

## Deep factorisation of the stable process\*

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### Abstract

The Lamperti–Kiu transformation for real-valued self-similar Markov processes (rssMp) states that, associated to each rssMp via a space-time transformation, there is a Markov additive process (MAP). In the case that the rssMp is taken to be an  $\alpha$ -stable process with  $\alpha \in (0, 2)$ , [16] and [24] have computed explicitly the characteristics of the matrix exponent of the semi-group of the embedded MAP, which we henceforth refer to as the *Lamperti-stable MAP*. Specifically, the matrix exponent of the Lamperti-stable MAP’s transition semi-group can be written in a compact form using only gamma functions.

Just as with Lévy processes, there exists a factorisation of the (matrix) exponents of MAPs, with each of the two factors uniquely characterising the ascending and descending ladder processes, which themselves are again MAPs. Although the case of MAPs with jumps in only one direction should be relatively straightforward to handle, to the author’s knowledge, not a single example of such a factorisation for two-sided jumping MAPs currently exists in the literature. In this article we provide a completely explicit Wiener–Hopf factorisation for the Lamperti-stable MAP.

The main value and novelty of exploring the matrix Wiener–Hopf factorisation of the underlying MAP comes about through style of the computational approach. Understanding the fluctuation theory of the underlying MAP offers new insight into different ways of analysing stable processes. Indeed, we obtain new space-time invariance properties of stable processes, as well as demonstrating examples how new fluctuation identities for stable processes can be developed as a consequence of the reasoning in deriving the matrix Wiener–Hopf factors. The methodology in this paper has already lead to new applications in [27] and [28].

**Keywords:** stable process; Wiener–Hopf factorisation; self-similar processes; Markov additive processes.

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### 1 Introduction

Let  $X := (X_t)_{t \geq 0}$  be a one-dimensional Lévy process, starting from zero, with law  $\mathbb{P}$ . The Lévy–Khintchine formula states that, for all  $\theta \in \mathbb{R}$ , the characteristic exponent  $\Psi(\theta) := -\log \mathbb{E}(e^{i\theta X_1})$  satisfies

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x| \leq 1)}) \Pi(dx), \quad \theta \in \mathbb{R}, \tag{1.1}$$

where  $a \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\Pi$  is a measure (the *Lévy measure*) concentrated on  $\mathbb{R} \setminus \{0\}$  such that  $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty$ . When analytical extension is possible, we refer to  $\psi(z) := -\Psi(-iz)$  as the Laplace exponent.

The process  $(X, \mathbb{P})$  is said to be a *strictly  $\alpha$ -stable process* (henceforth just written ‘stable process’) if it is an unkilled Lévy process which also satisfies the *scaling property*: under  $\mathbb{P}$ , for every  $c > 0$ , the process  $(cX_{tc^{-\alpha}})_{t \geq 0}$  has the same law as  $X$ . It is known that  $\alpha$  necessarily belongs to  $(0, 2]$ , and the case  $\alpha = 2$  corresponds to Brownian motion, which we exclude. The Lévy–Khintchine representation of such a process is as follows:  $\sigma = 0$ ,  $\Pi$  is absolutely continuous with density given by

$$\pi(x) := c_+ x^{-(\alpha+1)} \mathbf{1}_{(x>0)} + c_- |x|^{-(\alpha+1)} \mathbf{1}_{(x<0)}, \quad x \in \mathbb{R},$$

where  $c_+, c_- \geq 0$ , and  $a = (c_+ - c_-)/(\alpha - 1)$ .

The process  $X$  has the characteristic exponent

$$\Psi(\theta) = c|\theta|^\alpha (1 - i\beta \tan \frac{\pi\alpha}{2} \operatorname{sgn}(\theta)), \quad \theta \in \mathbb{R}, \tag{1.2}$$

where  $\beta = (c_+ - c_-)/(c_+ + c_-)$  and  $c = -(c_+ + c_-)\Gamma(-\alpha) \cos(\pi\alpha/2)$ . Self-similarity dictates that we must necessarily have  $\beta = 0$  when  $\alpha = 1$ , which is to say that the process is symmetric. For more details, see [33, Theorems 14.10 and 14.15].

For consistency with the literature that we shall appeal to in this article, we shall always parametrise our  $\alpha$ -stable process such that

$$c_+ = \Gamma(\alpha + 1) \frac{\sin(\pi\alpha\rho)}{\pi} \quad \text{and} \quad c_- = \Gamma(\alpha + 1) \frac{\sin(\pi\alpha\hat{\rho})}{\pi},$$

where  $\rho = \mathbb{P}(X_t \geq 0)$  is the positivity parameter, and  $\hat{\rho} = 1 - \rho$ . In that case, the constant  $c$  simplifies to just  $c = \cos(\pi\alpha(\rho - 1/2))$ . Moreover, we may also identify the exponent as taking the form

$$\Psi(\theta) = |\theta|^\alpha (e^{\pi i\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta>0)} + e^{-\pi i\alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(\theta<0)}), \quad \theta \in \mathbb{R}. \tag{1.3}$$

**Remark 1.1.** Different values of  $c$  in (1.2) correspond to different linear scalings of time of the stable process. The parametrisation we are taking therefore fixes a particular scaling of time.

With this normalisation, we take the point of view that the class of stable processes is parametrised by  $\alpha$  and  $\rho$ ; the reader will note that all the quantities above can be written in terms of these parameters. We shall restrict ourselves a little further within this class by excluding the possibility of having only one-sided jumps. In particular, this rules out the possibility that  $X$  is a subordinator or the negative of a subordinator, which occurs when  $\alpha \in (0, 1)$  and either  $\rho = 1$  or  $0$ . In the case of a subordinator, for future reference, we note that the characteristic exponent takes the form  $\Psi(\theta) = (-i\theta)^\alpha$ ,  $\theta \in \mathbb{R}$ , which is the analytic extension of the Bernstein function  $\lambda \mapsto \lambda^\alpha$ ,  $\lambda \geq 0$ .

A fascinating theoretical feature of all characteristic exponents of Lévy processes is that they can always be written in terms of the so-called *Wiener–Hopf factors*. That

is to say, for a given characteristic exponent of a Lévy process,  $\Psi$ , there exist unique Bernstein functions,  $\kappa$  and  $\hat{\kappa}$  such that, up to a multiplicative constant,

$$\Psi(\theta) = \hat{\kappa}(i\theta)\kappa(-i\theta), \quad \theta \in \mathbb{R}. \tag{1.4}$$

As Bernstein functions,  $\kappa$  and  $\hat{\kappa}$  can be seen as the exponents of (killed) subordinators. The probabilistic significance of these subordinators, known as the ascending and descending ladder height processes respectively, is that their range corresponds precisely to the range of the running maximum of  $X$  and of  $-X$  respectively. In this sense, they play an important role in understanding the path fluctuations of the underlying Lévy processes. In particular, a rich history of literature has shown their fundamental significance in the understanding of a variety first passage problems; see for example their extensive use the the development of fluctuation theory of Lévy processes in the texts [7] and [26] as well as [9].

