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DECREASING NETS OF σ-ALGEBRAS AND THEIR APPLICATIONS TO ERGODIC THEORY

BRUNON KAMIŃSKI

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Introduction. One of the important results in the classical ergodic theory is the following theorem of Rohlin and Sinai [10].

Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space and let T be a measure preserving automorphism of it.

THEOREM A. *There exists a sub-σ-algebra $\mathscr{A} \subset \mathscr{B}$ such that*

(a) $$T^{-1}\mathscr{A} \subset \mathscr{A} ,$$

(b) $$\bigvee_{n=-\infty}^{+\infty} T^n\mathscr{A} = \mathscr{B} ,$$

(c) $$\bigcap_{n=-\infty}^{+\infty} T^n\mathscr{A} = \pi(T) ,$$

(d) $$h(T) = H(\mathscr{A} | T^{-1}\mathscr{A})$$

where $\pi(T)$ and $h(T)$ denote the Pinsker σ-algebra and the entropy of T, respectively.

Every such σ-algebra is said to be perfect. Perfect σ-algebras have important applications to the investigations of mixing and spectral properties of automorphisms (cf. [9], [10]). Shimano [11], [12] investigated helices associated with a given perfect σ-algebra.

Theorem A has been generalized by the author in [3] as follows:

Let Z^d denote the group of d-dimensional integers, o the null vector of Z^d and $\prec$ the lexicographical ordering of Z^d, $d \geqslant 2$.

An ordered pair (A, B) of nonempty subsets of Z^d is called a cut if $A \cup B = Z^d$ and for every $g \in A$ and $h \in B$ it holds $g \prec h$.

A cut (A, B) is said to be a gap if A does not contain the greatest element and B does not contain the lowest element.

Let Φ be a Z^d-action on $(X, \mathscr{B}, \mu)$, i.e., Φ is a homomorphism of Z^d into the group of all measure preserving automorphisms of $(X, \mathscr{B}, \mu)$. We denote by Φ^g the automorphism of $(X, \mathscr{B}, \mu)$ which is the image of $g \in Z^d$ under Φ.

The following result, formulated in [3] in terms of measurable partitions, is an analogue of Theorem A for Z^d-actions.

THEOREM B. *There exists a sub-σ-algebra* $\mathscr{A} \subset \mathscr{B}$ *such that*

$$\Phi^g \mathscr{A} \subset \mathscr{A} \quad \text{for} \quad g \prec o, \tag{a$_1$}$$

$$\bigvee_{g \in \mathbf{Z}^d} \Phi^g \mathscr{A} = \mathscr{B}, \tag{b$_1$}$$

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathscr{A} = \pi(\Phi), \tag{c$_1$}$$

$$h(\Phi) = H(\mathscr{A} | \mathscr{A}^-) \quad \textit{where} \quad \mathscr{A}^- = \bigvee_{g \prec o} \Phi^g \mathscr{A}, \tag{d$_1$}$$

(e_1) *for every gap* (A, B) *of* $\mathbf{Z}^d$ *it holds*

$$\bigvee_{g \in A} \Phi^g \mathscr{A} = \bigcap_{g \in B} \Phi^g \mathscr{A},$$

where $\pi(\Phi)$ *and* $h(\Phi)$ *denote the Pinsker σ-algebra and the entropy of* Φ *respectively.*

Similarly as in the one-dimensional case $\mathscr{A}$ is called a perfect σ-algebra of Φ. The essential difference between the one-dimensional and multidimensional concept of a perfect σ-algebra is contained in the condition (e_1) which one may call a continuity condition. It is shown in [5] that there exist σ-algebras satisfying (a_1)–(d_1) but not (e_1).

The paper [3] also contains applications of Theorem B. It would be interesting to know whether the results of Shimano have multidimensional analogues.

The definition of a perfect σ-algebra admits a more accessible form if we represent the considered action Φ by a d-tuple of natural automorphisms associated with Φ. For simplicity we will do this only in the case $d=2$.

Let T and S be automorphisms which are images under Φ of the vectors (1, 0) and (0, 1), respectively. Hence for $g=(m, n) \in \mathbf{Z}^2$ we have $\Phi^g = T^m \circ S^n$. Obviously, T and S commute. Then the conditions (a_1)–(e_1) may be written as follows:

$$S^{-1}\mathscr{A} \subset \mathscr{A}, \qquad T^{-1}\mathscr{A}_S \subset \mathscr{A}, \tag{a$_2$}$$

$$\bigvee_{n=-\infty}^{+\infty} T^n \mathscr{A}_S = \mathscr{B}, \tag{b$_2$}$$

$$\bigcap_{n=-\infty}^{+\infty} T^n \mathscr{A}_S = \pi(\Phi), \tag{c$_2$}$$

$$h(\Phi) = H(\mathscr{A} | S^{-1}\mathscr{A}), \tag{d$_2$}$$

$$\bigcap_{n=-\infty}^{+\infty} S^n \mathscr{A} = T^{-1}\mathscr{A}_S, \tag{e$_2$}$$

where $\mathscr{A}_S = \bigvee_{n=-\infty}^{+\infty} S^n \mathscr{A}$.

Theorem B has been sharpened in [5] (see also [4]) in the following manner.

