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Deductive reasoning: in the eye of the beholder

Michal Ayalon · Ruhama Even

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Abstract This study examines ways of approaching deductive reasoning of people involved in mathematics education and/or logic. The data source includes 21 individual semi-structured interviews. The data analysis reveals two different approaches. One approach refers to deductive reasoning as a systematic step-by-step manner for solving problems, both in mathematics and in other domains. The other approach emphasizes formal logic as the essence of the deductive inference, distinguishing between mathematics and other domains in the usability of deductive reasoning. The findings are interpreted in light of theory and practice.

Keywords Deductive reasoning · Logical thinking · Mathematics educators · Approaches

Since the early days of Greek philosophical and scientific work, deductive reasoning has been considered as a high (and even the highest) form of human reasoning (Luria 1976; Sainsbury 1991). Specifically in mathematics, deductive reasoning has a most central role. Still, whereas mathematical proof has been, and still is, a central research focus in the field of mathematics education, views and approaches to deductive reasoning per se have received less attention. This study addresses this deficiency.

1 Introduction

There are various sorts of thinking and reasoning. Among them are association, creation, induction, plausible inference, and deduction (Johnson-Laird and Byrne 1991). Deductive reasoning is unique in that it is the process of inferring conclusions from known information (premises) based on formal logic rules, where conclusions are necessarily derived from the

M. Ayalon (✉) · R. Even
Weizmann Institute of Science, Rehovot, Israel
e-mail: michal.ayalon@weizmann.ac.il

given information and there is no need to validate them by experiments[1]. There are several forms of valid deductive argument, for example, *modus ponens* (If p then q; p; therefore q) and *modus tollens* (If p then q; not q; therefore not p). Valid deductive arguments preserve truth, in the sense that if the premises are true, then the conclusion is also true. However, the truth (or falsehood) of a conclusion or premises does not imply that an argument is valid (or invalid). Also, the premises and the conclusion of a valid argument may all be false.

Deductive reasoning is key to work in mathematics, because rigorous logical proof, which is a unique fundamental characteristic of mathematics, is constructed using deductive reasoning. Although there are some other accepted forms of mathematical validation, deductive proof is considered as the preferred tool in the mathematics community for verifying mathematical statements and showing their universality (Hanna 1990; Mariotti 2006; Yackel and Hanna 2003). And indeed, deductive reasoning is often used as a synonym for mathematical thinking.

In the pure formalist approach mathematical statements are neither true nor false (in the sense that they are not associated with meaningful interpretations) because they are about undefined terms. Instead, it is the logic basis of the inference that is important,

> All we can say in mathematics is that the theorem follows logically from the axioms. Thus the statements of mathematical theorems have no content at all; they are not *about* anything (Davis and Hersh 1981, p. 340).

Being free from the need to attend to the truth of mathematical statements enables mathematical explorations not available otherwise. Still, mathematics does not remain at the pure formal level. The undefined terms and axioms, which are the starting point of formal mathematical deductions, are often interpreted in connection to the world in which we live, and truth is associated with these interpretations. In this regard, the axioms of a specific mathematical theory are often said to be true and the theorems deduced from them are then also said to be true (Davis and Hersh 1981). The tension between the search for truth of mathematical statements (i.e., to associate them with meaningful interpretations) and the liberty to attend only to the validity of the deductive process and not to truth is at the basis of many developments in different domains in the field of mathematics (a nice illustration is the development of non-Euclidean geometries as the result of the attempts to prove Euclid's fifth postulate).

In the past there was a widespread belief that logical rules describe how people think (e.g., Inhelder and Piaget 1958), and many have considered deductive reasoning useful outside mathematics. Throughout human scientific development, great scientists, such as Descartes and Popper, emphasized the importance of this kind of reasoning to science. Johnson-Laird and Byrne (1991) emphasized its importance for work in science, technology, and the legal system, and Wu (1996) for facilitating wise decision making related to politics and the economy. Similarly, many mathematics curricula attribute importance to deductive reasoning, commonly associating it with the development of students' ability to reason logically both in mathematics and outside it (e.g., Australian Education Council 1990; Israel Ministry of Education and Culture 1990; National Council of Teachers of Mathematics 1989, 2000; The New Zealand Curriculum Framework 2001; Qualifications and Curriculum Authority 2006).