In the case of stable processes, the Wiener–Hopf factorisation takes a relatively straightforward form. Indeed, it is straightforward to argue that the ascending and descending ladder processes must necessarily be stable subordinators. One is therefore forced to take (up to a multiplicative constants)  $\kappa(\lambda) = \lambda^{\alpha_1}$ ,  $\lambda \geq 0$ , and  $\hat{\kappa}(\lambda) = \lambda^{\alpha_2}$ ,  $\lambda \geq 0$ , for some  $\alpha_1, \alpha_2 \in (0, 1)$ . Comparing (1.4) with (1.3), we must choose the parameters  $\alpha_1$  and  $\alpha_2$  such that, for example, when  $z > 0$ ,

$$z^\alpha e^{\pi i \alpha (\frac{1}{2} - \rho)} = z^{\alpha_1} e^{-\frac{1}{2} \pi i \alpha_1} \times z^{\alpha_2} e^{\frac{1}{2} \pi i \alpha_2}. \tag{1.5}$$

Matching radial and angular parts, we find that

$$\begin{cases} \alpha_1 + \alpha_2 = \alpha, \\ \alpha_1 - \alpha_2 = -\alpha(1 - 2\rho), \end{cases} \tag{1.6}$$

which gives us  $\alpha_1 = \alpha\rho$  and  $\alpha_2 = \alpha\hat{\rho}$ . As we have assumed that  $X$  does not have monotone paths, it is necessarily the case that  $0 < \alpha\rho \leq 1$  and  $0 < \alpha\hat{\rho} \leq 1$ . Note also that when  $\alpha\rho = 1$ , the ascending ladder height process is a pure linear drift. In that case, the range of the maximum process  $\bar{X}$  is  $[0, \infty)$ . This can only happen when  $X$  is spectrally negative which has been ruled out by assumption in the introduction. Similarly the case that  $\alpha\hat{\rho} = 1$  corresponds to spectral positivity which has also been ruled out by assumption. In conclusion,

$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \quad \lambda \geq 0$$

where  $0 < \alpha\rho, \alpha\hat{\rho} < 1$ .

This discussion also helps us justify that, taking account of all the special cases of stable processes that we have chosen to exclude, the set of admissible parameters we are left to work with is

$$\{(\alpha, \rho) : \alpha \in (0, 2), \rho \in (1 - 1/\alpha, 1/\alpha) \text{ and } \rho = 1/2 \text{ if } \alpha = 1\}.$$

In this article, we expose a second Wiener–Hopf factorisation which is ‘deeply’ embedded within the stable processes through its so-called Lamperti–Kiu representation. Like the factorisation (1.5), the ‘deep factorisation’ we will present has value in that it informs us about the fluctuations of the stable process. As we shall see, this gives us access to an array of new results for Lamperti stable processes, as well as raising a methodology which leads us to some new results for stable processes. In addition, the methodology developed here also plays an important role in forthcoming work concerning further ‘deep’ analysis of stable processes and general rssMp; see [27] and [28].

The Lamperti–Kiu representation is a pathwise decomposition that holds more generally for any real-valued self-similar Markov processes (rssMp) and shows that any such process can be written as a space-time changed Markov additive process (MAP). In a similar spirit to (1.1), the semi-group of a MAP can be characterised via an exponent, albeit that it now takes the form of a complex-valued matrix function. Moreover, just as with Lévy processes, there exists a factorisation of the aforesaid matrix exponent. Although the case of MAPs with jumps in only one direction should be relatively straightforward to handle, to the author’s knowledge, not a single example of such a factorisation for two-sided jumping MAPs currently exists in the literature. Our main objective here is to provide a completely explicit Wiener–Hopf factorisation for the MAP that underlies the stable process via the Lamperti–Kiu transform: the so-called *Lamperti-stable MAP*.

Our approach appeals to three main techniques. First, information about the respective individual entries in the matrix factors can be gleaned using asymptotic Markov additive renewal theory in the setting of excursion theory for MAPs. Second, the quantities that are identified in that way can be related to complex first passage problems for stable processes. Third, the aforesaid first entry problems can be simplified by appealing to a version of the Riesz–Bogdan–Żak transform, which relates the mapping of the path of a stable process via a Kelvin transform, together with an endogenous time change, to a Doob  $h$ -transform of the stable process. For  $\alpha \in (1, 2)$ , this Doob  $h$ -transform corresponds to conditioning the stable process to avoid the origin, as explored in [16]. For  $\alpha \in (0, 1)$  it corresponds to conditioning the stable process to being absorbed at the origin. Finally, for  $\alpha = 1$  there is, in effect, no  $h$ -transform as  $h \equiv 1$ .

The remainder of this article is structured as follows. In the next section we explain the nature of the Lamperti–Kiu representation for pssMps, due to [16], and its relation to MAPs. In particular we give the example of the Lamperti-stable MAP, also due to [16]. With this in hand, we are able to state our ‘deep’ Wiener–Hopf factorisation result for the stable process. In Section 3 we discuss the Riesz–Bogdan–Żak transform. In Sections 4, 5 and 6 we compute the first matrix factor of the Wiener–Hopf factorisation for the Lampert-stable MAP, which is the analogue of the contribution from the exponent of the ascending ladder height process in the Wiener–Hopf factorisation for Lévy processes. In Section 7, we compute the second Wiener–Hopf matrix factor, which is the analogue of the contribution from the ascending ladder height process of the dual in the Wiener–Hopf factorisation for Lévy processes. Finally, in Section 8, we outline how some Cramér-type asymptotics and explicit identities for the Lamperti-stable MAP and stable processes can be obtained from the methods that underly the deep factorisation.

## 2 MAPs and the Lamperti–Kiu transform

This section is laid out as follows. We devote the first two subsections to a discussion of Markov additive processes and real self-similar Markov processes via the Lamperti–Kiu representation. Finally, in the last subsection, give our main result, the deep Wiener–Hopf factorisation of the stable process.

### 2.1 Markov additive processes

Let  $E$  be a finite state space and  $(\mathcal{G}_t)_{t \geq 0}$  a standard filtration. A càdlàg process  $(\xi, J)$  in  $\mathbb{R} \times E$  with law  $\mathbf{P}$  is called a *Markov additive process (MAP)* with respect to  $(\mathcal{G}_t)_{t \geq 0}$  if  $(J(t))_{t \geq 0}$  is a continuous-time Markov chain in  $E$ , and the following property is satisfied, for any  $i \in E, s, t \geq 0$ :

$$\begin{aligned} &\text{given } \{J(t) = i\}, \text{ the pair } (\xi(t+s) - \xi(t), J(t+s)) \text{ is independent of } \mathcal{G}_t, \\ &\text{and has the same distribution as } (\xi(s) - \xi(0), J(s)) \text{ given } \{J(0) = i\}. \end{aligned} \quad (2.1)$$

Aspects of the theory of Markov additive processes are covered in a number of texts, among them [6] and [5]. We will mainly use the notation of [19], where it was principally assumed that  $\xi$  is spectrally negative; the results which we quote are valid without this hypothesis, however.

Let us introduce some notation. For  $x \in \mathbb{R}$ , write  $\mathbf{P}_{x,i} = \mathbf{P}(\cdot | \xi(0) = x, J(0) = i)$ . If  $\mu$  is a probability distribution on  $E$ , we write  $\mathbf{P}_{x,\mu} = \sum_{i \in E} \mu_i \mathbf{P}_{x,i}$ . We adopt a similar convention for expectations.

It is well-known that a Markov additive process  $(\xi, J)$  also satisfies (2.1) with  $t$  replaced by a stopping time, albeit on the event that the stopping time is finite. The following proposition gives a characterisation of MAPs in terms of a mixture of Lévy processes, a Markov chain and a family of additional jump distributions; see [5, §XI.2a], [19, Proposition 2.5] as well as more classical literature such as [14, 15, 4].