THEOREM C. *If $\mathscr{H} \subset \mathscr{B}$ is a sub-σ-algebra which is a factor of Φ, i.e., $\Phi^g \mathscr{H} = \mathscr{H}$, $g \in \boldsymbol{Z}^d$, then there exists a sub-σ-algebra $\mathscr{A} \supset \mathscr{H}$ satisfying* (a_1), (b_1), (e_1) *and*

$$(c_3) \qquad \bigcap_{g \in \boldsymbol{Z}^d} \Phi^g \mathscr{A} = \pi(\Phi | \mathscr{H}),$$

$$(d_3) \qquad h(\Phi | \mathscr{H}) = H(\mathscr{A} | \mathscr{A}^-),$$

where $\pi(\Phi|\mathscr{H})$ and $h(\Phi|\mathscr{H})$ denote the relative Pinsker σ-algebra and the relative entropy of Φ with respect to $\mathscr{H}$ (for the definitions see Section 2).

A sub-σ-algebra $\mathscr{A}$ satisfying the properties given in Theorem C is said to be relatively perfect with respect to $\mathscr{H}$. It is clear that a relatively perfect σ-algebra with respect to the trivial σ-algebra is perfect.

Theorem C has been used in [5] (see also [4]) to give an axiomatic definition of the entropy of a $\boldsymbol{Z}^d$-action.

In this paper we use this theorem to show that the concept of a relative K-action given by Thouvenot [13] is an extension of the concept of a K-action in the sense of Kolmogorov. Using this fact, we prove that if $\mathscr{H}$ is a factor having an independent complement $\mathscr{H}^c$ such that the restriction of Φ to the space $(X, \mathscr{H}^c, \mu)$ is a K-action, then Φ is a relative K-action with respect to $\mathscr{H}$. We also show a formula for the direct product of relative Pinsker σ-algebras which implies that the product of relative K-actions is a relative K-action. This formula is an extension of that of Pollit [9] to $\boldsymbol{Z}^d$-actions. These results are obtained due to the property of the exchangeability of the order of taking suprema and intersections of nets of σ-algebras.

1. Decreasing nets of σ-algebras. Let $(X, \mathscr{B}, \mu)$ be a probability space, $\operatorname{Sub}\mathscr{B}$ the family of all sub-σ-algebras of $\mathscr{B}$ and $\mathscr{N} = \mathscr{N}(X)$ the trivial sub-σ-algebra. All equalities between sets, functions, transformations and σ-algebras are to be interpreted up to a set of measure zero. For $\mathscr{A} \in \operatorname{Sub}\mathscr{B}$ we denote by $L^2(\mathscr{A})$ the subspace of $L^2(X, \mu)$ consisting of functions measurable with respect to $\mathscr{A}$. The conditional probability of a set $A \in \mathscr{B}$ with respect to $\mathscr{A}$ is denoted by $\mu(A|\mathscr{A})$. For $f \in L^1(X, \mu)$ we put

$$Ef = \int_X f d\mu \quad \text{and} \quad \|f\| = E|f|.$$

Now, let P be a countable measurable partition of X and let $\hat{P}$ be the sub-σ-algebra generated by P. We define the conditional entropy of P under $\mathscr{A}$ as

$$H(P|\mathscr{A}) = E\Big(-\sum_{A \in P} \mu(A|\mathscr{A}) \log \mu(A|\mathscr{A})\Big)$$

and the entropy of P as $H(P) = H(P|\mathscr{N})$.

If $\mathscr{C} \in \operatorname{Sub} \mathscr{B}$, then we define the conditional entropy of $\mathscr{C}$ under $\mathscr{A}$ by the formula

$$H(\mathscr{C} | \mathscr{A}) = \sup H(P | \mathscr{A}),$$

where the supremum is taken over all countable measurable partitions P such that $\hat{P} \subset \mathscr{C}$ and $H(P) < \infty$.

It is easy to check that the last definition is equivalent to that of Jacobs [2].

If $\mathscr{A}_1, \mathscr{A}_2 \in \operatorname{Sub} \mathscr{B}$, then the symbol $\mathscr{A}_1 \veebar \mathscr{A}_2$ ($\mathscr{A}_1 \vee \mathscr{A}_2$) means the smallest algebra (σ-algebra) containing $\mathscr{A}_1$ and $\mathscr{A}_2$.

In the sequel we use the following two elementary properties of the conditional probability.

Let $\mathscr{C} \in \operatorname{Sub} \mathscr{B}$ be fixed.

(1) For every $\mathscr{A} \in \operatorname{Sub} \mathscr{B}$ with $\mathscr{A} \supset \mathscr{C}$, $A \in \mathscr{B}$ and $C \in \mathscr{C}$, it holds

$$\mu(A \cap C | \mathscr{A}) = \mu(A | \mathscr{A}) \cdot \mathbf{1}_C .$$

(2) If $\mathscr{A}, \mathscr{D} \in \operatorname{Sub} \mathscr{B}$ are such that $\mathscr{A} \vee \mathscr{D}$ and $\mathscr{C}$ are independent, then for every $A \in \mathscr{A}$ it holds

$$\mu(A | \mathscr{C} \vee \mathscr{D}) = \mu(A | \mathscr{D}) .$$

Let I be a countable set directed by an ordering relation $<$. A net $(\mathscr{B}_t)_{t \in I}$ ($(\mathscr{B}_t)$ for short) in $\operatorname{Sub} \mathscr{B}$ is said to be decreasing (resp. increasing) if $\mathscr{B}_s \supset \mathscr{B}_t$ (resp. $\mathscr{B}_s \subset \mathscr{B}_t$) for $s < t$.