Still, reports of studies from the last decades challenge the claim that deductive reasoning plays an essential role outside mathematics. Psychologists like Wason (1968)

[1] This is the classic approach to deductive reasoning, which is also adopted in this paper. There are also other approaches; the main one is based not on formal rules of inference but on manipulations of mental models representing situations (Johnson-Laird 1999).

demonstrated that people make many logical errors, and that they are influenced by content and context. Toulmin (1969) asserted that rationality, in the sense of "taking the best choice out of a set of options whereby what counts as the best is a matter of negotiation" (Krummheuer 1995, p. 229), stands at the base of reasoning and communication in everyday activities, and that rationality is not bound to formal logic. Moreover, as Perelman and Olbrechts-Tyteca (1969) and Duval (2002) emphasized, in an attempt to convince others of the rationality of one's claims and choices, one uses various kinds of convincing arguments, which mostly do not have the logical rigidity of deductions that exist in mathematics. Also researchers as Perelman and Olbrechts-Tyteca (1969) and Polya (1954) pointed to other, less strict kinds of inferences, more of the plausible type (e.g., inductive, abductive), that are commonly used by people in non-mathematical situations.

Other researchers are more extreme in their view of the discrepancy between thinking in and outside mathematics. Researchers who adopt the evolutionary psychology point of view (Cosmides and Tooby 1997; Leron 2003, 2004) argue that a conflict exists between formal mathematical thinking and natural thinking. Evolutionary psychology researchers suggest that people do not naturally think in logical terms. On the other hand, people do reason naturally about social situations, using logics that are different from the formal one. For example, Leron (2004) pointed at a strong tendency of people to interpret "if… then" statements as if they were "if and only if" statements—a fallacy according to formal logic, but in line with the logic of social situations.

Whereas mathematical proof has been, and still is, a central research focus in the field of mathematics education (e.g., Fischbein and Kedem 1982; Duval 1991; Hanna 1990, 1996; Balacheff 1988; Mariotti 2006; Pedemonte 2007), conceptions, views and approaches to deductive reasoning per se have received less attention. How do people involved in mathematics education, such as, curriculum developers, teacher educators, teachers, and researchers approach deductive reasoning? What is their view on the usability of deductive reasoning in mathematics and outside it? The study reported here addresses these questions. The study examines the views and approaches to the meaning of deductive reasoning and its nature in mathematics and outside it of people involved in mathematics education. It is part of a larger study that investigates views of mathematics educators regarding the role of mathematics learning in the development of deductive reasoning. While working on the larger study we were surprised by unanticipated findings regarding approaches to deductive reasoning and decided to examine this issue more carefully. This is the focus of this paper.

2 Methodology

2.1 Research participants

Twenty-one people participated in the research. Most of them (17) belonged to different sub-communities in the field of mathematics education. This group was chosen to be as heterogeneous as possible in terms of the kind of involvement they had in the field of mathematics education, in order to increase the potential of diversity in their approaches. The group included mathematics teachers at various levels (from secondary school teachers to research mathematicians who teach undergraduate or graduate university mathematics), curriculum developers, teacher educators of prospective and practicing mathematics teachers, and researchers in mathematics education. Naturally, some of these participants belonged to several sub-communities (i.e. a curriculum developer who was also a teacher educator, and so on). All these participants had a reputation of being experienced and knowledgeable in their respective fields: well-known mathematicians and researchers in mathematics education from leading universities in Israel, prominent mathematics curriculum developers and teacher educators, and highly respected teachers, all had solid university or college education in mathematics; many also in mathematics education.

The four remaining participants out of the 21 were not connected to mathematics. They were chosen because their deep knowledge in issues related to logic and deductive reasoning. The aim was to enrich the data in order to contribute to the analysis and interpretation of the approaches of the math participants, which are the focus of the study. Two of these participants were logicians in the philosophy department of a leading university; the other two were university researchers in science education who had a long history of studying students' development of logical thinking.

The first three columns of Table 1 display the participants' main professional activities, as well as academic education in mathematics and mathematics education.

