Decoupling Based Cartesian Impedance Control of Flexible Joint Robots

Accepted for publication in the Proc. 2003 IEEE Int. Conf. on Robotics and Automation.

Christian Ott*, Alin Albu-Schäffer*, Andreas Kugi** and Gerd Hirzinger*

*German Aerospace Center (DLR) - Institute of Robotics and Mechatronics

**Saarland University - Chair of System Theory and Automatic Control

christian.ott@dlr.de

***Abstract*— This paper addresses the impedance control problem for flexible joint manipulators. An impedance controller structure is proposed, which is based on an exact decoupling of the torque dynamics from the link dynamics. A formal stability analysis of the proposed controller is presented for the general tracking case. Preliminary experimental results are given for a single flexible joint.**

I. Introduction

Whenever a robotic manipulator is supposed to get in contact with its environment in order to perform some manipulation tasks, a compliant behaviour of the manipulator is desired. The achievement of such a compliant behaviour by control therefore got a classical problem in robotics research [3]. For the case of a manipulator with rigid joints, various approaches to this problem have been studied in the literature and led to control techniques such as impedance control, admittance control or stiffness control. Compared to this, only little work has been spent on the compliant control problem for robotic manipulators with flexible joints.

Maybe the most obvious approach to the impedance control problem for a robot with flexible joints is based on a singular perturbation analysis of the flexible joint model. From this perspective an impedance controller may be designed in the same manner as for a robot with rigid joints. The flexibility of the joints is then treated in a sufficiently fast inner torque control loop. While this approach is very attractive at a first glance due to the simplicity of the resulting controllers, it has the conceptual problem that the singular perturbation approach does not admit a formal stability analysis for the complete model of the flexible joint robot without referring to an approximate consideration.

In contrast to this approach, in this paper an impedance controller is presented which fully accounts for the flexibility of the joints. The proposed controller structure is based on an internal torque controller which decouples the torque dynamics from the link dynamics exactly and thus leads to a cascaded structure. The desired torque for this inner loop results from a standard impedance control law. Stability results from the theory of cascaded systems [8] can then be used to prove the stability of the overall closed loop system.

Notice that the proposed combination of a decoupling based torque controller with an outer control law for the link positions is strongly related to the works of Lin and Goldenberg. While their design idea in [6] and [7] is similar to the one followed in this paper, their focus lies merely on the position control problem and leads to different controllers. Consequently, their stability analysis in [6] and [7] cannot be applied to the impedance control problem in a straightforward manner.

The paper is organized as follows. First, the considered model is given in Section II. In Section III the proposed torque controller is presented and compared to a simpler singular perturbation based controller. Next, the impedance control law is given in Section IV. Section V contains the stability proof of the decoupling based torque controller from Section III in combination with the outer impedance controller from Section IV. Finally, an experimental comparison of the proposed controller to a singular perturbation based controller is presented for a single flexible joint in Section VI.

II. Considered Model

In this paper a simplified model of a robot with n flexible joints is considered as proposed in [12]

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{ext} \ , \tag{1}$$

$$\boldsymbol{B}\ddot{\boldsymbol{\theta}} + \boldsymbol{\tau} = \boldsymbol{\tau}_m \ , \tag{2}$$

$$\boldsymbol{\tau} = \boldsymbol{K}(\boldsymbol{\theta} - \boldsymbol{q}) \ . \tag{3}$$

Herein, $\boldsymbol{q} \in \Re^n$ is the vector of link positions and $\boldsymbol{\theta} \in \Re^n$ the vector of motor positions. The vector of transmission torques is denoted by $\boldsymbol{\tau}$. Equation (1) contains the symmetric and positive definite mass matrix $\boldsymbol{M}(\boldsymbol{q})$, the vector of Coriolis and centripetal torques $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ and the vector of gravitational torques $\boldsymbol{g}(\boldsymbol{q})$. $\boldsymbol{B}$ and $\boldsymbol{K}$ are diagonal matrices containing the motor inertias and the stiffnesses for the individual joints. $\boldsymbol{\tau}_m$ is the vector of motor torques which will serve as the control input and $\boldsymbol{\tau}_{ext}$ is a vector of external torques which are exhibited by the manipulator's environment.

Herein it is furtheron assumed that the external torques

$\boldsymbol{\tau}_{ext}$ can be measured. This can be realized at least in applications where these torques at joint level result from forces and torques at the manipulator's endeffector and can therefore be measured e.g. by a 6DOF force/torque-sensor. For the further analysis, the model (1)-(2) may be rewritten by choosing $(\boldsymbol{q}^T, \dot{\boldsymbol{q}}^T, \boldsymbol{\tau}^T, \dot{\boldsymbol{\tau}}^T)^T$ as state variables

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{ext} \ , \qquad (4)$$

$$\boldsymbol{B}\boldsymbol{K}^{-1}\ddot{\boldsymbol{\tau}} + \boldsymbol{\tau} = \boldsymbol{\tau}_m - \boldsymbol{B}\boldsymbol{M}(\boldsymbol{q})^{-1}(\boldsymbol{\tau} + \boldsymbol{\tau}_{ext} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}))\ . \qquad (5)$$

Based on this model a cascaded control design procedure will be presented in the next sections. An outer loop impedance controller (treated in Section IV) generates a desired torque vector $\boldsymbol{\tau}_d$ for an inner torque control loop. The design of the torque controller is treated in the next section.