Let $(\mathscr{B}_t)$ be a decreasing net in $\operatorname{Sub} \mathscr{B}$ and let $\mathscr{B}_t \supset \mathscr{C}$ for all $t \in I$. Proceeding in the same way as in the proof of Lemma 2 in [6] we have:

LEMMA 1. *$\bigcap_{t \in I} \mathscr{B}_t = \mathscr{C}$ if and only if for every $B \in \bigvee_{t \in I} \mathscr{B}_t$ it holds*

$$\lim_{t \in I} \sup_{A \in \mathscr{B}_t} \| \mu(A \cap B | \mathscr{C}) - \mu(A | \mathscr{C}) \cdot \mu(B | \mathscr{C}) \| = 0 ,$$

i.e., for any $\varepsilon > 0$ there exists $t_0 \in I$ such that

$$\| \mu(A \cap B | \mathscr{C}) - \mu(A | \mathscr{C}) \mu(B | \mathscr{C}) \| < \varepsilon$$

for each $t < t_0$.

The following result is a sharpening of Theorem 2 in [6].

THEOREM 1. *If $(\mathscr{A}_t)$ is a decreasing net in $\operatorname{Sub} \mathscr{B}$ such that $\bigvee_{t \in I} \mathscr{A}_t$ and $\mathscr{C}$ are independent, then*

$$\bigcap_{t \in I} (\mathscr{A}_t \vee \mathscr{C}) = \bigcap_{t \in I} \mathscr{A}_t \vee \mathscr{C} .$$

PROOF. We define

$$\mathscr{A}_\infty = \bigcap_{t \in I} \mathscr{A}_t , \quad \mathscr{C}_\infty = \mathscr{A}_\infty \vee \mathscr{C}$$

and

$$\mathscr{B}_t^\circ = \mathscr{A}_t \underline{\vee} \mathscr{C}, \quad \mathscr{B}_t = \mathscr{A}_t \vee \mathscr{C}$$

for each $t \in I$. Let $s \in I$ be fixed. For any $B \in \mathscr{B}_s^\circ$, there exist sets $F_1, \ldots, F_q$ from $\mathscr{A}_s$ and pairwise disjoint sets $D_1, \ldots, D_q$ from $\mathscr{C}$ such that

$$B = \bigcup_{j=1}^{q} F_j \cap D_j .$$

First we shall prove that

$$\lim_{t \in I} \sup_{A \in \mathscr{B}_t^\circ} \| \mu(A \cap B | \mathscr{C}_\infty) - \mu(A | \mathscr{C}_\infty) \cdot \mu(B | \mathscr{C}_\infty) \| = 0 . \tag{3}$$

Let $t \in I$ be fixed. Similarly as above for any $A \in \mathscr{B}_t^\circ$ there exist sets $E_1, \ldots, E_p$ from $\mathscr{A}_t$ and pairwise disjoint sets $C_1, \ldots, C_p$ from $\mathscr{C}$ such that

$$A = \bigcup_{i=1}^{p} E_i \cap C_i .$$

It follows from (1), (2) and the independence assumption that

$$\begin{aligned}
&\| \mu(A \cap B | \mathscr{C}_\infty) - \mu(A | \mathscr{C}_\infty) \cdot \mu(B | \mathscr{C}_\infty) \| \\
&= \left\| \sum_{i=1}^{p} \sum_{j=1}^{q} \{ \mu(E_i \cap F_j \cap C_i \cap D_j | \mathscr{C}_\infty) - \mu(E_i \cap C_i | \mathscr{C}_\infty) \mu(F_j \cap D_j | \mathscr{C}_\infty \} \right\| \\
&= \left\| \sum_{i=1}^{p} \sum_{j=1}^{q} \{ \mu(E_i \cap F_j | \mathscr{A}_\infty) - \mu(E_i | \mathscr{A}_\infty) \cdot \mu(F_j | \mathscr{A}_\infty) \} \mathbf{1}_{C_i \cap D_j} \right\| \\
&\leqslant \sum_{i=1}^{p} \sum_{j=1}^{q} \| \mu(E_i \cap F_j | \mathscr{A}_\infty) - \mu(E_i | \mathscr{A}_\infty) \cdot \mu(F_j | \mathscr{A}_\infty) \| \, \mu(C_i \cap D_j) \\
&\leqslant \max_{1 \leqslant j \leqslant q} \sup_{E \in \mathscr{A}_t} \| \mu(E \cap F_j | \mathscr{A}_\infty) - \mu(E | \mathscr{A}_\infty) \cdot \mu(F_j | \mathscr{A}_\infty) \| .
\end{aligned}$$

Therefore for every $B \in \mathscr{B}_s^\circ$ and $t \in I$ it holds

$$\begin{aligned}
&\sup_{A \in \mathscr{B}_t^\circ} \| \mu(A \cap B | \mathscr{C}_\infty) - \mu(A | \mathscr{C}_\infty) \cdot \mu(B | \mathscr{C}_\infty) \| \\
&\qquad \leqslant \max_{1 \leqslant j \leqslant q} \sup_{E \in \mathscr{A}_t} \| \mu(E \cap F_j | \mathscr{A}_\infty) - \mu(E | \mathscr{A}_\infty) \cdot \mu(F_j | \mathscr{A}_\infty) \| .
\end{aligned}$$