Table 1 Participants' background and approaches

Interviewee #	Main activities	Academic education	Approach to deductive reasoning
1	Researcher in mathematics education	M.Sc. mathematics Ph.D. mathematics education	Logic—Radical
2	Researcher in mathematics education, Curriculum developer	B.Sc. mathematics Ph.D. mathematics education	Logic—Radical
3	Researcher in mathematics education	Ph.D. physics	Logic—Moderate
4	Researcher in mathematics education, Mathematician	Ph.D. mathematics	Logic—Radical
5	Curriculum developer	M.Sc. mathematics Ph.D. mathematics education	Systematic
6	Curriculum developer, Teacher educator	M.A. mathematics Ph.D. mathematics education	Systematic
7	Curriculum developer, Teacher educator	B.Sc. mathematics Ph.D. mathematics education	Logic—Radical
8	Curriculum developer, Teacher educator	B.Sc. mathematics M.Ed. mathematics education	Logic—Moderate
9	Curriculum developer, Teacher educator	B.Sc. mathematics Ph.D. mathematics education	Logic—Moderate
10	Teacher educator	Ph.D. mathematics	Logic—Moderate
11	Senior high school teacher	B.Sc. mathematics M.A. mathematics education	Systematic
12	Junior high school teacher	B.Ed. mathematics education	Logic—Moderate
13	Junior high school teacher	M.Sc. mathematics education	Systematic
14	Junior and senior high school teacher	B. Sc. Engineering M.A. mathematics education	Logic—Moderate
15	Mathematician	Ph.D. mathematics	Logic—Moderate
16	Mathematician	Ph.D. mathematics	Logic—Moderate
17	Mathematician	Ph.D. mathematics	Logic—Moderate
18	Logician in the philosophy department	Ph.D. philosophy	Logic—Moderate
19	Logician in the philosophy department	Ph.D. philosophy	Logic—Moderate
20	Researcher in science education	Ph.D. biology	Logic—Moderate
21	Researcher in science education	M.Sc. biology Ph.D. science education	Logic—Moderate

2.2 Data collection

The data sources are individual semi-structured interviews that lasted between one and two hours. They focused on different issues related to the role of learning mathematics in developing deductive reasoning, both in and outside mathematics. The interviews were conducted in no specific order (i.e., the interviewee # in Table 1 does not reflect the interviewing order). Examples of main interview questions are: What does the concept of deductive reasoning mean to you? Can you give me an example of deductive reasoning? An example of non-deductive reasoning? To what extent do you think deductive reasoning is significant in our lives? Where? Why? Can you give some examples? How do you perceive the connections between learning mathematics and the development of deductive reasoning? Probing during the interview aimed at elaboration and explanation of general statements, continuously asking the interviewees to give specific examples from their own experiences.

2.3 Data analysis

Data analysis has been based on the Grounded Theory method (Glaser and Strauss 1967; Sabar Ben-Yehoshua 2001; Strauss and Corbin 1990). Thus, no prior assumptions were made regarding the interviewees' opinions and approaches, nor regarding possible differences or similarities among different sub groups. The interviews were transcribed and read carefully several times in their entirety, in no specific order. We then used open coding (Strauss 1987) to generate initial categories. For example, the significance of deductive reasoning, the role and use of deductive reasoning in daily life, students' difficulties with deductive reasoning. The initial categories were constantly compared with new data from the interviews and refined. We identified relationships and hierarchies among the categories, and created core categories which are "the central phenomenon around which all the other categories are related" (Strauss and Corbin 1990, p. 116). We used the core categories as a source for theoretical constructs. One of the categories that was developed through this process and is discussed in this paper is the meaning of deductive reasoning.

3 Deductive reasoning: two approaches

Two different approaches regarding the meaning of deductive reasoning were identified among the interviewees. One approach referred to deductive reasoning as a systematic step-by-step manner for solving problems both in mathematics and in other domains. The other approach emphasized logic as the essence of the deductive inference, distinguishing between mathematics and other domains in the usability of deductive reasoning. The approach of each interviewee was consistent throughout the entire interview, and was continuously reflected in responses to different questions and probes—whether dealing with mathematics or not.