**Proposition 2.1.** *The pair  $(\xi, J)$  is a Markov additive process if and only if, for each  $i, j \in E$ , there exist a sequence of iid Lévy processes  $(\xi_i^n)_{n \geq 0}$  and a sequence of iid random variables  $(U_{i,j}^n)_{n \geq 0}$ , independent of the chain  $J$ , such that if  $\sigma_0 = 0$  and  $(\sigma_n)_{n \geq 1}$  are the jump times of  $J$ , the process  $\xi$  has the representation*

$$\xi(t) = \mathbf{1}_{(n > 0)}(\xi(\sigma_n -) + U_{J(\sigma_n -), J(\sigma_n)}^n) + \xi_{J(\sigma_n)}^n(t - \sigma_n), \quad t \in [\sigma_n, \sigma_{n+1}), \quad n \geq 0.$$

For each  $i \in E$ , it will be convenient to define, on the same probability space,  $\xi_i$  as a Lévy process whose distribution is the common law of the  $\xi_i^n$  processes in the above representation; and similarly, for each  $i, j \in E$ , define  $U_{i,j}$  to be a random variable having the common law of the  $U_{i,j}^n$  variables.

Henceforth, we confine ourselves to irreducible (and hence ergodic) Markov chains  $J$ . Let the state space  $E$  be the finite set  $\{1, \dots, N\}$ , for some  $N \in \mathbb{N}$ . Denote the transition rate matrix of the chain  $J$  by  $\mathbf{Q} = (q_{i,j})_{i,j \in E}$ . For each  $i \in E$ , the Laplace exponent of the Lévy process  $\xi_i$  will be written  $\psi_i$ . For each pair of  $i, j \in E$ , define the Laplace transform  $G_{i,j}(z) = \mathbf{E}(e^{zU_{i,j}})$  of the jump distribution  $U_{i,j}$ , where this exists. Write  $\mathbf{G}(z)$  for the  $N \times N$  matrix whose  $(i, j)$ th element is  $G_{i,j}(z)$ . We will adopt the convention that  $U_{i,j} = 0$  if  $q_{i,j} = 0$ ,  $i \neq j$ , and also set  $U_{i,i} = 0$  for each  $i \in E$ .

The multidimensional analogue of the Laplace exponent of a Lévy process is provided by the matrix-valued function

$$\mathbf{F}(z) = \text{diag}(\psi_1(z), \dots, \psi_N(z)) + \mathbf{Q} \circ \mathbf{G}(z), \tag{2.2}$$

for all  $z \in \mathbb{C}$  where the elements on the right are defined, where  $\circ$  indicates elementwise multiplication, also called Hadamard multiplication. It is then known that

$$\mathbf{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = (e^{\mathbf{F}(z)t})_{i,j}, \quad i, j \in E, t \geq 0,$$

for all  $z \in \mathbb{C}$  where one side of the equality is defined. For this reason,  $\mathbf{F}$  is called the *matrix exponent* of the MAP  $(\xi, J)$ .

Just as is the case with Lévy processes, the exponents of MAPs are also known to have a Wiener–Hopf factorisation. However, this time, the two factors correspond to the matrix exponent of the ascending (resp. descending) ladder processes. These are themselves MAPs with trajectories which agree with the range and state of the modulating chain at times of new maxima (resp. minima). In order to explain the nature of the Wiener–Hopf factorisation for MAPs, we need to introduce a little more notation.

Associated to the running maximum process  $(\sup_{s \leq t} \xi(s))_{t \geq 0}$  is a Markov additive subordinator. That is, a MAP, say  $(H^+(t), J^+(t))_{t \geq 0}$ , with the property that  $H^+$  is non-decreasing with the same range as the running maximum. Moreover, its exponent can be identified by  $-\kappa(-z)$ , where

$$\kappa(\lambda) = \text{diag}(\Phi_1(\lambda), \dots, \Phi_N(\lambda)) - \mathbf{\Lambda} \circ \mathbf{K}(\lambda), \quad \lambda \geq 0, \tag{2.3}$$

is a matrix analogue of a Bernstein function. Here, for  $i = 1, \dots, N$ ,  $\Phi_i$  are Bernstein functions (exponents of subordinators),  $\Lambda = (\Lambda_{i,j})_{i,j \in E}$  is the intensity matrix of  $J^+$  and  $\mathbf{K}(\lambda)_{i,j} = \mathbf{E}[e^{-\lambda U_{i,j}^+}]$ , where  $U_{i,j}^+ \geq 0$  are the additional discontinuities added to the path of  $\xi$  each time the chain  $J^+$  switches from  $i$  to  $j$ , and  $U_{i,i}^+ := 0$ ,  $i \in E$ . In Section 4 we explain how the exponents  $\kappa$  can only be identified up to pre-multiplication by a diagonal matrix with strictly positive diagonal entries. *Henceforth we shall refer to the latter as strictly positive diagonal matrix.*

We also need to talk about the same quantity but for the dual of  $(\xi, J)$ . Whilst the dual of a Lévy process is equal in law to nothing more than its negative, the situation for MAPs is a little more involved. First note that, thanks to irreducibility, the Markov chain  $J$  necessarily has a stationary distribution. We denote it by the vector  $\pi = (\pi_1, \dots, \pi_N)$ . The dual process that is the MAP with probabilities  $\hat{\mathbf{P}}_{x,i}$ ,  $x \in \mathbb{R}$ ,  $i \in E$ , whose matrix exponent, when it is defined, is given by,

$$\hat{\mathbf{E}}_{0,i}[e^{z\xi(t)}, J(t) = j] = (e^{\hat{\mathbf{F}}(z)t})_{i,j}, \quad i, j \in E,$$

where

$$\hat{\mathbf{F}}(z) := \text{diag}(\psi_1(-z), \dots, \psi_N(-z)) + \hat{\mathbf{Q}} \circ \mathbf{G}(-z)^T$$

and  $\hat{\mathbf{Q}}$  is the intensity matrix of the modulating Markov chain on  $E$  with entries given by

$$\hat{q}_{i,j} = \frac{\pi_j}{\pi_i} q_{j,i}, \quad i, j \in E.$$

Note that the latter can also be written  $\hat{\mathbf{Q}} = \Delta_\pi^{-1} \mathbf{Q}^T \Delta_\pi$ , where  $\Delta_\pi = \text{diag}(\pi)$ , the matrix with diagonal entries given by  $\pi$  and zeros everywhere else. Hence, when it exists,

$$\hat{\mathbf{F}}(z) = \Delta_\pi^{-1} \mathbf{F}(-z)^T \Delta_\pi, \tag{2.4}$$

showing that

$$\pi_i \hat{\mathbf{E}}_{0,i}[e^{z\xi(t)}, J(t) = j] = \pi_j \mathbf{E}_{0,j}[e^{-z\xi(t)}, J(t) = i]. \tag{2.5}$$

At the level of processes, one can understand (2.5) as saying the following: The time-reversed process  $\{(\xi((t-s)-) - \xi(t), J((t-s)-)) : s \leq t\}$  under  $\mathbf{P}_{0,\pi}$  is equal in law to  $\{(\xi(s), J(s)) : s \leq t\}$  under  $\hat{\mathbf{P}}_{0,\pi}$ .