Hence using Lemma 1 we get (3). Now, let $B \in \bigvee_{t \in I} \mathscr{B}_t$ and let $\varepsilon > 0$ be arbitrary. Then there exists $s \in I$ and a set $B_\varepsilon \in \mathscr{B}_s^\circ$ such that $\mu(B \div B_\varepsilon) < \varepsilon/5$. If follows from (3) that there exists $t_0 \in I$ such that for $t < t_0$ and any $E \in \mathscr{B}_t^\circ$ it holds

$$\| \mu(E \cap B_\varepsilon | \mathscr{C}_\infty) - \mu(E | \mathscr{C}_\infty) \cdot \mu(B_\varepsilon | \mathscr{C}_\infty) \| < \varepsilon/5 . \tag{4}$$

Let $t<t_0$ and let $A\in\mathscr{B}_t$. Then there exists $A_\varepsilon\in\mathscr{B}_t^\circ$ such that $\mu(A\div A_\varepsilon)<\varepsilon/5$. It follows from (4) and basic properties of the conditional probability that

$$\|\mu(A\cap B|\mathscr{C}_\infty)-\mu(A|\mathscr{C}_\infty)\cdot\mu(B|\mathscr{C}_\infty)\|\leqslant 2\mu(A\div A_\varepsilon)+2\mu(B\div B_\varepsilon)+\varepsilon/5<\varepsilon .$$

Thus for every $B\in\bigvee_{t\in I}\mathscr{B}_t$ it holds

$$\lim_{t\in I}\sup_{A\in\mathscr{B}_t}\|\mu(A\cap B|\mathscr{C}_\infty)-\mu(A|\mathscr{C}_\infty)\cdot\mu(B|\mathscr{C}_\infty)\|=0 .$$

Using again Lemma 1 we obtain the desired result.

REMARK 1. It is worth noting that Weizsacker [14] characterized decreasing sequences $(\mathscr{A}_n)$ in $\operatorname{Sub}\mathscr{B}$ for which it holds

$$\bigcap_{n=0}^{\infty}(\mathscr{A}_n\vee\mathscr{C})=\bigcap_{n=0}^{\infty}\mathscr{A}_n\vee\mathscr{C} .$$

Now, let $(X,\mathscr{B},\mu)$ and $(\tilde{X},\tilde{\mathscr{B}},\tilde{\mu})$ be probability spaces and let $(X\times\tilde{X},\mathscr{B}\otimes\tilde{\mathscr{B}},\mu\times\tilde{\mu})$ be the direct product.

COROLLARY 1. *If $(\mathscr{A}_t)$ is a decreasing net in $\operatorname{Sub}\mathscr{B}$ and $\tilde{\mathscr{C}}\in\operatorname{Sub}\tilde{\mathscr{B}}$, then*

$$\bigcap_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{C}})=\bigcap_{t\in I}\mathscr{A}_t\otimes\tilde{\mathscr{C}} .$$

PROOF. It is easy to see that the above equality is valid for $\tilde{\mathscr{C}}=\mathscr{N}(\tilde{X})$. Let now $\tilde{\mathscr{C}}$ be arbitrary. It follows from Theorem 1 that

$$\bigcap_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{C}})=\bigcap_{t\in I}(\mathscr{A}_t\otimes\mathscr{N}(\tilde{X})\vee\mathscr{N}(X)\otimes\tilde{\mathscr{C}})=\bigcap_{t\in I}(\mathscr{A}_t\otimes\mathscr{N}(\tilde{X}))\vee(\mathscr{N}(X)\otimes\tilde{\mathscr{C}})=\bigcap_{t\in I}\mathscr{A}_t\otimes\tilde{\mathscr{C}} .$$

REMARK 2. The results given in Corollary 1 and the following lemma are announced in [9] without proofs.

LEMMA 2. *For arbitrary $\mathscr{A}\in\operatorname{Sub}\mathscr{B}$ and $\tilde{\mathscr{A}}\in\operatorname{Sub}\tilde{\mathscr{B}}$ it holds $(\mathscr{A}\otimes\tilde{\mathscr{B}})\cap(\mathscr{B}\otimes\tilde{\mathscr{A}})=\mathscr{A}\otimes\tilde{\mathscr{A}}$.*

PROOF. It is enough to show that

$$L^2(\mathscr{A}\otimes\tilde{\mathscr{B}}\cap\mathscr{B}\otimes\tilde{\mathscr{A}})\subset L^2(\mathscr{A}\otimes\tilde{\mathscr{A}}) .$$

There exist orthonormal basis $(f_\alpha)_{\alpha\in I}$ in $L^2(X,\mu)$, $(g_\beta)_{\beta\in J}$ in $L^2(\tilde{X},\tilde{\mu})$ and subsets $I_0\subset I$, $J_0\subset J$ such that $(f_\alpha)_{\alpha\in I_0}$ is an orthonormal basis in $L^2(\mathscr{A})$, $(g_\beta)_{\beta\in J_0}$ is an orthonormal basis in $L^2(\tilde{\mathscr{A}})$.

We put $h_{\alpha\beta}=f_\alpha\cdot g_\beta$, $(\alpha,\beta)\in I\times J$. It is clear that the sets $(h_{\alpha\beta},\alpha\in I_0,\beta\in J)$, $(h_{\alpha\beta},\alpha\in I,\beta\in J_0)$ and $(h_{\alpha\beta},\alpha\in I_0,\beta\in J_0)$ are orthonormal basis in $L^2(\mathscr{A}\otimes\tilde{\mathscr{B}})$, $L^2(\mathscr{B}\otimes\tilde{\mathscr{A}})$ and $L^2(\mathscr{A}\otimes\tilde{\mathscr{A}})$, respectively.