III. Torque Controller

Obviously, some undesired terms of the torque dynamics equation (5) may be easily compensated with a feedback compensation of the form

$$\boldsymbol{\tau}_m = \boldsymbol{u} + \boldsymbol{B}\boldsymbol{M}(\boldsymbol{q})^{-1}(\boldsymbol{\tau} + \boldsymbol{\tau}_{ext} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}))\ , \quad (6)$$

with the new input variables $\boldsymbol{u}$. This leads to the system

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) &= \boldsymbol{\tau} + \boldsymbol{\tau}_{ext} \ , & (7) \\ \boldsymbol{B}\boldsymbol{K}^{-1}\ddot{\boldsymbol{\tau}} + \boldsymbol{\tau} &= \boldsymbol{u} \ . & (8) \end{aligned}$$

From replacing the torque variables[1] by introducing a *desired torque*-variable $\boldsymbol{\tau}_d$ and a torque error variable $\boldsymbol{z}$

$$\boldsymbol{z} = \boldsymbol{\tau} - \boldsymbol{\tau}_d \ , \qquad (9)$$

we obtain

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) &= \boldsymbol{z} + \boldsymbol{\tau}_d + \boldsymbol{\tau}_{ext} \\ \boldsymbol{B}\boldsymbol{K}^{-1}(\ddot{\boldsymbol{z}} + \ddot{\boldsymbol{\tau}}_d) + \boldsymbol{z} + \boldsymbol{\tau}_d &= \boldsymbol{u} \ . \end{aligned}$$

Based on this system two different torque controllers are given in the following. The first controller results from a singular perturbation approach and shall be used as a reference for the second controller later on in the experiments. The second controller achieves an exact decoupling of the torque dynamics from the link dynamics. This exact decoupling has the conceptual advantage that it admits, in combination with the impedance controller from the next section, a stability analysis for the complete flexible model without the need to refer to an approximate consideration as in the case of the singular perturbation based torque controller.

[1]which means shifting the steady state to $\boldsymbol{0}$

A. Singular Perturbation Based Controller

It shall not be in the scope of this paper to treat the singular perturbation analysis of flexible joint robots in detail. Instead we refer to [4] for a comprehensive treatment of the theoretical basis. In a singular perturbation analysis of (4)-(5), the flexible joint model is virtually split up into a fast and a slow subsystem for the joint torques $\boldsymbol{\tau}$ and the link positions $\boldsymbol{q}$ respectively. From these two subsystems it is then possible to design an inner loop controller for $\boldsymbol{\tau}$ and an outer loop controller for $\boldsymbol{q}$ separately.
For this study a singular perturbation based controller similar to the one in [9] is considered

$$\boldsymbol{u} = \boldsymbol{\tau}_d + \boldsymbol{B}\boldsymbol{K}^{-1}(-\boldsymbol{K}_s\dot{\boldsymbol{\tau}} - \boldsymbol{K}_t\boldsymbol{z})\ , \qquad (10)$$

which is (under a singular perturbation consideration) sufficient to stabilize the joint torque dynamics around the equilibrium point $\boldsymbol{\tau} = \boldsymbol{\tau}_d$. The matrices $\boldsymbol{K}_s$ and $\boldsymbol{K}_t$ herein are some positive definite controller gain matrices. Notice that[2], the controller (10) results in the following link dynamics:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}_d + \boldsymbol{\tau}_{ext} \ .$$

In the following the controller of (10) is extended by some additional terms, which achieve an exact decoupling of the torque dynamics but are not necessary from a singular perturbation point of view.

B. Decoupling Based Torque Controller

It is easy to see that an exact feedback decoupling of the torque dynamics may be obtained by a feedback law of the form

$$\boldsymbol{u} = \boldsymbol{\tau}_d + \boldsymbol{B}\boldsymbol{K}^{-1}(\ddot{\boldsymbol{\tau}}_d - \boldsymbol{K}_s\dot{\boldsymbol{z}} - \boldsymbol{K}_t\boldsymbol{z})\ . \qquad (11)$$

Again, the matrices $\boldsymbol{K}_s$ and $\boldsymbol{K}_t$ are chosen as some positive definite matrices. This leads to a system in cascaded form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{z} + \boldsymbol{\tau}_d + \boldsymbol{\tau}_{ext} \ , \qquad (12)$$

$$\ddot{\boldsymbol{z}} + \boldsymbol{K}_s\dot{\boldsymbol{z}} + (\boldsymbol{K}_t + \boldsymbol{K}\boldsymbol{B}^{-1})\boldsymbol{z} = \boldsymbol{0} \ . \qquad (13)$$

Notice that compared to (10) the controller in (11) also contains the time derivatives of the desired torque $\boldsymbol{\tau}_d$ up to the second order.
In the next section, the design of $\boldsymbol{\tau}_d$ is treated such that a desired impedance behaviour is achieved.