3.1 The *systematic* approach

Four participants described deductive reasoning as a process in which one develops a solution to a given problem in a systematic, step-by-step manner (see Table 1, column 4). No indication was given by these interviewees as to how a step is derived from its predecessor, i.e., the critical role of logic in deductive inferences was not mentioned nor the use of deductive reasoning in the validation of arguments. This attitude towards deductive reasoning reoccurred in different contexts throughout these participants' interviews—when describing the term 'deductive reasoning' and its role inside and outside mathematics, and when discussing aspects of deductive reasoning that can be developed by learning mathematics. The following response of an interviewee to the question what the term 'deductive reasoning' meant for her exemplifies this approach.

> Deductive reasoning, I am talking about being systematic in thinking, thinking and developing ideas in an organized way (interviewee no. 11).

The interviewees that expressed the *systematic* approach to deductive reasoning connected it to problem solving, as did the above interviewee at some point later in her interview:

> I think that deductive reasoning is, here, I found the word: being systematic in thinking. We have some problem that we need to solve; I want the student to have a systematic way of thinking that is built layer upon layer—he thinks about something, he draws a conclusion, which brings him to the next thing… Logic is the procedural, algorithmic structure of things, and logical thinking is the ability to construct these processes or to activate them in situations of solving problems (interviewee no. 11).

Although this interviewee used the terms 'logic' and 'logical thinking', she did not refer explicitly to formal logic rules, but rather emphasized procedural and algorithmic aspects of problem solving in general terms.

The interviewees expressing the *systematic* approach did not relate deductive reasoning to problem solving in mathematics only, but also to problem solving in non-mathematical situations[2], not distinguishing between the use of deductive reasoning in mathematics and outside it. When people address a problem, these interviewees claimed, whether in mathematics or in other domains, they have to think systematically in order to progress towards its solution:

> Look, it is not only about mathematics. A person with good deductive reasoning can handle the demands of life more easily. I mean, for example, in organizing things, coping with problems at work, at home, and also in your profession. Like a teacher planning a lesson, an editor working on a book, and also a psychologist who reads the material of a patient and has to decide on the appropriate treatment. They all have to examine data, establish a working strategy and work towards achieving their goals. One who has a developed deductive ability would know how to approach tasks in a more systematic and organized manner. He would systematically gather the data he needs, the questions the task raises, and then will progress step by step towards the solution (interviewee no. 5).

Going even further with not distinguishing between the use of deductive reasoning in mathematics and outside it, the *systematic* approach interviewees stated that there is

[2] When referring to non-mathematical situations the interviewees (of both approaches) did not differentiate among different domains. Instead, they talked in general terms and illustrated their claims with examples, usually from daily life situations. Some interviewees did refer to specific domains, like art, science, literature, and law in order to illustrate a general point. However, these references were not presented as specific to the referred domain. Rather, all examples were treated as instances of general principles, and no comparisons were made among the nature of logical rules in different domains.

similarity between the rules of reasoning used inside and outside mathematics. For example:

> It is the same. It is the systematic thinking, to progress systematically. It is the use of the same logical rules—walk step by step, go forward, be organized in your thinking. These are the logical rules inside mathematics and also in other domains (interviewee no. 5).

Again, the use of the term 'logical rules' does not refer to formal logic rules, but rather to systematic and organized way of work.

As illustrated in the above excerpts, the vocabulary consistently and continuously used by the *systematic* approach interviewees included corresponding terms: order, systematically, organized way, thinking methodically, step-by-step, thinking straightforwardly without messing around, one thing after another, sequential order, solving problems.

3.2 The *logic* approach

Unlike the former approach, which focuses on deductive reasoning as a systematic process, the other approach identified focuses on the logic essence of the inference. The 17 interviewees expressing the *logic* approach (see Table 1, column 4) described deductive reasoning as an action of inference based on the rules of formal logic. For example, when an interviewee was asked what the term 'deductive reasoning' meant, he answered:

> Logical thinking – I've got no other definition – drawing conclusions according to the rules of logic... When a set of premises is given, deductive reasoning leads to solid conclusions because of the strict logical structure in which we are (interviewee no. 15).

