2} \psi^!(K_{p+1}, \dots, K_{p+q}; o) \\
&\quad \times \rho_{l_1+p}(K_1, \dots, K_p, L_1, \dots, L_{l_1}) \rho_{l_2+q}(K_{p+1}, \dots, K_{p+q}, N_1, \dots, N_{l_2}) \\
&\quad \times \mu^{l_1}(\text{d}(L_1, \dots, L_{l_1})) \mu^{l_2}(\text{d}(N_1, \dots, N_{l_2})) \\
&= \sum_{l_1, l_2=0}^{\infty} \frac{1}{l_1! l_2!} \int_{\Psi_{l_1, l_2}} \sum_{J_1 \subseteq [l_1], J_2 \subseteq [l_2]} (-1)^{l_1+l_2-|J_1|-|J_2|} \\
&\quad \times \psi^!\left(K_1, \dots, K_p; \sum_{i \in J_1} \delta_{L_i}\right) \psi^!\left(K_{p+1}, \dots, K_{p+q}; \sum_{i \in J_2} \delta_{N_i}\right) \\
&\quad \times \rho_{l_1+p}(K_1, \dots, K_p, L_1, \dots, L_{l_1}) \rho_{l_2+q}(K_{p+1}, \dots, K_{p+q}, N_1, \dots, N_{l_2}) \\
&\quad \times \mu^{l_1}(\text{d}(L_1, \dots, L_{l_1})) \mu^{l_2}(\text{d}(N_1, \dots, N_{l_2})) \\
&= \sum_{l=0}^{t(k-1)} \sum_{j=0}^l \frac{1}{j!(l-j)!} \int_{\Psi_{j, l-j}} \sum_{J_1 \subseteq [j], J_2 \subseteq [l] \setminus [j]} (-1)^{l-|J_1|-|J_2|} \\
&\quad \times \psi^!\left(K_1, \dots, K_p; \sum_{i \in J_1} \delta_{L_i}\right) \psi^!\left(K_{p+1}, \dots, K_{p+q}; \sum_{i \in J_2} \delta_{L_i}\right) \\
&\quad \times \rho_{j+p}(K_1, \dots, K_p, L_1, \dots, L_j) \rho_{l-j+q}(K_{p+1}, \dots, K_{p+q}, L_{j+1}, \dots, L_l) \\
&\quad \times \mu^l(\text{d}(L_1, \dots, L_l)) \\
&= \sum_{l=0}^{t(k-1)} \sum_{j=0}^l \frac{1}{j!(l-j)!} \int_{\Psi_{j, l-j}} \sum_{J \subseteq [l]} (-1)^{l-|J|} \\
&\quad \times \psi^!\left(K_1, \dots, K_{p+q}; \sum_{i \in J} \delta_{L_i}\right) \rho_{j+p}(K_1, \dots, K_p, L_1, \dots, L_j) \\
&\quad \times \rho_{l-j+q}(K_{p+1}, \dots, K_{p+q}, L_{j+1}, \dots, L_l) \mu^l(\text{d}(L_1, \dots, L_l)).
\end{aligned}$$

Altogether we have, using c_1, c_2 from (3.5), that

$$\begin{aligned}
& \left| m^{(k_1, \dots, k_{p+q})}(K_1, \dots, K_{p+q}; n) - m^{(k_1, \dots, k_p)}(K_1, \dots, K_p; n) m^{(k_{p+1}, \dots, k_{p+q})}(K_{p+1}, \dots, K_{p+q}; n) \right| \\
&\leq \sum_{l=0}^{t(k-1)} \sum_{j=0}^l \sum_{J \subseteq [l]} \frac{(-1)^{l-|J|}}{j!(l-j)!} \int_{\Psi_{j, l-j}} \psi^!\left(K_1, \dots, K_{p+q}; \sum_{i \in J} \delta_{L_i}\right)
\end{aligned}$$