One final piece of information we will need later concerning MAPs is their long term behaviour. This can be gleaned from the spectral properties of the matrix  $\mathbf{F}$ . For  $z$  such that  $\mathbf{F}(z)$  is well defined, there exists a leading real-valued eigenvalue of the matrix  $\mathbf{F}(z)$ , also called the *Perron-Frobenius eigenvalue*; see [5, §XI.2c] and [19, Proposition 2.12]. If we denote this eigenvalue by  $\chi(z)$ , then it turns out that it is larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector  $\mathbf{v}(z)$  has strictly positive entries, and can be normalised such that  $\pi \cdot \mathbf{v}(z) = 1$ , where we recall that  $\pi$  is the stationary distribution of the underlying chain  $J$ . Assuming it exists, the derivative  $\chi'(0)$  dictates the long term behaviour of the process  $\xi$ . Indeed, a trichotomy similar in spirit to the one that describes the long term behaviour of Lévy processes exists, which states that either  $\lim_{t \rightarrow \infty} \xi(t) = \infty$ ,  $\lim_{t \rightarrow \infty} \xi(t) = -\infty$  or  $\limsup_{t \rightarrow \infty} \xi(t) = -\liminf_{t \rightarrow \infty} \xi(t) = \infty$  accordingly as  $\chi'(0) > 0$ ,  $\chi'(0) < 0$  or  $\chi'(0) = 0$ , respectively.

We are now ready to state the Wiener-Hopf factorisation for MAPs. Whilst some results in this direction exist in classical literature, see for example Chapter XI of [5] or Theorem 3.28 of [21]. Several other references can be cited in this respect, for example [22] and [4]. None of them are in an appropriate form for our purposes. We have lifted the following result from the Appendix of the recent article [18].

**Theorem 2.2.** For  $\theta \in \mathbb{R}$ , up to the pre-multiplication of  $\kappa$  (or  $\hat{\kappa}$ ) by a strictly positive diagonal matrix,

$$-F(i\theta) = \Delta_\pi^{-1} \hat{\kappa}(i\theta)^T \Delta_\pi \kappa(-i\theta), \tag{2.6}$$

where  $\hat{\kappa}$  plays the role of  $\kappa$ , but for the dual MAP to  $(\xi, J)$ .

This Theorem is consistent with the Wiener–Hopf factorisation for Lévy processes (1.4) as, in that setting, the dual process is its negative. In the Lévy setting the factorisation is valid up to a meaningless multiplicative constant. This is due to the fact that the ladder height exponents appearing in the factorisation can only be found up to a multiplicative constants, corresponding to an arbitrary linear scaling in local time (which does not affect the range of the ladder height processes themselves).

An equivalent way of stating Theorem 2.2 is to say that the Wiener–Hopf factorisation in (2.6) can only be identified up to pre-multiplication of  $F$  by a strictly positive diagonal matrix. (Note that pre-multiplying  $F$  by a strictly positive diagonal matrix produces the matrix exponent of another MAP that corresponds to a constant multiplicative time change of the original MAP, albeit that the constant depends on the state of  $J$ . This is discussed in more detail later in Remark 2.5.)

To see why this is the case, suppose, on the right hand side of (2.6), we write  $\kappa = \Delta_a \kappa'$  and  $\hat{\kappa} = \Delta_b \hat{\kappa}'$ , where  $\Delta_a$  and  $\Delta_b$  are two strictly positive diagonal matrices. Let  $\pi'$  be the vector with entries  $a_i \pi_i b_i / \sum_{j \in E} a_j \pi_j b_j$ , for  $i \in E$ . Then we have

$$-F(i\theta) = \Delta_\pi^{-1} \hat{\kappa}'(i\theta)^T \Delta_b \Delta_\pi \Delta_a \kappa'(-i\theta) = (\Delta_a \Delta_b) \Delta_\pi^{-1} \hat{\kappa}'(i\theta)^T \Delta_{\pi'} \kappa'(-i\theta), \tag{2.7}$$

for  $\theta \in \mathbb{R}$ . Note that

$$(\pi')^T [(\Delta_a \Delta_b)^{-1} F(0)] = \pi^T F(0) = 0,$$

showing that  $\pi'$  is the stationary distribution of  $J$  following the multiplicative time change corresponding to pre-multiplication of  $F$  by  $\Delta_{ab}^{-1} := \Delta_a^{-1} \Delta_b^{-1}$ . We now see that (2.7) identifies the factorisation of  $\Delta_{ab}^{-1} F$ .

## 2.2 Real self-similar Markov processes

The structure of real self-similar Markov processes has been investigated by [17] in the symmetric case, and [16] in general. Here, we give an interpretation of these authors' results in terms of a two-state Markov additive process. We begin with some relevant definitions and introductory results.

A *real self-similar Markov process* (rssMp) with *self-similarity index*  $\alpha > 0$  is a standard (in the sense of [12]) Markov process  $X = (X_t)_{t \geq 0}$  with probability laws  $(\mathbb{P}_x)_{x \in \mathbb{R}}$  which satisfies the *scaling property* that for all  $x \in \mathbb{R} \setminus \{0\}$  and  $c > 0$ ,

$$\text{the law of } (cX_{tc^{-\alpha}})_{t \geq 0} \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}. \tag{2.8}$$

In [16] the authors confine their attention to processes in 'class **C.4**'. An rssMp  $X$  is in this class if, for all  $x \neq 0$ ,  $\mathbb{P}_x(\exists t > 0 : X_t X_{t-} < 0) = 1$ ; that is, with probability one, the process  $X$  changes sign infinitely often. Define

$$\tau^{\{0\}} = \inf\{t \geq 0 : X_t = 0\},$$

the time to first hitting the origin. Such processes may be identified with a MAP via a deformation of space and time known as the *Lamperti–Kiu representation* of  $X$ . The following result is a simple corollary of [16, Theorem 6] and, although it was originally stated for the class **C.4**, it is valid for all rssMp. We will comment further after the statement of the result.

**Proposition 2.3.** *Let  $X$  be an rssMp in class **C.4** and fix  $x \neq 0$ . Define the symbol*

$$[y] = \begin{cases} 1, & y > 0, \\ 2, & y < 0. \end{cases}$$

*Then there exists a time-change  $\sigma$ , adapted to the filtration of  $X$ , such that, under the law  $\mathbb{P}_x$ , the process*

$$(\xi(t), J(t)) = (\log|X_{\sigma(t)}|, [X_{\sigma(t)}]), \quad t \geq 0,$$

*is a MAP with state space  $E = \{1, 2\}$  under the law  $\mathbf{P}_{\log|x|, [x]}$ . Furthermore, the process  $X$  under  $\mathbb{P}_x$  has the representation*

$$X_t = \exp(\xi(\varphi(t)) + i\pi(J(\varphi(t)) + 1)), \quad 0 \leq t < \tau^{\{0\}}, \tag{2.9}$$

*where  $\varphi$  is the inverse of the time-change  $\sigma$ , and may be given by*

$$\varphi(t) = \inf \left\{ s > 0 : \int_0^s \exp(\alpha\xi(u)) du > t \right\}, \quad t < \tau^{\{0\}}, \tag{2.10}$$

*such that  $(\xi, J)$  has law  $\mathbf{P}_{\log|x|, [x]}$ .*

**Remark 2.4.** The requirement that  $X$  belongs to class **C.4** is not really necessary and merely restricts matters to the case that  $J$  is an irreducible Markov chain. Without it, the Lampert-Kiu transformation is still valid, providing we allow the notion of a MAP to include the possibility that each of the Lévy processes underlying the MAP allow for killing at an exponential rate (which is dependent on the state of  $J$ ) with cemetery state  $-\infty$ .