IV. Task Space Impedance Controller

In this section an impedance controller is presented which can be used in combination with the two torque controllers from the last section.
It is assumed that the desired behaviour of the manipulator can be described in task-space coordinates[3] $\boldsymbol{x} \in \Re^m$ and

[2]again under a singular perturbation consideration
[3]e.g. describing the endeffector movement

the mapping between these coordinates and the link angles is known $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q})$. The relevant coordinate mappings for the first and the second derivatives can then be computed via the Jacobian $\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}$ as

$$\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \; , \quad (14)$$

$$\ddot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \; . \quad (15)$$

For reasons of simplicity only the nonredundant and nonsingular case shall be treated herein, thus it is assumed that $m = n$ and that the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ has full rank in the considered region of the workspace. A description of an appropriate singularity treatment can be found in [5]. It is furtheron assumed that in the considered workspace the vector function $\boldsymbol{f}(\boldsymbol{q})$ is a one-to-one mapping. Under these assumptions the coordinates $\boldsymbol{x}$ completely describe the rigid-body-behaviour of the robot and can be used as generalized coordinates.

Notice that, while the above-mentioned assumptions on the coordinates $\boldsymbol{x}$ are trivially fulfilled for a joint space consideration (with $\boldsymbol{x} = \boldsymbol{q}$) in the whole workspace, in the case of a desired impedance behaviour in Cartesian coordinates it is generally not possible to find coordinates which fulfill the assumptions globally [13].

However, the analysis in this paper focuses on globally valid statements and is therefore formally based on a globally valid set of coordinates $\boldsymbol{x}$.

With these assumptions, the external torques $\boldsymbol{\tau}_{ext}$ can also be written in task space coordinates as $\boldsymbol{F}_{ext}$ via the well known relationship

$$\boldsymbol{\tau}_{ext} = \boldsymbol{J}(\boldsymbol{q})^T \boldsymbol{F}_{ext} \quad (16)$$

and equation (12) may also be rewritten as[4]

$$\boldsymbol{\Lambda}(\boldsymbol{x})\ddot{\boldsymbol{x}} + \boldsymbol{\mu}(\boldsymbol{x}, \dot{\boldsymbol{x}})\dot{\boldsymbol{x}} + \boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{q})^{-T}(\boldsymbol{z} + \boldsymbol{\tau}_d) + \boldsymbol{F}_{ext} \; , \quad (17)$$

with the equivalent task space mass matrix

$$\boldsymbol{\Lambda}(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{q})^{-T}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})^{-1} \; , \quad (18)$$

$$\boldsymbol{\mu}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{J}(\boldsymbol{q})^{-T}\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{J}(\boldsymbol{q})^{-1} - \boldsymbol{\Lambda}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})^{-1} \quad (19)$$

and $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{q})^{-T}\boldsymbol{g}(\boldsymbol{q})$.

It is well known that in the case of a rigid robot an arbitrary second order impedance behaviour can be (at least theoretically) realized by feedback. It can be shown that for a flexible joint robot this is not possible exactly, therefore only an approximation can be expected here. By comparing the structure of equation (17) to the dynamical equations of a robot with rigid joints one can see that the only difference is the occurrence of the torque error term $\boldsymbol{J}(\boldsymbol{q})^{-T}\boldsymbol{z}$ in (17). If for the design of the impedance controller this term is neglected, then $\boldsymbol{\tau}_d$ can be chosen according to an impedance controller for a robot with rigid joints.

In our previous experiments with singular perturbation based controllers the realization of a desired second order impedance behaviour with an arbitrary inertia turned out to be very difficult [1]. Also, in many applications the main focus merely lies on the realization of a desired stiffness and damping behaviour. Consequently, it is also considered herein that the desired behaviour in the task space can be characterized by a positive definite damping matrix $\boldsymbol{D}_d$ and a positive definite stiffness matrix $\boldsymbol{K}_d$, while the manipulator's mass matrix shall be maintained. Thus, for a given trajectory $\boldsymbol{x}_d(t)$, the desired behaviour of the manipulator with respect to an external force $\boldsymbol{F}_{ext}$ is given by

$$\boldsymbol{\Lambda}(\boldsymbol{x})\ddot{\boldsymbol{e}}_x + (\boldsymbol{\mu}(\boldsymbol{x}, \dot{\boldsymbol{x}}) + \boldsymbol{D}_d)\dot{\boldsymbol{e}}_x + \boldsymbol{K}_d\boldsymbol{e}_x = \boldsymbol{F}_{ext} \quad (20)$$

$$\boldsymbol{e}_x = \boldsymbol{x} - \boldsymbol{x}_d \; .$$

The desired trajectory $\boldsymbol{x}_d(t)$ is assumed to be continuously differentiable up to the order two. For the desired impedance behaviour, and under the above assumption on $\boldsymbol{J}(\boldsymbol{q})$ and $\boldsymbol{x}_d(t)$, three important properties shall be mentioned:

Property 1: *For $\boldsymbol{F}_{ext} = \boldsymbol{0}$, the system (20) with the positive definite matrices $\boldsymbol{K}_d$ and $\boldsymbol{D}_d$ is uniformly globally asymptotically stable.*

Property 2: *For $\dot{\boldsymbol{x}}_d(t) = \boldsymbol{0}$, the system (20) with the positive definite matrices $\boldsymbol{K}_d$ and $\boldsymbol{D}_d$ gets time-invariant and represents a passive mapping from the external force $\boldsymbol{F}_{ext}$ to the velocity error $\dot{\boldsymbol{e}}_x$.*