$$\begin{aligned}
& \times \left| \rho_{j+p}(K_1, \dots, K_p, L_1, \dots, L_j) \rho_{l-j+q}(K_{p+1}, \dots, K_{p+q}, L_{j+1}, \dots, L_l) \right. \\
& \left. - \rho_{l+p+q}(K_1, \dots, K_{p+q}, L_1, \dots, L_l) \right| \mu^l(d(L_1, \dots, L_l)) \\
& \leq \sum_{l=0}^{t(k-1)} \sum_{j=0}^l \sum_{J \subseteq [l]} \frac{(-1)^{l-|J|}}{j!(l-j)!} \int_{\Psi_{j,l-j}} \psi^l \left(K_1, \dots, K_{p+q}; \sum_{i \in J} \delta_{L_i} \right) \\
& \quad \times \lambda^{l+p+q} \min(j+p, l-j+q) c_1 \\
& \quad \times \exp(-c_2 d(\{K_1, \dots, K_p, L_1, \dots, L_j\}, \{K_{p+1}, \dots, K_{p+q}, L_{j+1}, \dots, L_l\})) \\
& \quad \times \mu^l(d(L_1, \dots, L_l)).
\end{aligned}$$

Using

$$T(K, \xi) \mathbf{1}\{\xi(\mathcal{C}^{(d)}) = n\} \leq \frac{n^{k-1}}{k} \|h\|_\infty,$$

we have

$$\sum_{J \subseteq [l]} \left| \psi^l \left(K_1, \dots, K_{p+q}; \sum_{i \in J} \delta_{L_i} \right) \right| \leq 2^l \left(\|h\|_\infty \frac{|p+q+l|^{k-1}}{k} \right)^t.$$

The difference of weighted mixed moments is finally bounded by

$$\begin{aligned}
& \left| m^{(k_1, \dots, k_{p+q})}(K_1, \dots, K_{p+q}; n) - m^{(k_1, \dots, k_p)}(K_1, \dots, K_p; n) m^{(k_{p+1}, \dots, k_{p+q})}(K_{p+1}, \dots, K_{p+q}; n) \right| \\
& \leq \sum_{l=0}^{t(k-1)} \sum_{j=0}^l \left(\|h\|_\infty \frac{|p+q+l|^{k-1}}{k} \right)^t \frac{\exp(-8u) 2^l (-1)^{l-|J|}}{j!(l-j)!} \mu(B(\{\mathbf{0}\}, 3u))^l \\
& \quad \times \lambda^{l+p+q} \min(j+p, l-j+q) c_1 \exp(-c_2 d(\{K_1, \dots, K_p\}, \{K_{p+1}, \dots, K_{p+q}\})).
\end{aligned}$$

As $\min(j+p, l-j+q) \leq l+p+q \leq kt$, we obtain the desired constants for exponential decay depending only on t and the attributes of the admissible pair. \square

4.3 Limit theorems

In this subsection we prove mean and variance asymptotics of admissible U -statistics $F_n := F_h(\Xi_n)$, $n \in \mathbb{N}$, as well as a central limit theorem. The approaches to the proofs come from [1] and they can be generalized from point processes in \mathbb{R}^d to particle processes. In Theorem 4.10 this generalization is possible thanks to our version of the p -moment condition in Proposition 4.3. The method of the proof of Theorem 4.11 is standard and its step by step generalization to particle processes is omitted.

To proceed we need good versions of the correlation functions ρ_1 and ρ_2 and the first and second order Palm distributions. The measure $\mathbb{E}[\int \mathbf{1}\{(K, \Xi) \in \cdot\} \Xi(dK)]$ is invariant under (joint) translations. By [16, Theorem 7.6] there exists a translation invariant version of the correlation function ρ_1 . Moreover, the first order Palm distributions can be chosen in an invariant way, that is

$$\mathbb{P}_{K-x}(\theta_x \Xi \in \cdot) = \mathbb{P}_K(\Xi \in \cdot), \quad (K, x) \in \mathcal{C}^{(d)} \times \mathbb{R}^d. \quad (4.20)$$