**Remark 2.5.** In light of the multiplicative strictly positive diagonal matrix appearing in the statement of Theorem 2.2, it is interesting to ask what happens if we change the MAP  $(\xi, J)$  by scaling time by a constant in such a way that constant may depend on the state  $J$ . More precisely, what is the effect through the Lamperti transform if we replace  $(\xi, J)$  by the pair  $\xi'(t) := \xi(\int_0^t a(J_s) ds)$  and  $J'(t) = J(\int_0^t a(J_s) ds)$ ,  $t \geq 0$ , where  $\mathbf{a} := (a(1), a(-1))$  has strictly positive entries? Note that this is equivalent to premultiplying the matrix exponent of  $(\xi, J)$  by  $\mathbf{\Delta}_a$ . By considering the Lamperti representation of  $X$  until each crossing time of the origin, we note that this alteration of time scales has the effect that the Lamperti transform of  $(\xi', J')$ , say  $X'$  satisfies

$$X'_t = X_{\int_0^t a(\text{sign}(X_s)) ds}, \quad t \geq 0.$$

In the case that  $X$  is a stable processes, we have taken a particular linear scaling of time (cf. Remark 1.1). A different scaling would correspond to multiplication of  $\mathbf{F}$  by a scalar constant (or equivalently premultiplication by  $\mathbf{\Delta}_a$ , where  $a(1) = a(-1) > 0$ ).

### 2.3 The Lamperti-Stable MAP and deep factorisation

Now let us return to the case that  $X$  is a stable process as described in the introduction, which is also an rssMp. In [16, §4.1], the authors note that stable processes satisfy **C.4** and proceed to calculate the characteristics of the Lamperti-Kiu representation for  $X$  until first hitting the origin; that is, they compute the characteristics of the processes  $\xi_i$ , the jump distributions  $U_{i,j}$  and rates the  $q_{i,j}$ , for  $i, j \in \{1, 2\}$ . Using this information, and the representation (2.2), it was shown in [24] that the MAP  $(\xi, J)$  has matrix exponent



$$F(z) = \begin{bmatrix} -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1 - \alpha\hat{\rho} + z)} & \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\ \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha\rho + z)} \end{bmatrix}, \quad (2.11)$$

for  $\text{Re}(z) \in (-1, \alpha)$ . In the spirit of [26, Chapter 13.4] we refer to this process as a *Lamperti-stable MAP*.

We should also note that the diagonal terms have entries which are characteristic exponents which belong to the class of so-called *hypergeometric Lévy processes*. Moreover, up to a multiplicative constant, the off-diagonal terms can be shown to be the Laplace transforms of distributions, which possess a density with respect to Lebesgue measure that can be written in terms of the classical hypergeometric  ${}_2F_1$  function (see Chapter 13 of [26]). The matrix exponent (2.11) could, in theory, be shown to belong to a bigger family of MAPs which are in some sense a natural generalisation of the class of hypergeometric Lévy processes (cf. [23]). Indeed, we shall see other MAPs in the forthcoming analysis which are different to the Lamperti-stable MAP but clearly are close relatives with a common analytic structure. We shall explore this remark in more detail in future work however.

Our main results, below, give the explicit deep factorisation of (2.11) as predicted by Theorem 2.2. To our knowledge this is the first time that an example of the Wiener-Hopf factorisation for a MAP has been detailed explicitly. It is necessary to split the presentation of the deep factorisation in two cases accordingly as  $\alpha \in (0, 1)$  and  $\alpha \in [1, 2)$ . As will become clear in the proofs, this boils down whether  $X$  can hit the origin or not, which is distinguished by these two parameter regimes. Specifically, if  $\alpha \in (0, 1)$ , then  $X$  never visits the origin. In terms of the Lamperti-Kiu transform, this is synonymous with the fact that  $\lim_{t \rightarrow \infty} \xi(t) = \infty$ . On the other hand, if  $\alpha \in [1, 2)$ , then  $X$  will almost surely touch the origin in a finite time and this is synonymous with the fact that  $\lim_{t \rightarrow \infty} \xi(t) = -\infty$ . The latter observation is particularly worthy of note at this stage as Proposition 2.3 makes clear, the Lamperti-Kiu transform can only describe  $X$  up to first hitting of the origin.

To state our main results, we first need to introduce some notation. Of use will be the family of Bernstein functions

$$\kappa_{q,p}(\lambda) := \int_0^\infty (1 - e^{-\lambda x}) \frac{(q \vee p) - 1}{(1 - e^{-x})^q (1 + e^{-x})^p} e^{-\alpha x} dx, \quad \lambda \geq 0, \quad (2.12)$$

where  $q, p \in \{\alpha\rho \pm 1, \alpha\hat{\rho} \pm 1\}$  such that  $q + p = \alpha + 1$ . Note that it is easy to verify that the above expression is indeed a Bernstein function as the associated Lévy density behaves like either  $x^{-\alpha\rho-1}$  or  $x^{-\alpha\hat{\rho}-1}$  as  $x \downarrow 0$  and like  $e^{-\alpha x}$  as  $x \uparrow \infty$ . Accordingly, it is also straightforward to verify that the mean value  $\kappa'_{q,p}(0+)$  is finite.

**Theorem 2.6.** *When  $\alpha \in (0, 1]$ , We have the following two components to the factorisation in Theorem 2.2.*

(i) *Up to pre-multiplication by a strictly positive diagonal matrix, the ascending ladder MAP exponent is given by*

$$\kappa(\lambda) = \begin{bmatrix} \kappa_{\alpha\rho+1, \alpha\hat{\rho}}(\lambda) + \frac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha\rho)} \kappa'_{\alpha\hat{\rho}, \alpha\rho+1}(0+) & -\frac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha\rho)} \frac{\kappa_{\alpha\hat{\rho}, \alpha\rho+1}(\lambda)}{\lambda} \\ -\frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})} \frac{\kappa_{\alpha\rho, \alpha\hat{\rho}+1}(\lambda)}{\lambda} & \kappa_{\alpha\hat{\rho}+1, \alpha\rho}(\lambda) + \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})} \kappa'_{\alpha\rho, \alpha\hat{\rho}+1}(0+) \end{bmatrix},$$

for  $\lambda \geq 0$ .

(ii) Up to pre-multiplication by a strictly positive diagonal matrix, the dual ascending ladder MAP exponent is given by

$$\hat{\kappa}(\lambda) = \begin{bmatrix} \kappa_{\alpha\hat{\rho}+1,\alpha\rho}(\lambda + 1 - \alpha) + \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})}\kappa'_{\alpha\rho,\alpha\hat{\rho}+1}(0+) & -\frac{\kappa_{\alpha\rho,\alpha\hat{\rho}+1}(\lambda + 1 - \alpha)}{\lambda + 1 - \alpha} \\ -\frac{\kappa_{\alpha\hat{\rho},\alpha\rho+1}(\lambda + 1 - \alpha)}{\lambda + 1 - \alpha} & \kappa_{\alpha\rho+1,\alpha\hat{\rho}}(\lambda + 1 - \alpha) + \frac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha\rho)}\kappa'_{\alpha\hat{\rho},\alpha\rho+1}(0+) \end{bmatrix},$$

for  $\lambda \geq 0$ .