Let $f\in L^2(\mathscr{A}\otimes\tilde{\mathscr{B}}\cap\mathscr{B}\otimes\tilde{\mathscr{A}})$ and let $c_{\alpha\beta}$ denote the Fourier coefficient of f with respect to $h_{\alpha\beta}$, i.e.,

$$c_{\alpha\beta}=E(f\cdot\bar{h}_{\alpha\beta}), \qquad (\alpha, \beta)\in I\times J.$$

Since $f\in L^2(\mathscr{A}\otimes\tilde{\mathscr{B}})\cap L^2(\mathscr{B}\otimes\tilde{\mathscr{A}})$ we have $c_{\alpha\beta}=0$ for $\alpha\notin I_0$, $\beta\in J$ or $\alpha\in I$, $\beta\notin J_0$, and f has the following Fourier expansions:

$$\sum c_{\alpha\beta}\cdot h_{\alpha\beta}=f=\sum c_{\alpha\beta}\cdot h_{\alpha\beta}$$

where the first sum is taken over all $(a, b)\in I_0\times J$ and the second over all $(\alpha, \beta)\in I\times J_0$. Hence $c_{\alpha\beta}=0$ for $(\alpha, \beta)\notin I_0\times J_0$ and so $f=\sum c_{\alpha\beta}\cdot h_{\alpha\beta}$, where the sum is taken over all $(\alpha, \beta)\in I_0\times J_0$. This means that $f\in L^2(\mathscr{A}\otimes\tilde{\mathscr{A}})$ and the lemma is proved.

COROLLARY 2. *If* $(\mathscr{A}_t)$ *and* $(\tilde{\mathscr{A}}_t)$ *are decreasing nets in* Sub $\mathscr{B}$ *and* Sub $\tilde{\mathscr{B}}$, *respectively, then*

$$\bigcap_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{A}}_t)=\bigcap_{t\in I}\mathscr{A}_t\otimes\bigcap_{t\in I}\tilde{\mathscr{A}}_t.$$

PROOF. If follows from Corollary 1 that

$$\bigcap_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{A}}_t)\subset\bigcap_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{B}})=\bigcap_{t\in I}\mathscr{A}_t\otimes\tilde{\mathscr{B}}$$

and similarly

$$\bigcap_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{A}}_t)\subset\mathscr{B}\otimes\bigcap_{t\in I}\tilde{\mathscr{A}}_t.$$

Hence using Lemma 2 we get

$$\bigcap_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{A}}_t)\subset\bigcap_{t\in I}\mathscr{A}_t\otimes\bigcap_{t\in I}\tilde{\mathscr{A}}_t.$$

Since the opposite inclusion is obvious, we get the desired result.

REMARK 3. It is not difficult to show that Corollary 2 is also valid for increasing nets of sub-σ-algebras.

We will also use in the sequel a property analogous to that given in Corollary 2, for increasing nets of σ-algebras.

REMARK 4. If $(\mathscr{A}_t)$ and $(\tilde{\mathscr{A}}_t)$ are increasing nets in Sub $\mathscr{B}$ and Sub $\tilde{\mathscr{B}}$ respectively, then

$$\bigvee_{t\in I}(\mathscr{A}_t\otimes\tilde{\mathscr{A}}_t)=\bigvee_{t\in I}\mathscr{A}_t\otimes\bigvee_{t\in I}\tilde{\mathscr{A}}_t,$$

as can be easily proved.

2. Relative Kolmogorov Z^d-actions. Investigating measure preserving automorphisms of a Lebesgue space with the strong Pinsker property, Thouvenot has introduced in [13] an interesting class of factors of automorphisms—the so called entropy maximal factors. These factors have been also objects of investigations of Ornstein [8]. Lind used them in [7] for studying skew products on compact groups.

Now we recall the definition of these factors. Let T be an automorphism of a Lebesgue space $(X, \mathscr{B}, \mu)$ with $h(T) < \infty$.

A factor $H_T = \bigvee_{n=-\infty}^{+\infty} T^n H$ of T, where H is a finite partition of X, is said to be entropy maximal if for every finite partition P of X, the conditions

$$P_T \supset H_T \quad \text{and} \quad h(P, T) = h(H, T)$$

imply $P_T = H_T$.

Instead of saying that H_T is entropy maximal one says in [13] that T is a relative K-system with respect to H_T.

At first glance the concept of a relative K-system seems to have no connection with the traditional meaning of a Kolmogorov system (automorphism) for which there should exist some special exhaustive σ-algebras for the automorphism.

However, we will show that such a connection exists, not only for single automorphisms with finite entropy, but also for arbitrary $\boldsymbol{Z}^d$-actions. In order to do so we introduce some concepts concerning $\boldsymbol{Z}^d$-actions.

Let Φ be a $\boldsymbol{Z}^d$-action on a Lebesgue space $(X, \mathscr{B}, \mu)$. Let $\mathscr{H} \in \operatorname{Sub} \mathscr{B}$ be a factor of Φ. For a countable measurable partition P of X with $H(P) < \infty$ we put

$$h(P, \Phi \,|\, \mathscr{H}) = H(P \,|\, P^- \vee \mathscr{H}),$$

where $P^- = \hat{P}^-$.