By the same argument we can assume that ρ_2 is translation invariant and

$$\mathbb{P}_{K-x, L-x}(\theta_x \Xi \in \cdot) = \mathbb{P}_{K, L}(\Xi \in \cdot), \quad (K, L, x) \in \mathcal{C}^{(d)} \times \mathcal{C}^{(d)} \times \mathbb{R}^d. \quad (4.21)$$

For the weighted mixed moments in (4.7) we denote some special cases as follows. For $K, L \in \mathcal{C}^{(d)}$ we set

$$\begin{aligned} m_{(1)}(K) &:= \mathbb{E}_K[T(K, \Xi)]\rho_1(K), \\ m_{(2)}(K, L) &:= \mathbb{E}_{K, L}[T(K, \Xi)T(L, \Xi)]\rho_2(K, L), \\ m_{(1,2)}(K) &:= \mathbb{E}_K[T^2(K, \Xi)]\rho_1(K). \end{aligned}$$

Further for $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ we abbreviate

$$\begin{aligned} m_{(1)}(K; n, x) &:= \mathbb{E}_K[T(K, \Xi_n^x)]\rho_1(K), \\ m_{(2)}(K, L; n, x) &:= \mathbb{E}_{K, L}[T(K, \Xi_n^x)T(L, \Xi_n^x)]\rho_2(K, L), \end{aligned}$$

where $\Xi_n^x := \Xi \cap (W_n - n^{\frac{1}{d}}x)$.

Theorem 4.10. *Let (T, Ξ) be an admissible pair. Then it holds that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}F_n = \int_{\mathcal{C}_0^d} m_{(1)}(K) \mathbb{Q}(dK), \quad (4.22)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \text{var } F_n &= \int_{\mathcal{C}_0^d} m_{(1,2)}(K) \mathbb{Q}(dK) \\ &+ \int_{(\mathcal{C}_0^d)^2} \int_{\mathbb{R}^d} (m_{(2)}(K, L+x) - m_{(1)}(K)m_{(1)}(L)) dx \mathbb{Q}(dK) \mathbb{Q}(dL) < \infty. \end{aligned} \quad (4.23)$$

Proof. From (2.5), (2.11) and (2.2), we deduce that

$$\begin{aligned} \mathbb{E}F_n &= \int_{\mathcal{C}_n^d} \mathbb{E}_K T(K, \Xi_n) \rho_1(K) \mu(dK) \\ &= \int_{\mathcal{C}_0^{(d)}} \int_{W_n} \mathbb{E}_{K+x} [T(K+x, \Xi_n)] \rho_1(K+x) dx \mathbb{Q}(dK). \end{aligned}$$

By equation (4.20) and translation invariance of T ,

$$\mathbb{E}_{K+x} [T(K+x, \Xi)] = \mathbb{E}_K [T(K, \Xi)].$$

Together with the translation invariance of ρ_1 this gives

$$\int_{\mathcal{C}_0^{(d)}} \int_{W_n} \mathbb{E}_{K+x} [T(K+x, \Xi)] \rho_1(K+x) dx \mathbb{Q}(dK) = n \int_{\mathcal{C}^{(d)}} m_{(1)}(K) \mathbb{Q}(dK).$$

To prove (4.22) it remains to show that

$$\frac{1}{n} \int_{\mathcal{C}_0^{(d)}} \int_{W_n} |\mathbb{E}_{K+x} [T(K+x, \Xi_n)] - \mathbb{E}_{K+x} [T(K+x, \Xi)]| dx \rho_1(K) \mathbb{Q}(dK)$$

tends to zero as $n \rightarrow \infty$. The function ρ_1 is bounded. For $K \in \mathcal{C}_0^d$, $K \subseteq B(\mathbf{0}, R)$ fixed and $x \in W_n$, we use (4.3) to obtain $T(K+x, \Xi_n) = T(K+x, \Xi)$ whenever $d(x, \partial W_n) \geq 2R$ for the distance from x to the boundary of W_n holds. The 1-moment condition (4.6) implies the existence of some $0 < a < \infty$ such that $\mathbb{E}_{K+x}[|T(K+x, \Xi_n)|] \leq a$ and $\mathbb{E}_{K+x}[|T(K+x, \Xi)|] \leq a$, uniformly in $n \in \mathbb{N}$ and $K+x \in \mathcal{C}_n^d$. Since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{L}_d \{x \in W_n \mid d(x, \partial W_n) \leq 2R\} = 0,$$

the first assertion of the theorem is proven.