The next theorem deals with the case that  $\alpha \in (1, 2)$ . For this we need to introduce another family of Bernstein functions. Define

$$\phi_{q,p}(\lambda) = \int_0^\infty (1 - e^{-\lambda u}) \left\{ \frac{(q \vee p) - 1}{(1 - e^{-u})^q(1 + e^{-u})^p} - \frac{(\alpha - 1)}{2(1 - e^{-u})^q(1 + e^{-u})^p} \right\} e^{-u} du,$$

for  $\lambda \geq 0$ ,  $q, p \in \{\alpha\rho \pm 1, \alpha\hat{\rho} \pm 1\}$  such that  $q + p = \alpha + 1$ . Note, again, that the density in curly brackets can easily be verified to be positive in all cases and is a Bernstein function since, as before, the associated Lévy density behaves like either  $x^{-\alpha\rho-1}$  or  $x^{-\alpha\hat{\rho}-1}$  as  $x \downarrow 0$  and like  $e^{-x}$  as  $x \uparrow \infty$ . Once again, it is also subsequently straightforward to verify that the mean value  $\kappa'_{q,p}(0+)$  is finite.

**Theorem 2.7.** When  $\alpha \in (1, 2)$ , we have the following two components to the factorisation in Theorem 2.2.

(i) Up to pre-multiplication by a strictly positive diagonal matrix, the ascending ladder MAP exponent is given by

$$\kappa(\lambda) = \begin{bmatrix} \frac{\sin(\pi\alpha\rho)\phi_{\alpha\rho+1,\alpha\hat{\rho}}(\lambda + \alpha - 1)}{\sin(\pi\alpha\rho)\phi'_{\alpha\hat{\rho},\alpha\rho+1}(0+)} & -\sin(\pi\alpha\hat{\rho})\frac{\phi_{\alpha\hat{\rho},\alpha\rho+1}(\lambda + \alpha - 1)}{\lambda + \alpha - 1} \\ -\sin(\pi\alpha\rho)\frac{\phi_{\alpha\hat{\rho},\alpha\rho+1}(\lambda + \alpha - 1)}{\lambda + \alpha - 1} & \frac{\sin(\pi\alpha\hat{\rho})\phi_{\alpha\hat{\rho}+1,\alpha\rho}(\lambda + \alpha - 1)}{\sin(\pi\alpha\hat{\rho})\phi'_{\alpha\rho,\alpha\hat{\rho}+1}(0+)} \end{bmatrix},$$

for  $\lambda \geq 0$ .

(ii) Up to pre-multiplication by a strictly positive diagonal matrix, the dual ascending ladder MAP exponent is given by

$$\hat{\kappa}(\lambda) = \begin{bmatrix} \sin(\pi\alpha\hat{\rho})\phi_{\alpha\hat{\rho}+1,\alpha\rho}(\lambda) + \sin(\pi\alpha\hat{\rho})\phi'_{\alpha\rho,\alpha\hat{\rho}+1}(0+) & -\sin(\pi\alpha\hat{\rho})\frac{\phi_{\alpha\rho,\alpha\hat{\rho}+1}(\lambda)}{\lambda} \\ -\sin(\pi\alpha\rho)\frac{\phi_{\alpha\hat{\rho},\alpha\rho+1}(\lambda)}{\lambda} & \sin(\pi\alpha\rho)\phi_{\alpha\rho+1,\alpha\hat{\rho}}(\lambda) + \sin(\pi\alpha\rho)\phi'_{\alpha\hat{\rho},\alpha\rho+1}(0+) \end{bmatrix},$$

for  $\lambda \geq 0$ .

The two main results above, and in particular the techniques used to prove them, offer many new insights into the analysis of stable processes. Classically, the Wiener-Hopf factorisation of Lévy processes provides the basis of many proofs for fluctuation identities, both exact and asymptotic. Historically there has been less exploration in this respect for the case of MAPs. But, nonetheless, the same importance of the role of the Wiener-Hopf factorisation applies, with many proofs following analogous lines of reasoning to the Lévy case. See for example the Appendix in [18]. When one now takes account of the degree of explicit detail that we offer here with regard to the Lamperti-stable MAP Wiener-Hopf factorisation, as well as the pathwise embedding of the fluctuations of this MAP into the fluctuations of stable process, one should expect to gain new results for the latter family of processes. Based on the computations derived in obtaining the matrix factorisations above, we offer some results in this respect at the end of this paper. Moreover, the robustness and applicability of the techniques we develop in

proving the above two theorems also plays an important role in forthcoming work; see [28] and [27]. In particular, [27] explores the matrix potential measures associated to  $\kappa$  and  $\hat{\kappa}$  and how they can be used to study the stationary distribution of the stable process reflected in its previous radial maximum; [28] makes rigorous the notion of conditioning a stable process with index  $\alpha \in (0, 1)$  to continuously absorb at the origin.

More generally, the deep factorisation can be seen as the very beginnings of what, in higher dimensions, we will call *radial excursion theory*. Suppose that  $X$  is a self-similar Markov process (ssMp) in  $\mathbb{R}^d$ , that is to say, a regular strong Markov property which satisfies (2.8). A Lamperti–Kiu-type decomposition still exists which allows us to study the process from the point of view of generalised polar coordinates,  $X_t = (|X_t|, \text{Arg}(X_t))$ ,  $t \geq 0$ . Note  $|X|$  is the radial distance of  $X$  from the origin and  $\text{Arg}(X)$  is its ‘angular’ displacement on  $\mathbb{S}_d := \{x \in \mathbb{R}^d : |x| = 1\}$ . The excursion theory that we consider here for MAPs of one-dimensional stable processes extends in  $\mathbb{R}^d$  to the setting of excursions from the point of last radial maximum of general ssMps. In further work, in the spirit of [13], we hope to offer a rigorous treatment of radial excursion theory for ssMps, thereby providing an additional rare example of an excursion theory from a set; that is to say the excursion theory of the Markov process  $(X_t / \sup_{s \leq t} |X_s|, \sup_{s \leq t} |X_s|)$ ,  $t \geq 0$ , from  $\mathbb{S}_d \times (0, \infty)$ . Speculating, at least two very important results will follow from this radial excursion theory and that is the existence of entrance laws and recurrent extensions from the origin for ssMps in  $\mathbb{R}^d$ .

### 3 Non-symmetric Riesz–Bogdan–Žak transform

A key component in proving Theorem 2.6 and 2.7 will be the use of the so-called *Riesz–Bogdan–Žak* transform which we now outline.

**Theorem 3.1** (Riesz–Bogdan–Žak transform). *Suppose that  $X$  is a stable process as outlined in the introduction.*

(i) *Define*

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}, \quad t \geq 0.$$

*Then, for all  $x \in \mathbb{R} \setminus \{0\}$ ,  $(-1/X_{\eta(t)})_{t \geq 0}$  under  $\mathbb{P}_x$  a rssMp with underlying MAP via the Lamperti-Kiu transform given by*

$$\mathbf{F}^\circ(z) = \begin{bmatrix} -\frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & \frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ \frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & -\frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix}, \quad (3.1)$$

*for  $\text{Re}(z) \in (-\alpha, 1)$ .*

(ii) *Moreover, for all  $x \in \mathbb{R} \setminus \{0\}$ ,  $(-1/X_{\eta(t)})_{t \geq 0}$  under  $\mathbb{P}_x$  is equal in law to  $(X, \mathbb{P}_{-1/x}^\circ)$ , where*

$$\frac{d\mathbb{P}_x^\circ}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \left( \frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\text{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\text{sgn}(x)} \right) \left| \frac{X_t}{x} \right|^{\alpha-1} \mathbf{1}_{(t < \tau^{(0)})} \quad (3.2)$$

*and  $\mathcal{F}_t := \sigma(X_s : s \leq t)$ ,  $t \geq 0$ .*

In the case that  $X$  is a symmetric stable process (i.e.  $c_+ = c_-$ , equivalently  $\rho = 1/2$ ) the result is contained in the result of Bogdan and Žak [11], who deal with isotropic stable processes in one or more dimensions. One may see the work of Bogdan and Žak, specifically the idea of spatial inversion through a sphere (here a symmetric interval), as

building on original results of M. Reisz, who used this technique to analyse potentials, cf. [30, pp. 13-171], [31] as well as the discussion in Section 3 of [10].