Property 3: *The matrix $\dot{\boldsymbol{\Lambda}}(\boldsymbol{x}) - 2\boldsymbol{\mu}(\boldsymbol{x}, \dot{\boldsymbol{x}})$ is skew symmetric.*

The proof of property 1 can be found, e.g., in [10]. Although it is drawn therein only for the case of joint space coordinates ($\boldsymbol{x} = \boldsymbol{q}$), it is obviously also valid for general coordinates $\boldsymbol{x}$ under the above-mentioned assumptions. Property 2 can be shown easily with the storage function $V_s = \frac{1}{2}\dot{\boldsymbol{e}}_x^T\boldsymbol{\Lambda}(\boldsymbol{x})\dot{\boldsymbol{e}}_x + \frac{1}{2}\boldsymbol{e}_x^T\boldsymbol{K}_d\boldsymbol{e}_x$.

Notice that property 2 is very important from a practical point of view. For cases when the robot is to be expected to get in contact with an unknown environment, a common assumption is that the environment can be represented by a passive[5] system which is in feedback interconnection with the robot. Then the above passivity property is sufficient in order to preserve the stability of the whole system.

Obviously, for the case of a rigid joint robot (with $\boldsymbol{z} = \boldsymbol{0}$), the desired behaviour (20) can be achieved for the system (17) by the following control law

$$\boldsymbol{\tau}_d = \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}(\boldsymbol{q})^T(\boldsymbol{\Lambda}(\boldsymbol{x})\ddot{\boldsymbol{x}}_d + \boldsymbol{\mu}(\boldsymbol{x}, \dot{\boldsymbol{x}})\dot{\boldsymbol{x}}_d - \boldsymbol{D}_d\dot{\boldsymbol{e}}_x - \boldsymbol{K}_d\boldsymbol{e}_x) \; . \quad (21)$$

[4]The substitution $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{x})$ may be obmitted in the following.

[5]with respect to the input/output-pair $(\dot{\boldsymbol{e}}_x, -\boldsymbol{F}_{ext})$

In the case of the flexible model, the controller (21) in combination with the torque controller (11) leads to the following closed loop dynamics

$$\boldsymbol{\Lambda}(\boldsymbol{x})\ddot{\boldsymbol{e}}_x + (\boldsymbol{\mu}(\boldsymbol{x},\dot{\boldsymbol{x}}) + \boldsymbol{D}_d)\dot{\boldsymbol{e}}_x + \boldsymbol{K}_d\boldsymbol{e}_x = \boldsymbol{F}_{ext} + \boldsymbol{J}(\boldsymbol{q})^{-T}\boldsymbol{z}\ , \tag{22}$$

$$\ddot{\boldsymbol{z}} + \boldsymbol{K}_s\dot{\boldsymbol{z}} + (\boldsymbol{K}_t + \boldsymbol{K}\boldsymbol{B}^{-1})\boldsymbol{z} = \boldsymbol{0}\ . \tag{23}$$

While the desired impedance characteristics is realized only approximately for the flexible joint robot with the described controller, as one can see from (22), property 1 and property 2 still hold for the system (22)-(23). This is shown in detail in the next section.

V. Stability Analysis

First we formulate the main result of this paper in form of two propositions.

Proposition 1: *For $\boldsymbol{F}_{ext} = \boldsymbol{0}$, the system (22)-(23) with the positive definite matrices $\boldsymbol{K}_s$, $\boldsymbol{K}_t$, $\boldsymbol{K}_d$ and $\boldsymbol{D}_d$ is uniformly globally asymptotically stable.*

Proposition 2: *For $\dot{\boldsymbol{x}}_d(t) = \boldsymbol{0}$, the system (22)-(23) with the positive definite matrices $\boldsymbol{K}_s$, $\boldsymbol{K}_t$, $\boldsymbol{K}_d$ and $\boldsymbol{D}_d$ gets time-invariant and represents a passive mapping from the external force $\boldsymbol{F}_{ext}$ to the velocity error $\dot{\boldsymbol{e}}_x$.*

A. Proof of Prop. 1

For the stability analysis of the system (22)-(23) it is important to notice that the system is time-variant due to the occurence of $\boldsymbol{x}_d(t)$ in the equations of motion. In order to rewrite the system (22)-(23) for $\boldsymbol{F}_{ext} = \boldsymbol{0}$ in the state variables $\boldsymbol{e}_x, \dot{\boldsymbol{e}}_x, \boldsymbol{z}, \dot{\boldsymbol{z}}$ only, it is convenient to make the following substitutions: $\boldsymbol{J}(\boldsymbol{e}_x, t) = \boldsymbol{J}(\boldsymbol{f}^{-1}(\boldsymbol{x}))$, $\boldsymbol{\Lambda}(\boldsymbol{e}_x, t) = \boldsymbol{\Lambda}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{e}_x, \dot{\boldsymbol{e}}_x, t) = \boldsymbol{\mu}(\boldsymbol{x}, \dot{\boldsymbol{x}})$. Also for the linear part of the system the substitutions $\boldsymbol{w} = (\boldsymbol{w}_1^T, \boldsymbol{w}_2^T)^T = (\boldsymbol{z}^T, \dot{\boldsymbol{z}}^T)^T$ and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{K}_s & -(\boldsymbol{K}_t + \boldsymbol{K}\boldsymbol{B}^{-1}) \end{bmatrix}$$