Furthermore, while the *systematic* approach related deductive reasoning to problem solving, the *logic* approach related it to deducing valid inferences or to the examination of the validity of arguments:

> Inference and derivation are the essence of the work of logic. Deductive reasoning is seen when we make the transition from assumptions to a necessary conclusion, or when we examine the validity of an inference… The inference is deductively valid when its conclusion can be derived from its premises by means of formal logic rules of inference (interviewee no. 19).

Similarly, another interviewee talked about the "good" deductive reasoner, emphasizing the validation of arguments:

> A person who has good deductive reasoning knows how to examine the logical connections among propositions, whether propositions are derived one from another according to rules of logic. Suppose he sees two propositions, A and B; he would know to say: Is B derived from A? Can A also be derived from B? Is there any logical connection at all? (interviewee no. 10).

Thus, while the *systematic* approach to deductive reasoning described deductive reasoning as a systematic, step-by-step process used for solving problems, the *logic* approach focused on the logical essence of the inference as a means for validation and argumentation. As before, this attitude towards deductive reasoning reoccurred in different contexts throughout the interviews: when discussing the term 'deductive reasoning' and its role inside and outside mathematics, and when discussing aspects of deductive reasoning that can be developed by learning mathematics.

The differences between the *systematic* approach and the *logic* approach were expressed also in the different vocabularies used. Whereas the vocabulary of the *systematic* approach included terms, such as, systematic, order, step-by-step, and problem solving, the vocabulary of the *logic* approach comprised of terms, such as, formal logic rules, logical connections, inference, validity, necessity, derivation, argumentation, justification.

In contrast with the *systematic* approach the interviewees expressing the *logic* approach made a distinction between the usability of deductive reasoning in mathematics and outside it. Still, not all of them held the same view. Most of these interviewees (13 out of 17) were rather moderate in their approach. They claimed that while deductive reasoning is essential in mathematics, in non-mathematical situations we frequently apply other rules of inference in addition to the formal ones. The other four interviewees were more radical and claimed that we do not, or even cannot, use deductive reasoning in non-mathematical contexts.

3.2.1 The moderate logic approach

The 13 interviewees expressing the *moderate logic* approach asserted that deductive reasoning has certain use not only in mathematics, but also outside mathematics (see Table 1, column 4), for example, when trying to analyze insurance rights according to different levels of payment. Yet, these interviewees claimed that different kinds of factors affect reasoning outside mathematics; thus people apply other, usually 'softer', rules of inference, in addition to the rigorous ones.

Some of those expressing the *moderate logic* approach claimed that uncertainty and complexity of phenomena in nature and society distract deductive reasoning outside mathematics. They explained that real-life phenomena cannot always be dealt with using logic. Therefore, one uses less strict inference tools to understand these phenomena; probability and common sense were mentioned as frequent tools used to make an inference outside mathematics. For example:

> The problem is that in life it is not always possible to use all these logical inferences. Sometimes the situations are very complicated and not always one thing is derived deductively from the other. Besides, in mathematics there is no such thing as an exception, because then it is actually a counter example that refutes the argument. In life there are sometimes situations that do not conform to the rule and then we refer to them as exceptions. This means that it is impossible to apply deductions to them… In life we use the mathematical deductive rules, but because it is not always possible, we sometimes use logic that is less strict, something like common sense (interviewee no. 12).

Other interviewees expressing the *moderate logic* approach argued that in real life, where thinking depends to some extent on emotions and beliefs, it is more difficult to use logic:

> The emotions distract their logical reasoning from the rigors that exist in mathematics, so it is sometimes difficult for them to keep the logic of their claims and thought (interviewee no. 9).

Another frequent claim raised was that outside mathematics, convincing others of the truth of one's statement, and not validity, is most significant. Thus, deductive reasoning, which is central to validation, turns to be less relevant:

> In mathematics, because it is mathematics, one has to build his claims according to the rules of logic. It is a matter of derivation of theorems, one from another, in a deductive method. The final conclusion must be logically valid. But in non-mathematical situations there are other factors that are more important than this validity. In fact, the aims are changed. It is more a matter of how much your claims are convincing or can stand against other claims (interviewee no. 16).