It is straightforward to deduce that  $(X, \mathbb{P}_x^\circ, x \in \mathbb{R} \setminus \{0\})$  is a rssMp, inheriting the index of self-similarity  $\alpha$  from  $(X, \mathbb{P}_x), x \in \mathbb{R} \setminus \{0\}$ . When  $\alpha \in (1, 2)$ , [16] have identified  $(X, \mathbb{P}_x^\circ)$  to be the law of a stable process conditioned to avoid the origin when issued from  $x \in \mathbb{R} \setminus \{0\}$ . Their requirement that  $\alpha \in (1, 2)$  pertains to the fact that points are polar for  $\alpha \in (0, 1]$  and, accordingly, conditioning to avoid the origin makes no sense in the latter parameter regime. Nonetheless, the change of measure (3.2) is still meaningful and gives preference to paths that approach the origin closely, penalising paths that wander far from the origin. Indeed, [28] have recently showed that (3.2) corresponds to conditioning the stable process to continuously absorb at the origin; see also the forthcoming Remark 7.1. In this sense,  $(X, \mathbb{P}_x^\circ), x \in \mathbb{R} \setminus \{0\}$ , may be considered to be the stable process conditioned to be absorbed at the origin. When  $\alpha = 1$ , one easily sees that  $\mathbb{P}_x^\circ = \mathbb{P}_x, x \in \mathbb{R}$ .

The leading eigenvalue  $\chi$  of  $\mathbf{F}$  and its associated right eigenfunction  $v$  feature in the following probabilistic result, which identifies a martingale (the analogue of the Wald martingale), a change of measure and the analogue of the Esscher transformation for exponents of Lévy processes; cf. [5, Proposition XI.2.4, Theorem XIII.8.1]. (Note that the result is still true for general MAPs as introduced in Section 2.2.)

**Proposition 3.2.** *Let  $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}, t \geq 0$ , and*

$$M(t, \gamma) = e^{\gamma(\xi(t) - \xi(0)) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \quad t \geq 0, \tag{3.3}$$

for some  $\gamma$  such that  $\chi(\gamma)$  is defined. Then,  $M(\cdot, \gamma)$  is a unit-mean martingale with respect to  $(\mathcal{G}_t)_{t \geq 0}$ . Moreover, under the change of measure

$$\left. \frac{d\mathbb{P}_{x,i}^\gamma}{d\mathbb{P}_{x,i}} \right|_{\mathcal{G}_t} = M(t, \gamma), \quad t \geq 0,$$

the process  $(\xi, J)$  remains in the class of MAPs and, where defined, its characteristic exponent given by

$$\mathbf{F}_\gamma(z) = \mathbf{\Delta}_v(\gamma)^{-1} \mathbf{F}(z + \gamma) \mathbf{\Delta}_v(\gamma) - \chi(\gamma) \mathbf{I}, \tag{3.4}$$

where  $\mathbf{I}$  is the identity matrix and  $\mathbf{\Delta}_v(\gamma) = \text{diag}(v(\gamma))$ . (The latter matrix we understand to mean the diagonal matrix with entries of  $v(\gamma)$  loaded on to its diagonal.)

*Proof of Theorem 3.1.* First note that, if  $X$  is an  $(\alpha, \rho)$  stable process, then  $-X$  is a  $(\alpha, \hat{\rho})$  stable process. Next, we show that  $(-1/X_{\eta(t)})_{t \geq 0}$  is a rssMp with index  $\alpha$  by analysing its Lamperti–Kiu decomposition.

To this end, note that, if  $(\xi^*, J^*)$  is the MAP that underlies  $X^* := -X$ , then its matrix exponent, say  $\mathbf{F}^*(z)$ , is equal to (2.11) with the roles of  $\rho$  and  $\hat{\rho}$  interchanged. As  $X^*$  is a rssMp, we have

$$X_t^* = \exp \{ \xi^*(\varphi^*(t)) + i\pi(J^*(\varphi^*(t)) + 1) \}, \quad t < \tau^{\{0\}},$$

where

$$\int_0^{\varphi^*(t)} e^{\alpha \xi^*(s)} ds = t.$$

Noting that

$$\int_0^{\eta(t)} e^{-2\alpha \xi^*(\varphi^*(u))} du = t, \quad \eta(t) < \tau^{\{0\}},$$

a straightforward differentiation of the last two integrals shows that, respectively,

$$\frac{d\varphi^*(t)}{dt} = e^{-\alpha\xi^*(\varphi^*(t))} \text{ and } \frac{d\eta(t)}{dt} = e^{2\alpha\xi^*(\varphi^*\circ\eta(t))}, \quad \eta(t) < \tau^{\{0\}}.$$

The chain rule now tells us that

$$\frac{d(\varphi^*\circ\eta)(t)}{dt} = \left. \frac{d\varphi^*(s)}{ds} \right|_{s=\eta(t)} \frac{d\eta(t)}{dt} = e^{\alpha\xi^*(\varphi^*\circ\eta(t))},$$

and hence,

$$\int_0^{\varphi^*\circ\eta(t)} e^{-\alpha\xi^*(u)} du = t, \quad \eta(t) < \tau^{\{0\}}.$$

The qualification that  $\eta(t) < \tau^{\{0\}}$  only matters when  $\alpha \in (1, 2)$ . In that case, the fact that  $\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1$  for all  $x \in \mathbb{R}$  implies that  $\lim_{t \rightarrow \infty} \xi_t^* = -\infty$  almost surely. As a consequence, it follows that  $\int_0^\infty e^{-\alpha\xi^*(u)} du = \infty$  and hence  $\lim_{t \rightarrow \infty} \varphi^* \circ \eta(t) = \infty$ . That is to say, we have  $\lim_{t \rightarrow \infty} \eta(t) = \tau^{\{0\}}$ . Noting that for  $k \in \mathbb{N}$ ,  $e^{-i\pi k} = e^{i\pi k}$ , it now follows that

$$\frac{1}{X_{\eta(t)}^*} = \exp\{-\xi^*(\varphi^* \circ \eta(t)) + i\pi(J^*(\varphi^* \circ \eta(t)) + 1)\}, \quad t < \tau^{\{0\}}$$

is the representation of a rssMp whose underlying MAP has matrix exponent given by  $\mathbf{F}^*(-z)$ , whenever it is well defined. Recalling the definition of  $\mathbf{F}^*(z)$ , we see that the MAP that underlies  $(-1/X_{\eta(t)})_{t \geq 0}$  via the Lamperti-Kiu transform is identically equal in law to the MAP with matrix exponent

$$\mathbf{F}_{\alpha-1}(z) = \begin{bmatrix} -\frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & \frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ \frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & -\frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix},$$

for  $\text{Re}(z) \in (-\alpha, 1)$ .