For a given factor $\mathscr{C} \supset \mathscr{H}$ we define

$$h(\Phi/\mathscr{C} \,|\, \mathscr{H}) = \sup h(P, \Phi \,|\, \mathscr{H}),$$

where the supremum is taken over all partitions P with $\hat{P} \subset \mathscr{C}$ and $H(P) < \infty$.

By the relative entropy of Φ with respect to $\mathscr{H}$ we mean $h(\Phi \,|\, \mathscr{H}) = h(\Phi/\mathscr{B} \,|\, \mathscr{H})$.

The smallest sub-σ-algebra containing all sub-σ-algebras $\hat{P}$, where P is a countable measurable partition such that $H(P) < \infty$ and $h(P, \Phi \,|\, \mathscr{H}) = 0$ is called the relative Pisker σ-algebra with respect to $\mathscr{H}$ and is denoted by $\pi(\Phi \,|\, \mathscr{H})$.

It is clear that the sub-σ-algebra $\pi(\Phi \,|\, \mathscr{H})$ is a factor of Φ with $h(\Phi/\pi(\Phi \,|\, \mathscr{H}) \,|\, \mathscr{H}) = 0$.

We shall use in the sequel the following two results.

LEMMA 3 (cf. [5]). *For every factor $\mathscr{C} \supset \mathscr{H}$ it holds*

$$h(\Phi/\mathscr{C}) = h(\Phi/\mathscr{H}) + h(\Phi/\mathscr{C} \,|\, \mathscr{H}).$$

DEFINITION 1. A σ-algebra $\mathscr{A} \in \operatorname{Sub} \mathscr{B}$ is said to be exhaustive if it satisfies the properties (a_1), (b_1) and (e_1) of perfect σ-algebras.

LEMMA 4. *If $\mathscr{A} \supset \mathscr{H}$ is exhaustive, then*

$$\bigcap_{g\in \boldsymbol{Z}^d} \Phi^g \mathscr{A} \supset \pi(\Phi|\mathscr{H}).$$

If, in addition, $H(\mathscr{A}|\mathscr{A}^-)=h(\Phi|\mathscr{H})<\infty$, *then*

$$\bigcap_{g\in \boldsymbol{Z}^d} \Phi^g \mathscr{A} = \pi(\Phi|\mathscr{H}).$$

PROOF. The first part of the lemma is proved in [5]. Now let us suppose $h(\Phi|\mathscr{H})<\infty$. Let P, Q be countable measurable partitions of X with $H(P)<\infty$, $H(Q)<\infty$, $\hat{P}\subset\mathscr{A}$ and $\hat{Q}\subset\bigcap_{g\in\boldsymbol{Z}^d}\Phi^g\mathscr{A}$. The following equality is a relativized version of the Pinsker formula for $\boldsymbol{Z}^d$-actions.

$$h(P\vee Q,\Phi|\mathscr{H})=h(Q,\Phi|\mathscr{H})+H(P|P^-\vee Q_\Phi\vee\mathscr{H}), \tag{5}$$

where $Q_\Phi=\bigvee_{g\in\boldsymbol{Z}^d}\Phi^g Q$. The proof of (5) is analogous to that given in [1] in the case $\mathscr{H}=\mathscr{N}$. Using (5) and the inclusions $\hat{P}\subset\mathscr{A}$, $Q_\Phi\subset\mathscr{A}$, $\mathscr{H}\subset\mathscr{A}$ we get

$$\begin{aligned} h(\Phi|\mathscr{H})\geqslant h(P\vee Q,\Phi|\mathscr{H})&=h(Q,\Phi|\mathscr{H})+H(P|P^-\vee Q_\Phi\vee\mathscr{H})\\ &\geqslant h(Q,\Phi|\mathscr{H})+H(P|\mathscr{A}^-).\end{aligned}$$

Hence

$$h(\Phi|\mathscr{H})\geqslant h(Q,\Phi|\mathscr{H})+H(\mathscr{A}|\mathscr{A}^-).$$

It follows from our assumption that $h(Q,\Phi|\mathscr{H})=0$, i.e., $\hat{Q}\subset\pi(\Phi|\mathscr{H})$. Thus we have shown the inclusion

$$\bigcap_{g\in \boldsymbol{Z}^d} \Phi^g \mathscr{A} \subset \pi(\Phi|\mathscr{H})$$

which completes the proof.

Now we formulate an extension of the definition of Thouvenot to $\boldsymbol{Z}^d$-actions in our notation.

DEFINITION 2. A $\boldsymbol{Z}^d$-action Φ is called a relative Kolmogorov action (K-action for short) with respect to a factor $\mathscr{H}$ of Φ (or $\mathscr{H}$ is entropy maximal) if for every factor $\mathscr{C}\supset\mathscr{H}$ with $h(\Phi/\mathscr{C}|\mathscr{H})=0$ it holds $\mathscr{C}=\mathscr{H}$.

It follows immediately from Lemma 3 that in the case $d=1$ and $h(\Phi)<\infty$ our definition reduces to that of Thouvenot. It is clear that in the absolute case ($\mathscr{H}=\mathscr{N}$) it coincides with the definition of a $\boldsymbol{Z}^d$-action ($d\geqslant 1$) with completely positive entropy and therefore (see [10] for $d=1$ and [3] for $d\geqslant 2$) with the definition of a K-action.