are made. This leads to the system:

$$\boldsymbol{\Lambda}(\boldsymbol{e}_x, t)\ddot{\boldsymbol{e}}_x + (\boldsymbol{\mu}(\boldsymbol{e}_x, \dot{\boldsymbol{e}}_x, t) + \boldsymbol{D}_d)\dot{\boldsymbol{e}}_x + \boldsymbol{K}_d\boldsymbol{e}_x = \boldsymbol{J}(\boldsymbol{e}_x, t)^{-T}\boldsymbol{w}_1\ ,$$

$$\dot{\boldsymbol{w}} = \boldsymbol{A}\boldsymbol{w}\ .$$

Notice that this system has a cascaded structure because the linear system $\dot{\boldsymbol{w}} = \boldsymbol{A}\boldsymbol{w}$ does not depend on the state variables $\boldsymbol{e}_x$ and $\dot{\boldsymbol{e}}_x$. For a nonlinear time-invariant system in such a cascaded form to be asymptotically stable it is necessary to show that all solutions of the coupled system remain bounded and the uncoupled subsystems are asymptotically stable [11]. Loria extended this result to the time-variant case in [8]. In order to apply this result to the system (22)-(23), the following theorem of [8] is reproduced:

Theorem 1: *Consider the system*

$$\dot{\boldsymbol{y}}_1 = \boldsymbol{f}_1(\boldsymbol{y}_1, t) + \boldsymbol{h}(\boldsymbol{y}, t)\boldsymbol{y}_2 \tag{24}$$

$$\dot{\boldsymbol{y}}_2 = \boldsymbol{f}_2(\boldsymbol{y}_2, t) \tag{25}$$

with $\boldsymbol{y} = (\boldsymbol{y}_1^T, \boldsymbol{y}_2^T)^T$. The functions $\boldsymbol{f}_1(\boldsymbol{y}_1, t)$, $\boldsymbol{f}_2(\boldsymbol{y}_2, t)$ and $\boldsymbol{h}(\boldsymbol{y}, t)$ are continuous in their arguments, locally Lipschitz in $\boldsymbol{y}$, uniformly in t, and $\boldsymbol{f}_1(\boldsymbol{y}_1, t)$ is continuously differentiable in both arguments. This system is uniformly globally asymptotically stable if and only if the following assumptions hold:

- *There exists a nondecreasing function $H(\cdot)$ such that*

$$||\boldsymbol{h}(\boldsymbol{y}, t)|| \le H(||\boldsymbol{y}||)\ . \tag{26}$$

- *The systems*

$$\dot{\boldsymbol{y}}_1 = \boldsymbol{f}_1(\boldsymbol{y}_1, t)$$
$$\dot{\boldsymbol{y}}_2 = \boldsymbol{f}_2(\boldsymbol{y}_2, t)$$

  are uniformly globally asymptotically stable
- *The solutions of (24)-(25) are uniformly globally bounded.*

The proof of this theorem can be found in [8].
Notice that for the system (22)-(23) the existence of a nondecreasing function $H(\cdot)$ for which (26) holds is fulfilled due to the assumption that the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is nonsingular. Thus, there exists a $\delta \in \Re$, $0 < \delta < \infty$, such that

$$||\boldsymbol{J}(\boldsymbol{e}_x, t)^{-T}|| \le \sup_{t\in[0,\infty[} \sqrt{\lambda_{max}(\boldsymbol{J}(\boldsymbol{e}_x, t)^{-1}\boldsymbol{J}(\boldsymbol{e}_x, t)^{-T})} < \delta$$

with $\lambda_{max}(\boldsymbol{A}(t))$ as the maximum eigenvalue of $\boldsymbol{A}(t)$ at the time t.
Uniformly globally asymptotic stability of the two uncoupled subsystems is given by property 1 and the fact that the linear system $\dot{\boldsymbol{w}} = \boldsymbol{A}\boldsymbol{w}$ is even globally exponentially stable for positive definite matrices $\boldsymbol{K}_s$ and $\boldsymbol{K}_t$.
Hence it is sufficient to show that all solutions of the coupled system are uniformly globally bounded. Before this is shown, two well known matrix lemmas, which will be used in the following, shall be given without proof ([6], [4]).

Lemma 1: *Suppose that a symmetric matrix $\boldsymbol{A}$ is partitioned as*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} \\ \boldsymbol{A}_{1,2}^T & \boldsymbol{A}_{2,2} \end{bmatrix} \tag{27}$$

where $\boldsymbol{A}_{1,1}$ and $\boldsymbol{A}_{2,2}$ are square. Then the matrix $\boldsymbol{A}$ is positive definite if and only if $\boldsymbol{A}_{1,1}$ is positive definite and $\boldsymbol{A}_{2,2} > \boldsymbol{A}_{1,2}^T\boldsymbol{A}_{1,1}^{-1}\boldsymbol{A}_{1,2}$.