By a standard point process calculation we obtain for the second moment

$$\begin{aligned} \mathbb{E}F_n^2 &= J_1 + J_2 := \int_{\mathcal{C}_0^d} \int_{W_n} \mathbb{E}_{K+u} T^2(K+u, \Xi_n) \rho_1(K+u) du \mathbb{Q}(dK) \\ &+ \int_{(\mathcal{C}_0^d)^2} \int_{(W_n)^2} \mathbb{E}_{K+u, L+v} [T(K+u, \Xi_n) T(L+v, \Xi_n)] \rho_2(K+u, L+v) du dv \mathbb{Q}(dK) \mathbb{Q}(dL). \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} \frac{J_1}{n} = \int_{\mathcal{C}^d} m_{(1,2)}(K) \mathbb{Q}(dK)$$

is obtained analogously to the mean value asymptotics above using the 2-moment condition (4.6).

In the second term J_2 we use the substitutions $x = n^{-\frac{1}{d}}u$ and $z = v - u$, obtaining

$$\begin{aligned} \frac{J_2}{n} &= \int_{(\mathcal{C}_0^d)^2} \int_{W_1} \int_{W_n - n^{-\frac{1}{d}}x} \mathbb{E}_{K+n^{-\frac{1}{d}}x, L+z+n^{-\frac{1}{d}}x} [T(K+n^{-\frac{1}{d}}x, \Xi_n) T(L+z+n^{-\frac{1}{d}}x, \Xi_n)] \\ &\times \rho_2(K+n^{-\frac{1}{d}}x, L+z+n^{-\frac{1}{d}}x) dz dx \mathbb{Q}(dK) \mathbb{Q}(dL) \\ &= \int_{(\mathcal{C}_0^d)^2} \int_{W_1} \int_{W_n - n^{-\frac{1}{d}}x} \mathbb{E}_{K, L+z} [T(K, \Xi_n^x) T(L+z, \Xi_n^x)] \rho_2(K, L+z) dz dx \mathbb{Q}(dK) \mathbb{Q}(dL), \end{aligned}$$

where we have used the invariance of ρ_2 , (4.21), the invariance of T and $(\Xi + n^{-\frac{1}{d}}x)_n - n^{-\frac{1}{d}}x = \Xi_n^x$. Since $\text{var } F_n = \mathbb{E}F_n^2 - (\mathbb{E}F_n)^2$, we investigate the expression $\frac{1}{n}(J_2 - (\mathbb{E}F_n)^2)$. It takes the form

$$\int_{(\mathcal{C}_0^d)^2} \int_{W_1} \int_{W_n - n^{-\frac{1}{d}}x} (m_{(2)}(K, L+z; n, x) - m_{(1)}(K; n, x)m_{(1)}(L+z; n, x)) dz dx \mathbb{Q}(dK) \mathbb{Q}(dL). \quad (4.24)$$

Splitting the innermost integral in (4.24) into the two terms

$$\int_{W_n - n^{-\frac{1}{d}}x} (\dots) dz = \int_{W_n - n^{-\frac{1}{d}}x} (\dots) \mathbf{1}\{|z| \leq M\} dz + \int_{W_n - n^{-\frac{1}{d}}x} (\dots) \mathbf{1}\{|z| > M\} dz, \quad (4.25)$$

for an arbitrary $M > 0$, we observe that the part of (4.24) corresponding to the first term of (4.25), i.e.