THEOREM 2. *Φ is a relative* K*-action with respect to $\mathscr{H}$ if and only if there exists an exhaustive sub-σ-algebra $\mathscr{A}\supset\mathscr{H}$ with $\bigcap_{g\in\boldsymbol{Z}^d}\Phi^g\mathscr{A}=\mathscr{H}$.*

PROOF. Since $h(\Phi/\pi(\Phi|\mathscr{H})|\mathscr{H})=0$ our assumption implies $\pi(\Phi|\mathscr{H})=\mathscr{H}$. It follows from Theorem C that there exists an exhaustive sub-σ-algebra $\mathscr{A} \supset \mathscr{H}$ with

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathscr{A} = \pi(\Phi | \mathscr{H}) .$$

Now, let us suppose $\mathscr{A} \supset \mathscr{H}$ is an exhaustive sub-σ-algebra with

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathscr{A} = \mathscr{H} .$$

It follows from Lemma 4 that

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathscr{A} \supset \pi(\Phi | \mathscr{H}) ,$$

i.e., $\pi(\Phi | \mathscr{H}) = \mathscr{H}$. If $\mathscr{C} \supset \mathscr{H}$ is a factor such that $h(\Phi/\mathscr{C} | \mathscr{H}) = 0$, then $\mathscr{C} \subset \pi(\Phi | \mathscr{H})$ and therefore $\mathscr{C} \subset \mathscr{H}$, i.e., $\mathscr{C} = \mathscr{H}$. Thus Φ is a relative K-action with respect to $\mathscr{H}$.

COROLLARY 1. *Every $\mathbf{Z}^d$-action Φ is a relative* K*-action with respect to the Pinsker σ-algebra $\pi(\Phi)$.*

It is enough to take as $\mathscr{A}$ in Theorem 2 an arbitrary perfect σ-algebra of Φ.

It is shown in [3] that if $h(\Phi) = 0$ then $\mathscr{B}$ is the only exhaustive σ-algebra. Therefore in this case there are no nontrivial factors with respect to which Φ is a relative K-action.

Now, let Φ be a $\mathbf{Z}^d$-action with $h(\Phi) > 0$.

COROLLARY 2. *If $\mathscr{H}$ is a factor such that there exists a factor $\mathscr{H}^c$ independent of $\mathscr{H}$, $\mathscr{H} \vee \mathscr{H}^c = \mathscr{B}$ and the action Φ restricted to the space $(X, \mathscr{H}^c, \mu)$ is a* K*-action, then Φ is a relative* K*-action with respect to $\mathscr{H}$.*

PROOF. It follows from Theorem 2 that there exists $\mathscr{C} \in \operatorname{Sub} \mathscr{B}$ which is exhaustive in $(X, \mathscr{H}^c, \mu)$ and such that

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathscr{C} = \mathscr{N} . \tag{6}$$

Let $\mathscr{A} = \mathscr{C} \vee \mathscr{H}$. It is clear that

$$\Phi^g \mathscr{A} \subset \mathscr{A} \quad \text{for} \quad g \prec o \quad \text{and} \quad \bigvee_{g \in \mathbf{Z}^d} \Phi^g \mathscr{A} = \mathscr{B} .$$

Let a cut (A, B) of $\mathbf{Z}^d$ be a gap. Applying Theorem 1 in the preceding section to the directed set $(B, \prec)$ and to the net $(\Phi^g \mathscr{A})_{g \in B}$ we get

$$\bigcap_{g \in B} \Phi^g \mathscr{A} = \bigcap_{g \in B} (\Phi^g \mathscr{C} \vee \mathscr{H}) = \bigcap_{g \in B} \Phi^g \mathscr{C} \vee \mathscr{H} = \bigvee_{g \in A} \Phi^g \mathscr{C} \vee \mathscr{H} = \bigvee_{g \in A} \Phi^g \mathscr{A} .$$

This means that $\mathscr{A}$ is an exhaustive sub-σ-algebra. In the same way using (6), we get

$$\bigcap_{g\in \boldsymbol{Z}^d} \Phi^g \mathscr{A} = \bigcap_{g\in \boldsymbol{Z}^d} \Phi^g \mathscr{C} \vee \mathscr{H} = \mathscr{H} .$$

It follows from Thorem 2 that Φ is a relative K-action with respect to $\mathscr{H}$.

Now, let $(X, \mathscr{B}, \mu)$, $(\tilde{X}, \tilde{\mathscr{B}}, \tilde{\mu})$ be Lebesgue probability spaces and let Φ, $\tilde{\Phi}$ be $\boldsymbol{Z}^d$-actions on $(X, \mathscr{B}, \mu)$ and $(\tilde{X}, \tilde{\mathscr{B}}, \tilde{\mu})$, respectively. We denote by $\Phi\times\tilde{\Phi}$ the product action of Φ and $\tilde{\Phi}$, i.e.,

$$(\Phi\times\tilde{\Phi})^g(x, y) = (\Phi^g x, \tilde{\Phi}^g y) , \qquad (x, y)\in X\times\tilde{X} , \quad g\in \boldsymbol{Z}^d .$$

Our next aim is to show the following:

THEOREM 3. *For any factors $\mathscr{H}$ and $\tilde{\mathscr{H}}$ of Φ and $\tilde{\Phi}$, respectively, it holds*

$$\pi(\Phi\times\tilde{\Phi} \,|\, \mathscr{H}\times\tilde{\mathscr{H}}) = \pi(\Phi\,|\,\mathscr{H})\otimes\pi(\tilde{\Phi}\,|\,\tilde{\mathscr{H}}) .$$