> I'm saying that we use logical rules in life. However, there are things in life that can have an effect on logical thinking. For example, sometimes people want to convince others about the correctness of their arguments. A politician who wants the public to follow his claims can intentionally build these claims illogically. The most important issue for him would be to make the claims sound good. It can also happen to him without intention (interviewee no. 19).

Insufficient knowledge was also mentioned as a source for distracting deductive reasoning in life, in a sense that lack of information (i.e., premises) makes it impossible to use logical rules to reach conclusions outside mathematics. For example,

> Does one have in life enough information? Did you try to evaluate Netanyahu's [former Minister of the Treasury] economic plan? You may know simple logical rules, but not always will you have enough knowledge. In mathematics it is easier to collect the required information… When information is missing, it is more difficult to apply deduction, and we tend to use more plausible rules of inference, that can be applied under the conditions of uncertainty (interviewee no. 21).

3.2.2 The radical logic approach

Four interviewees of the *logic* approach took the previous approach to an extreme, and claimed that outside mathematical context, we do not, or even cannot, use deductive reasoning (see Table 1, column 4).

> Deductive reasoning is rigorous and is not suitable for daily life… I think that deductive reasoning serves only the needs of theoretical mathematicians, not even those of applied mathematicians or physicists. All of them do not really use deductive reasoning. In daily life, most of the population and even people who deal with very deep and important things do not really need deductive reasoning in the mathematical sense. Most people deal with things that are not certain, that cannot be measured with deductive tools (interviewee no. 4).

> In mathematics one does not look at the world. A mathematician defines something, and from that moment on he derives a theorem from a theorem, a statement from a statement… It is all logic. All the new statements are obtained only by manipulations of propositions. This is not the case in daily life where things cannot be derived logically one from the other (interviewee no. 1).

As in the case of the *moderate logic* approach, some of those expressing the *radical logic* approach supported their claim by referring to the uncertainty and complexity of phenomena in nature and society. Taking an extreme stand they claimed that whereas in mathematics it is necessary to use rigorous rules, in life one never encounters suitable circumstances for using them. For example:

> As I said, only theoretical mathematicians use deductive reasoning. In daily life or in matters that other people deal with, it is not possible to use mathematical deductive logic. These things cannot be measured, be analyzed in a deductive manner. One thing is not derived from another thing. It is all much more diffuse, complex, unclear (interviewee no. 4).

Other explanations given by those expressing the *radical logic* approach were different from those given by the interviewees holding the *moderate logic* approach. Whereas the *moderate logic* approach interviewees mentioned emotions and beliefs, aim of convincing, and lack of knowledge, the *radical logic* approach interviewees argued that the very nature of human thinking is in opposition to deductive reasoning. For example:

> To a large extent, deductive reasoning is against human nature from many points of view… I emphasize the evolutionary aspect. Formal thinking is in conflict with natural thinking… In daily life people do not think in a deductive way. They do not need it and they also do not have the skills for it. The thinking is, there are many other kinds of logics, not deductive mathematical logic, which our thinking follows them (interviewee no. 4).

And he continues, emphasizing how difficult it is for people in general to think in a deductive way:

> Deductive reasoning is not something that is meaningful to people with a humanistic way of thinking, soft thinking, open thinking. The need to be accurate is not really meaningful to them (interviewee no. 4).

Related to that was the claim raised by the *radical logic* approach interviewees that outside mathematics, the argumentative norms are such that the logic of an argument one builds is neither a necessary condition for understanding nor for accepting the argument. The following excerpt illustrates this idea.

> If you had taken segments from an everyday discourse in which people do derive things, and analyzed them according to logical rules that you know from standard mathematical discourse, you would have said 'Oh my god'. There are infinitely many examples. A mother says to a child: 'If you don't eat, then you won't get sweets'. The child says: 'I ate, so I deserve some sweets'. It is obvious that that was the mother's intention. She meant to say that if he eats he will get some sweets. But it is not equivalent. It is a different logical phrase. And you know what, as a logician, even I could say to my child: 'if you don't do X you won't get Y'… and I would mean that if she does X she will get Y. These rules are not those rules… The whole thing is that logical rules are logical rules, but in daily life people understand each other even in the case of a rule that is wrong in the logical sense, like the example of the mother and the dessert (interviewee no. 1).