Lemma 2: *Given an arbitrary positive definite matrix $\boldsymbol{Q}$, one can find a unique positive definite solution $\boldsymbol{P}$ of the Lyapunov equation $\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$ if and only if the matrix $\boldsymbol{A}$ is Hurwitz.*

Consider the positive definite function[6]

$$V_c = \frac{1}{2}\dot{\boldsymbol{e}}_x^T \boldsymbol{\Lambda}(\boldsymbol{e}_x,t)\dot{\boldsymbol{e}}_x + \frac{1}{2}\boldsymbol{e}_x^T \boldsymbol{K}_d \boldsymbol{e}_x + \frac{1}{2}\boldsymbol{w}^T \boldsymbol{P}\boldsymbol{w} \qquad (28)$$

with a positive definite matrix $\boldsymbol{P}$. Under consideration of the well known skew symmetry property 3 the time derivative of V_c along the solutions of (22)-(23) is given by

$$\dot{V}_c \quad = \quad -\dot{\boldsymbol{e}}_x^T \boldsymbol{D}_d \dot{\boldsymbol{e}}_x - \frac{1}{2}\boldsymbol{w}^T \boldsymbol{Q}\boldsymbol{w} + \dot{\boldsymbol{e}}_x^T \boldsymbol{J}(\boldsymbol{e}_x,t)^{-T}\boldsymbol{w}_1$$

where $\boldsymbol{Q} = -(\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P})$ can be an arbitrary positive definite matrix, because $\boldsymbol{A}$ is Hurwitz for positive definite matrices $\boldsymbol{K}_s$ and $\boldsymbol{K}_t$ and the matrix $\boldsymbol{P}$ in V_c is positive definite (see Lemma 2).
Obviously, $\dot{V}_c$ can be written in matrix form

$$\dot{V}_c \quad = \quad - \begin{bmatrix} \dot{\boldsymbol{e}}_x \\ \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix}^T \boldsymbol{N} \begin{bmatrix} \dot{\boldsymbol{e}}_x \\ \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix}$$

with

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{D}_d & \begin{bmatrix} -\frac{1}{2}\boldsymbol{J}(\boldsymbol{e}_x,t)^{-T} & \boldsymbol{0} \end{bmatrix} \\ \begin{bmatrix} -\frac{1}{2}\boldsymbol{J}^{-1}(\boldsymbol{e}_x,t) \\ \boldsymbol{0} \end{bmatrix} & \boldsymbol{Q} \end{bmatrix}$$

From Lemma 1 it follows that a necessary and sufficient condition for $\boldsymbol{N}$ to be positive definite[7] is

$$\boldsymbol{J}^T(\boldsymbol{e}_x,t)\boldsymbol{D}_d\boldsymbol{J}(\boldsymbol{e}_x,t) > \frac{1}{4}\boldsymbol{Q}^{-1} \ ,$$

which can be fulfilled for every positive definite matrix $\boldsymbol{D}_d$, because by assumption $\boldsymbol{J}(\boldsymbol{e}_x,t)$ does not get singular and the matrix $\boldsymbol{Q}$ is some positive definite matrix which may be chosen arbitrarily. Hence, one can conclude that

$$\dot{V}_c(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w}, t) \leq 0 \ .$$

At this point it is worth mentioning that V_c is bounded from above and below by some time-invariant, radially unbounded and positive definite functions $W_1(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w})$ and $W_2(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w})$

$$\begin{aligned} W_1(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w}) \leq V_c(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w}, t) \leq W_2(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w}) \\ W_1(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w}) = \tfrac{1}{2}\lambda_1 ||\dot{\boldsymbol{e}}_x||_2^2 + \tfrac{1}{2}\boldsymbol{e}_x^T \boldsymbol{K}_d \boldsymbol{e}_x + \tfrac{1}{2}\boldsymbol{w}^T\boldsymbol{P}\boldsymbol{w} \\ W_2(\dot{\boldsymbol{e}}_x, \boldsymbol{e}_x, \boldsymbol{w}) = \tfrac{1}{2}\lambda_2 ||\dot{\boldsymbol{e}}_x||_2^2 + \tfrac{1}{2}\boldsymbol{e}_x^T \boldsymbol{K}_d \boldsymbol{e}_x + \tfrac{1}{2}\boldsymbol{w}^T\boldsymbol{P}\boldsymbol{w} \end{aligned}$$

where

$$\begin{aligned} 0 < \lambda_1 < \inf_{t\in[0,\infty[} \quad & \lambda_{min}(\boldsymbol{\Lambda}(\boldsymbol{e}_x,t)) \quad < \\ \sup_{t\in[0,\infty[} \quad & \lambda_{max}(\boldsymbol{\Lambda}(\boldsymbol{e}_x,t)) \quad < \lambda_2 < \infty \end{aligned}$$

[6]Notice that the positive definiteness of V_c is ensured by the fact that the eigenvalues of the matrix $\boldsymbol{\Lambda}(\boldsymbol{e}_x,t)$ are bounded from above and below by some positive constants for all $t \in \Re$ and all $\boldsymbol{e}_x \in \Re^n$

[7]Notice also that, in addition to Lemma 1, it is also possible to show that all eigenvalues of $\boldsymbol{N}$ are bounded from above and below by some positive constants, because $\boldsymbol{Q}$ can be chosen arbitrarily and the matrix $\boldsymbol{J}(\boldsymbol{e}_x,t)$ is nonsingular.

with $\lambda_{min}(\boldsymbol{A}(t))$ and $\lambda_{max}(\boldsymbol{A}(t))$ as the minumum and maximum eigenvalue of $\boldsymbol{A}(t)$ at the time t.
From this property and the fact that $\dot{V}_c \leq 0$, it can be shown that the solutions of (22)-(23) are globally uniformly bounded. Proposition 1 follows then from Theorem 1.
Notice that the need to refer to Theorem 1 in this stability proof results from the facts that on the one hand the considered system is time-varying and on the other hand the time derivative of the chosen function V_c is not negative definite but only negative semidefinite. This fact, together with the remark that $\boldsymbol{Q}$ can be arbitrarily chosen, are the most important differences to the proofs in [6] and [7].