PROOF. Let $\mathscr{A}$ and $\tilde{\mathscr{A}}$ be relatively perfect sub-σ-algebras for Φ and $\tilde{\Phi}$ with respect to $\mathscr{H}$ and $\tilde{\mathscr{H}}$, respectively. It is clear by Theorem C that

$$\mathscr{H}\otimes\tilde{\mathscr{H}} \subset (\Phi\times\tilde{\Phi})^g(\mathscr{A}\otimes\tilde{\mathscr{A}}) \subset \mathscr{A}\otimes\tilde{\mathscr{A}} , \qquad g \prec o$$

and

$$\bigvee_{g\in \boldsymbol{Z}^d} (\Phi\times\tilde{\Phi})^g(\mathscr{A}\otimes\tilde{\mathscr{A}}) = \mathscr{B}\otimes\tilde{\mathscr{B}} .$$

Let (A, B) be a cut of $\boldsymbol{Z}^d$ which is a gap. It follows from Corollary 2 to Theorem 1 and Remark 4 that

$$\bigcap_{g\in B} (\Phi\times\tilde{\Phi})^g(\mathscr{A}\otimes\tilde{\mathscr{A}}) = \bigcap_{g\in B} (\Phi^g\mathscr{A}\otimes\tilde{\Phi}^g\tilde{\mathscr{A}}) = \bigcap_{g\in B} \Phi^g\mathscr{A}\otimes\bigcap_{g\in B}\tilde{\Phi}^g\tilde{\mathscr{A}}$$

$$= \bigvee_{g\in A} \Phi^g\mathscr{A}\otimes\bigvee_{g\in A}\tilde{\Phi}^g\tilde{\mathscr{A}} = \bigvee_{g\in A} (\Phi\times\tilde{\Phi})^g(\mathscr{A}\otimes\tilde{\mathscr{A}}) .$$

This means that $\mathscr{A}\otimes\tilde{\mathscr{A}}$ is exhaustive. Therefore using Lemma 4 and the fact that $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are relatively perfect, we get

$$\pi(\Phi\times\tilde{\Phi}\,|\,\mathscr{H}\otimes\tilde{\mathscr{H}}) \subset \bigcap_{g\in \boldsymbol{Z}^d} (\Phi\times\tilde{\Phi})^g(\mathscr{A}\otimes\tilde{\mathscr{A}}) = \bigcap_{g\in \boldsymbol{Z}^d} \Phi^g\mathscr{A}\otimes\bigcap_{g\in \boldsymbol{Z}^d}\tilde{\Phi}^g\tilde{\mathscr{A}} = \pi(\Phi\,|\,\mathscr{H})\otimes\pi(\tilde{\Phi}\,|\,\tilde{\mathscr{H}}) .$$

In order to show the opposite inclusion let us suppose that $E\in\pi(\Phi\,|\,\mathscr{H})$ and $\tilde{E}\in\pi(\tilde{\Phi}\,|\,\tilde{\mathscr{H}})$. We consider the partitions P and $\tilde{P}$ *of* X and $\tilde{X}$ respectively such that

$$P = \{E, E^c\} , \quad \tilde{P} = \{\tilde{E}, \tilde{E}^c\}$$

where $E^c = X\setminus E$ and $\tilde{E}^c = \tilde{X}\setminus\tilde{E}$. It follows from the definition of the relative Pinsker σ-algebra that $h(P, \Phi\,|\,\mathscr{H}) = 0$ and $h(\tilde{P}, \tilde{\Phi}\,|\,\tilde{\mathscr{H}}) = 0$. It is easy to check that

$$h(P\times\tilde{P},\,\Phi\times\tilde{\Phi}\,|\,\mathscr{H}\otimes\tilde{\mathscr{H}})=h(P,\,\Phi\,|\,\mathscr{H})+h(\tilde{P},\,\tilde{\Phi}\,|\,\tilde{\mathscr{H}})$$

where $P\times\tilde{P}$ is the partition of $X\times\tilde{X}$ defined by

$$P\times\tilde{P}=\{E\times\tilde{E},\,E^c\times\tilde{E},\,E\times\tilde{E}^c,\,E^c\times\tilde{E}^c\}\,.$$

Therefore we have $h(P\times\tilde{P},\,\Phi\times\tilde{\Phi}\,|\,\mathscr{H}\otimes\tilde{\mathscr{H}})=0$ which implies $A\times\tilde{A}\in\pi(\Phi\times\tilde{\Phi}\,|\,\mathscr{H}\otimes\tilde{\mathscr{H}})$. Hence, by the definition of a product σ-algebra, we get the desired inclusion.

COROLLARY. *If Φ and $\tilde{\Phi}$ are relative* K*-actions with respect to $\mathscr{H}$ and $\tilde{\mathscr{H}}$, respectively, then $\Phi\times\tilde{\Phi}$ is a relative* K*-action with respect to $\mathscr{H}\otimes\tilde{\mathscr{H}}$.*

Assuming $\mathscr{H}=\mathscr{N}(X)$ and $\tilde{\mathscr{H}}=\mathscr{N}(\tilde{X})$ we get an extension of the Pollit formula (cf. [9]) to $\boldsymbol{Z}^d$-actions.

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INSTITUTE OF MATHEMATICS
NICHOLAS COPERNICUS UNIVERSITY
UL. CHOPINA 12/18
87–100 TORUŃ
POLAND