4 Discussion

Two different approaches regarding the nature of deductive reasoning were identified in this study. One, which we expected, describes deductive reasoning as an action of inference based on the rules of formal logic. The other approach, which we did not anticipate when starting the study, describes deductive reasoning as a systematic step-by-step manner for solving problems, with no attention to issues of validity, formal logic rules, or necessity—the very essence of deductive reasoning.

Moreover, whereas all study participants agreed that deductive reasoning is essential to mathematics, different approaches regarding the usability of deductive reasoning outside mathematics were identified. Those approaching deductive reasoning as a systematic step-by-step manner for solving problems considered the use of deductive reasoning in mathematics to be the same as its use in other domains or in daily life. In contrast, those emphasizing logic as the essence of deductive reasoning, distinguished between mathematics and other domains in the usability of deductive reasoning. Some were moderate in their approach, claiming that in non-mathematical situations we apply other rules of inference in addition to the formal ones. Their claims are compatible with those raised in the argumentation literature (e.g., Duval 2002; Krummheuer 1995; Mariotti 2006; Toulmin 1969), which also points at the common use of plausible inferences outside mathematics, and at various factors (among them content and aims of convincing) that affect the use of deductive reasoning outside mathematics. Other interviewees were more radical and claimed that we do not, or even cannot, use deductive reasoning in non-mathematical contexts. The example of the “mother and sweets” episode, for instance, which is “logically wrong” but, on the other hand, compatible with norms of argumentation in everyday discourse, expresses the sizeable discrepancy between formal thinking and natural thinking suggested by Leron (2003, 2004) and Cosmides and Tooby (1997), as well as Toulmin (1969) and Duval (2002).

This distinction between the usability of deductive reasoning in and outside mathematics might help to explain the intriguing finding of this study: the *systematic* approach to deductive reasoning which describes deductive reasoning as a systematic step-by-step manner for solving problems, whether in mathematics or in other domains, with no attention to the logic essence of deductive reasoning. In a way, the assertion that there is a distinction between the usability of deductive reasoning in and outside mathematics contradicts a different long-established prevalent assertion that learning mathematics develops the ability to use deductive reasoning in non-mathematical situations (e.g., Australian Education Council 1990; Israel Ministry of Education and Culture 1990; National Council of Teachers of Mathematics 1989, 2000; The New Zealand Curriculum Framework 2001; Qualifications and Curriculum Authority 2006). One way to resolve the tension created by the seemingly contradiction between the two assertions is to attend to aspects of reasoning and thinking that are important to both mathematical and non-mathematical problem solving situations. For example, being systematic and organized, working step-by-step. It might be that this is what the *systematic* group did.

Views of and approaches to deductive reasoning have seldom been the explicit focus of research in mathematics education. Our initial study contributes to raising several issues for future research. For example, are there additional approaches to deductive reasoning and its usability in mathematics and outside it among mathematics educators? Among other professionals (e.g., scientists, economists, lawyers)? The last question is related to another important matter. Table 1 shows that the *systematic* group includes two teachers and two curriculum developers, one of which is also a teacher educator. There are several other teachers, teacher educators and curriculum developers in the *logic* group as well. However, all the mathematicians and the researchers in mathematics education (as well as the researchers in science education and the logicians) belonged to the *logic* group. Do specific sub-communities in the community of mathematics educators tend to approach deductive reasoning in a particular way? Do institutional positions which different people hold predispose them to different biases regarding deductive reasoning? The limited scope of this study does not allow us to answer such questions, and it seems worthwhile to study the subject more thoroughly. Another issue for future research has to do with the connection between approaches and practice. For example, what would happen when a teacher with a systematic approach teaches a logic-based curriculum? Or vice versa? Likewise, it is recommended to conduct comparative studies in other countries and cultures to examine whether approaches to deductive reasoning are rooted in a specific culture or are more general.

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References

Australian Education Council (1990). *A national statement on mathematics for Australian schools*. Carlton, Victoria: Curriculum Corporation.