$$\begin{aligned} &\int_{(\mathcal{C}_0^d)^2} \int_{W_1} \int_{W_n - n^{-\frac{1}{d}}x} (m_{(2)}(K, L+z; n, x) - m_{(1)}(K; n, x)m_{(1)}(L+z; n, x)) \\ &\times \mathbf{1}\{|z| \leq M\} dz dx \mathbb{Q}(dK) \mathbb{Q}(dL), \end{aligned}$$

converges to

$$\int_{(C_0^d)^2} \int_{\mathbb{R}^d} (m_{(2)}(K, L+x) - m_{(1)}(K)m_{(1)}(L)) dx \mathbb{Q}(dK) \mathbb{Q}(dL),$$

when first $n \rightarrow \infty$ and then $M \rightarrow \infty$. Using (4.8), the absolute value of the second term in (4.25) can be bounded uniformly in n by

$$b_2 \int_{|z|>M} \exp(-a_2 d_H(K, L+z)) dz,$$

which tends to zero when $M \rightarrow \infty$. Thus the part of (4.24) corresponding to the second term of (4.25), i.e.

$$\begin{aligned} & \int_{(C_0^d)^2} \int_{W_1} \int_{W_{n-n\frac{1}{d}}x} (m_{(2)}(K, L+z; n, x) - m_{(1)}(K; n, x)m_{(1)}(L+z; n, x)) \\ & \quad \times \mathbf{1}\{|z| > M\} dz dx \mathbb{Q}(dK) \mathbb{Q}(dL), \end{aligned}$$

converges to zero. We can justify these limits analogously to Lemma 4.1 in [1]. The boundedness in (4.23) follows from the 2-moment condition (4.6) for the first term and from (4.8) for the second term. \square

Theorem 4.11. *Let (T, Ξ) be an admissible pair. If, for some $\beta \in (0, \infty)$,*

$$\liminf_{n \rightarrow \infty} \frac{\text{var } F_n}{n^\beta} > 0, \tag{4.26}$$

then we have the CLT

$$\frac{F_n - \mathbb{E}F_n}{(\text{var } F_n)^{1/2}} \xrightarrow[n \rightarrow \infty]{d} N(0, 1). \tag{4.27}$$

Proof. Denote $\bar{F}_n := F_n - \mathbb{E}F_n$. The idea is to prove that the l^{th} order cumulants of $(\text{var } F_n)^{-1/2} \bar{F}_n$ vanish as $n \rightarrow \infty$ and l large. This follows by showing that (4.6) and (4.8) imply volume order growth (i.e., of order $O(n)$) for the l^{th} order cumulant of \bar{F}_n , $l \geq 2$, and using the assumption (4.26). Then (4.27) holds. The details are analogous to [1, p. 881-886], with the difference that deal with measures defined on C_n^d . \square

Remark 4.12. Checking assumption (4.26) is a problem in its own right. In view of [30, Theorem 1.1] it can be expected that (4.26) holds, whenever the activity λ is smaller than some critical branching intensity. In particular this should apply to the facet processes from Example 2.10 and Example 4.2. Remark 3.5 shows that this critical branching intensity can be smaller than the critical intensity of our dominating Boolean model.

Remark 4.13. It might be conjectured that (4.26) holds for any admissible pair (Ξ, T) (see Definition 4.4), provided that T is non-degenerate in a suitable sense. A proof should be based on the specific properties of disagreement percolation. We leave this as an interesting open problem, which is beyond the scope of our paper.

5 Concluding remarks

The results of this paper have the potential for several extensions. The disagreement coupling and its consequences, for instance, can probably be derived for other potentials (without a deterministic range) and other spaces. A similar comment applies to our results for admissible U -statistics. Moreover, it can be expected that these results can be extended to stabilizing functionals, as studied in [1]. Again one would then obtain an improvement in the range of possible activities; cf. (3.6) and (3.7).

Acknowledgement: This research started when then Ph.D. student J. Večeřa visited G. Last in late 2016, the support of KIT Karlsruhe and of the DFG-Forschergruppe FOR 1548 “Geometry and Physics of Spatial Random Systems” for this stay is highly appreciated. Further the authors were supported as follows: V. Beneš by Czech Science Foundation, project 19-04412S, J. Večeřa by Charles University, project SVV260454. C. Hofer-Temmel thanks the Charles University, Faculty of Mathematics and Physics, and the KIT Karlsruhe for their hospitality.

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