B. Proof of Prop. 2

Choosing V_c as the considered storage function yields for $\dot{V}_c$ in the case of $\boldsymbol{F}_{ext} \neq \boldsymbol{0}$

$$\dot{V}_c = - \begin{bmatrix} \dot{\boldsymbol{e}}_x \\ \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix}^T \boldsymbol{N} \begin{bmatrix} \dot{\boldsymbol{e}}_x \\ \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix} + \dot{\boldsymbol{e}}_x^T \boldsymbol{F}_{ext} \ , \qquad (29)$$

The matrix $\boldsymbol{N}$ has already be shown to be positive definite. From this one can conclude the passivity property from Proposition 2 easily.

VI. Experimental results

In order to show the advantage of the proposed controller compared to a simpler singular perturbation based controller an experimental comparison with a single flexible joint is presented. The chosen hardware setup is shown

Fig. 1. Hardware setup

in figure 1 and consists of a joint as used in the DLR light weight robots [2] with a mass of approx. 10 kg attached to it as a load. These joints have torque sensors in addition to the common motor position sensors. In order to get a full state measurement, these signals are differentiated numerically. The link position q and its first derivative are then computed from the motor position θ and the joint torque τ via the known joint stiffness k:

$$q = \theta - \tau/k \tag{30}$$

$$\dot{q} = \dot{\theta} - \dot{\tau}/k \tag{31}$$

Notice that the second and third order derivatives of q, which are necessary for the implementation of the impedance controller with the decoupling based torque controller, can be computed from (1) if the external torque τ_{ext} can be measured.

In the experiment only the regulation case shall be considered, and the desired link position is given by q_d. Notice that for a single flexible joint with a constant link side inertia m the only nonlinearity results from the effects of gravity. Therefore, in this experiment the desired stiffness and damping values for the desired behaviour of the link position with respect to external forces can be characterized with a chosen cutoff frequency $w_{bd,q}$ and a damping factor ξ_q. The desired behaviour, from which the controller gains k_d and d_d can be computed, is then given by the linear system:

$$m\ddot{q} + m(2\xi_q w_{bd,q})\dot{q} + m w_{bd,q}^2 (q - q_d) = \tau_{ext}$$

Also for the design of the torque controller gains k_s and k_t the desired torque dynamics are characterized by a linear behaviour of the form

$$\ddot{z} + 2\xi_z w_{bd,z}\dot{z} + w_{bd,z}^2 z = 0$$

with a cutoff frequency $w_{bd,z}$ and a damping factor ξ_z. The chosen values of these parameters are given in table I. While the cutoff frequency $w_{bd,z}$ of the torque dynamics has been chosen as high as possible, only a low damping factor has been chosen. On the other hand, the desired impedance is well damped. In the experiment, a step for the desired position q_d from 9 to 10 degrees was commanded in the absence of external torques. The same impedance controller was used in combination with the two torque controllers from Section III. Figure 2 shows the comparison of the ideal step response to the measured result with the decoupling based torque controller. In Fig. 3 the result with the singular perturbation based torque controller is given. Notice that in this experiment both $w_{bd,q}$ and $w_{bd,z}$ were chosen quite high. Clearly, for impedance controllers with a considerably lower bandwidth and better damped torque control loop, the difference of the step responses of the two controllers would be smaller. But in this experiment it is shown that, for cases where the singular perturbation based controller reaches its limit, the proposed decoupling based controller behaves much better due to the exact decoupling of the torque dynamics.

The same can be seen from the comparison of the torque error. This comparison is given in Fig. 4. Here the inital torque error due to the step of the desired position q_0 is diminished considerably faster in case of the decoupling based controller.

TABLE I

CHOSEN PARAMETERS FOR THE EXPERIMENT

$w_{bd,q}$	$2\pi 3.5$ rad/s
ξ_q	0.7
$w_{bd,z}$	$2\pi 18$ rad/s
ξ_z	0.2

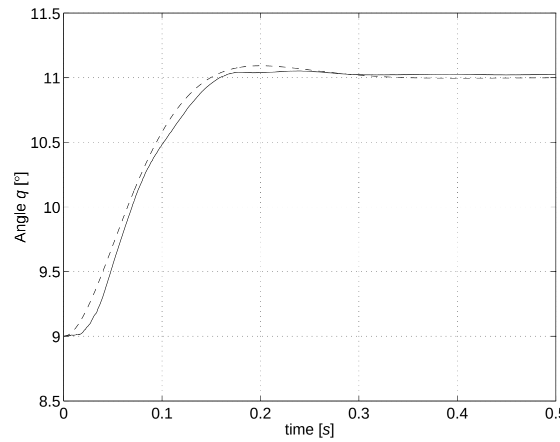

Fig. 2. Step response for the decoupling based controller: ideal (dashed) and measured (solid) response