Balacheff, N. (1988). 'A study of students' proving processes at the junior high school level'. *Paper presented at the 66th Annual Meeting of the National Council of Teachers of Mathematics*, Chicago, USA.

Cosmides, L., & Tooby, J. (1997). Evolutionary Psychology: A primer. http://philosophy.wisc.edu/shapiro/Phil553/EvPsych%20Primer.pdf.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience*. Boston: Birkhäuser.

Duval, R. (1991). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*, 103–131 doi:10.1007/s10649-006-0400-z.

Duval, R. (2002). 'Proof understanding in mathematics', *Proceedings of 2002 International Conference on Mathematics: Understanding proving and proving to understand* (pp. 23–44). National Taiwan Normal University. http://www.math.ntnu.edu.tw/~cyc/_private/mathedu/me1/me1_2002_1/duval.doc.

Fischbein, E., & Kedem, I. (1982). 'Proof and certitude in the development of mathematical thinking'. In A. Vermandel (Ed.), *Proceedings of the sixth international conference for the psychology of mathematics education* (pp. 128–131). Antwerp: Universitaire Instelling Antwerpen.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, *21*(1), 6–13 doi:10.1007/BF01809605.

Hanna, G. (1996). 'The ongoing value of proof'. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th conference of the international group for the psychology of mathematics education* (pp. 21–35). Valencia, Spain.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence: An essay on the construction of formal operational structures*. London: Basic Books.

Israel Ministry of Education and Culture (1990). *The mathematics curriculum in grades* (pp. 7–9). Jerusalem: Israel Ministry of Education and Culture.

Johnson-Laird, P. N. (1999). Deductive reasoning. *Annual Review of Psychology*, *50*, 109–135 http://arjournals.annualreviews.org/doi/abs/10.1146/annurev.psych.50.1.109.

Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction*. Hillsdale, NJ: Erlbaum.

Krummheuer, G. (1995). 'The ethnography of argumentation'. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.

Leron, U. (2003). *Origins of mathematical thinking: A synthesis*. http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG1/TG1_leron_cerme3.pdf.
Leron, U. (2004). *Mathematical thinking and human nature: Consonance and conflict*. http://edu.technion.ac.il/Faculty/uril/Papers/Leron_ESM_%20Human_Nature.pdf.
Luria, A. R. (1976). *Cognitive development: Its cultural and social foundations*. Cambridge, MA: Harvard University Press.
Mariotti, M. A. (2006). 'Proof and proving in mathematics education'. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 173–203). Rotterdam: Sense.
National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
Pedemonte, B. (2007). How can the relationships between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*, 23–41 doi:10.1007/s10649-006-9057-x.
Perelman, C., & Olbrechts-Tyteca, L. (1969). *The new rhetoric: A treatise on argumentation*. Notre Dame, IN: University of Notre Dame Press.
Polya, G. (1954). *Induction and analogy in mathematics*. Princeton, NJ: Princeton University Press.
Qualifications and Curriculum Authority (2006). National curriculum online. http://www.nc.uk.net/index.html.
Sabar Ben-Yehoshua, N. (2001). *Genres and tradition in qualitative research*. Lod: Dvir (in Hebrew).
Sainsbury, M. (1991). *Logical forms: An introduction to philosophical logic*. Oxford: Blackwell.
Strauss, A. (1987). *Qualitative analysis for social scientists*. Cambridge: Cambridge University Press.
Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Thousand Oaks, CA: Sage.
The New Zealand Curriculum Framework (2001) http://www.tki.org.nz/r/maths/curriculum/statement/p8_e.php.
Toulmin, S. E. (1969). *The uses of arguments*. Cambridge: Cambridge University Press.
Wason, P. (1968). Reasoning about a rule. *The Quarterly Journal of Experimental Psychology*, *20*, 273–281 doi:10.1080/14640746808400161.
Wu, H. (1996). The role of Euclidean geometry in high school. *The Journal of Mathematical Behavior*, *15*, 221–237 doi:10.1016/S0732-3123(96)90002-4.
Yackel, E., & Hanna, G. (2003). 'Reasoning and proof'. In J. Kilpatrick, W. G. Martin, & D. E. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 227–236). Reston, VA: National Council of Teachers of Mathematics.