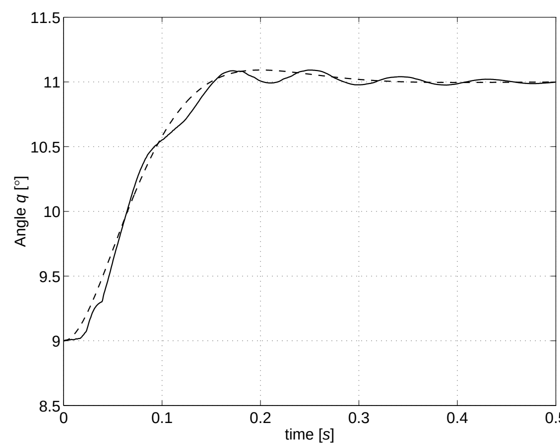

Fig. 3. Step response for the singular perturbation based controller: ideal (dashed) and measured (solid) response

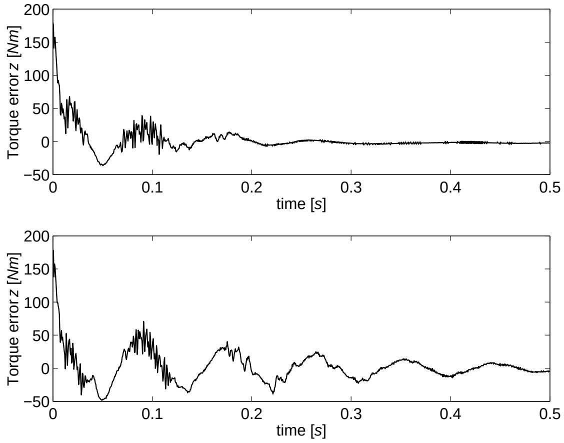

Fig. 4. Torque error z for both controllers: decoupling based controller (above), singular perturbation based controller (below)

VII. CONCLUSIONS

In this paper an impedance controller for flexible joint robots has been presented which is based on an exact decoupling of the torque dynamics. A stability analysis was given for the case of an impedance behaviour in which a desired stiffness and damping matrix can be chosen while the manipulator's inertial behaviour keeps unchanged. Finally, an experimental comparison of the proposed controller structure with a simpler singular perturbation based controller was given. The implementation of the proposed controller on the 7DOF DLR-light-weight-robot (see Fig. 5) is topic of our current research activity.

Fig. 5. 7DOF DLR-light-weight-robot (third generation)

ACKNOWLEDGMENT

This work was supported by the German Federal Ministry of Education and Research ”BMBF-Leitprojekt MORPHA” number ITL 0902E.

VIII. REFERENCES

[1] A. Albu-Schäffer, C. Ott, U. Frese, and G. Hirzinger. Cartesian Impedance control of redundant robots: Recent results with the dlr-light-weight-arms. *IEEE Int. Conf. on Robotics and Automation*, 2003.

[2] G. Hirzinger, A. Albu-Schäffer, M. Hähnle, I. Schaefer, and N. Sporer. On a new generation of torque controlled light-weight robots. In *IEEE International Conference of Robotics and Automation*, pages 3356–3363, 2001.

[3] N. Hogan. Impedance control: An approach to manipulation, part I - theory, part II - implementation, part III - applications. *Journ. of Dyn. Systems, Measurement and Control*, 107:1–24, 1985.

[4] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, second edition, 1996.

[5] O. Khatib. A unified approach for motion and force control of robot manipulators: The operational space formulation. *IEEE Journal of Robotics and Automation*, 3(1):1115–1120, February 1987.

[6] T. Lin and A. Goldenberg. Robust adaptive control of flexible joint robots with joint torque feedback. *IEEE Int. Conf. on Robotics and Automation*, pages 1229–1234, 1995.

[7] T. Lin and A. Goldenberg. A unified approach to motion and force control of flexible joint robots. *IEEE Int. Conf. on Robotics and Automation*, pages 1115–1120, 1996.

[8] A. Loria. Cascaded nonlinear time-varying systems: analysis and design. *Minicourse at the Congreso Internacional de Computacion, Cd. Mexico*, 2001.

[9] C. Ott, A. Albu-Schäffer, and G. Hirzinger. Comparison of adaptive and nonadaptive tracking control laws for a flexible joint manipulator. *IEEE Int. Conf. on Intelligent Robots and Systems*, 2002.

[10] V. Santibanez and R. Kelly. Strict lyapunov functions for control of robot manipulators. *Automatica*, 33(4):675–682, 1997.

[11] P. Seibert and R. Suarez. Global stabilization of nonlinear cascade systems. *Systems and Control Letters 14*, pages 347–352, 1990.

[12] M. Spong. Modeling and control of elastic joint robots. *IEEE Journal of Robotics and Automation*, RA-3:291–300, 1987.

[13] S. Stramigioli and H. Bruyninckx. Geometry and screw theory for constrained and unconstrained robot. *Tutorial at ICRA*, 2001.