Next we prove the second part. When we take  $\mathbf{F}$  to be given by (2.11), we can compute explicitly the quantity  $\pi$  as well as  $\mathbf{v}(\gamma)$  for a particular value of  $\gamma$  that is of interest. We are interested in the case that  $\gamma := \alpha - 1$ . Note that  $\gamma \in (-1, \alpha)$ . A straightforward computation shows that, for  $\text{Re}(z) \in (-1, \alpha)$ ,

$$\det \mathbf{F}(z) = \frac{\Gamma(\alpha-z)^2 \Gamma(1+z)^2}{\pi^2} \{ \sin(\pi(\alpha\rho-z)) \sin(\pi(\alpha\hat{\rho}-z)) - \sin(\pi\alpha\rho) \sin(\pi\alpha\hat{\rho}) \},$$

which has a root at  $z = \alpha - 1$ . In turn, this implies that  $\chi(\alpha - 1) = 0$ . One also easily checks with the help of the reflection formula for gamma functions that

$$\mathbf{v}(\alpha - 1) \propto \begin{bmatrix} \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) \end{bmatrix}$$

and, by considering  $\mathbf{F}(0) = \mathbf{Q}$ ,

$$\pi \propto \begin{bmatrix} \sin(\pi\alpha\rho) \\ \sin(\pi\alpha\hat{\rho}) \end{bmatrix}. \tag{3.5}$$

We see that with  $\gamma = \alpha - 1$ , the change of measure (3.3) corresponds precisely to (3.2) when  $(\xi, J)$  is the MAP underlying the stable process. To see this, first note that the time change  $\varphi(t)$  is a stopping time and so we consider the change of measure (3.3) at this stopping time. In this respect, appealing to (2.9), we use  $\exp\{\xi(\varphi(t))\} = |X_t|$ . Moreover, the rather large ratio in the expression for (3.3) matches the term  $v_{J(\varphi(t))}(\gamma)/v_{J(0)}(\gamma)$ .

We can now say that the MAP associated to the process  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ , formally named  $\mathbf{F}^\circ(z)$ , is equal to  $\mathbf{F}_{\alpha-1}(z)$ , for  $\operatorname{Re}(z) \in (-\alpha, 1)$ , where we have again used the reflection formula for the gamma function to deal with the terms coming from  $\Delta_\nu(\alpha - 1)$  in (3.4). The proof is now complete.  $\square$

**Remark 3.3.** Recall that for a different linear normalisation of time in the definition of the stable process, that is to say when  $(X_t)_{t \geq 0}$ , is replaced by  $X' := (X_{ct})_{t \geq 0}$ , for some  $c > 0$ , we have that the matrix  $\mathbf{F}$  in (2.11) is replaced by  $c\mathbf{F}$  (cf. Remark 1.1). If one works with this different scaling of time in the context of Theorem 3.1, then it can be easily verified that the matrix in (3.1) is also changed by multiplying it by the constant  $c$ . In turn, this matrix exponent correctly identifies with the process  $(X', \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ . Later in this text, when applying Theorem 3.1, the specific time-scaling of the stable process made in (1.3) is of no importance because it is the *range* of the stable process and the conditioned process  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ , that matters.

### 4 The ascending ladder MAP

We shall derive the Matrix exponent  $\kappa$  by deriving each and every component of the matrices  $\operatorname{diag}(\Phi_1(\lambda), \Phi_2(\lambda))$ ,  $\mathbf{\Lambda}$  and  $\mathbf{K}(\lambda)$ , for  $\lambda \geq 0$ . In order to do this, we will make use of the Riesz-Bogdan-Żak transform from the previous section as well as some classical Markov additive renewal theory. In order to understand how the latter bears relevance, we need to briefly recall how the ascending ladder process  $(H^+, J^+)$  emerges as a consequence of excursion theory and accordingly is a non-decreasing MAP.

Let  $Y_t^{(x)} = (x \vee \xi(t)) - \xi(t)$ ,  $t \geq 0$ , where  $\xi(t) = \sup_{s \leq t} \xi(s)$ ,  $t \geq 0$ . Following ideas that are well known from the theory of Lévy processes, it is straightforward to show that, as a pair, the process  $(Y^{(x)}, J)$  is a strong Markov process. For convenience, write  $Y$  in place of  $Y^{(0)}$ . We know by standard theory (c.f. Chapter IV of [7]) there exists a local time of  $(Y, J)$  at the point  $(0, i)$ , which we henceforth denote by  $\{L_t^{(i)} : t \geq 0\}$ . Now consider the process

$$L_t := \sum_{i \in E} L_t^{(i)}, \quad t \geq 0.$$

Note that the individual local times can be constructed uniquely up to independent multiplicative constants, which will turn out to be of pertinence later. Since, almost surely, for each  $i \neq j$  in  $E$ , the points of increase of  $L^{(i)}$  and  $L^{(j)}$  are disjoint, it follows that  $(L^{-1}, H^+, J^+) := \{(L_t^{-1}, H^+(t), J^+(t)) : t \geq 0\}$  is a (possibly killed) *Markov additive bivariate subordinator*, where

$$H^+(t) := \xi(L_t^{-1}) \text{ and } J^+(t) := J(L_t^{-1}), \quad \text{if } L_t^{-1} < \infty,$$

and  $H^+(t) := \infty$  and  $J^+(t) := \dagger$  (a cemetery state) otherwise. Note, as a Markov additive subordinator,  $(L^{-1}, H^+, J^+)$  is a Markov additive process with co-ordinatewise non-decreasing paths such that, in each of the states of  $J^+$ , the process  $H^+$  evolves as a subordinator possibly killed at an independent and exponentially distributed time, whereupon it is sent to the cemetery state  $\infty$  and  $J^+$  is sent to  $\dagger$ . Killing rates may depend on the state of  $J^+$ .

If we define

$$\epsilon_t = \{\epsilon_t(s) := \xi(L_{t-s}^{-1} + s) - \xi(L_{t-s}^{-1}) : s \leq \Delta L_t^{-1}\}, \quad \text{if } \Delta L_t^{-1} > 0,$$

and  $\epsilon_t = \partial$ , some artificial isolated state, otherwise, then it turns out that the process  $\{\epsilon_t : t \geq 0\}$  is a (killed) Cox process. Henceforth, write  $n_i$  for the intensity measure of this Cox process when the underlying modulating chain  $J^+$  is in state  $i \in E$ . As a Markov

additive subordinator, the process  $(H^+, J^+)$  has a matrix exponent given by

$$\mathbf{E}_{0,i}[e^{-\lambda H^+(t)}, J^+(t) = j] = (e^{-\kappa(\lambda)t})_{i,j}, \quad \lambda \geq 0,$$

where  $\kappa(\lambda)$  was given in (2.3). Note in particular that, for  $i = 1, 2$ ,  $\Phi_i(\lambda)$  is the subordinator Bernstein exponent that describes the movement of  $H^+$  when the modulating chain  $J^+$  is in state  $i$ . Moreover,  $\mathbf{\Lambda}$  is the intensity of  $J^+$  and the matrix  $\mathbf{K}(\lambda) = (\mathbf{K}(\lambda))_{i,j}$  is such that, for  $i \neq j$  in  $E$ , its  $(i, j)$ -th entry is the Laplace transform of the additional jump incurred by  $H^+$  when the modulating chain changes state from  $i$  to  $j$ . The diagonal elements of  $\mathbf{K}(\lambda)$  are set to unity. In general, we can write

$$\Phi_i(\lambda) = n_i(\zeta = \infty) + \mathbf{b}_i \lambda + \int_0^\infty (1 - e^{-\lambda x}) n_i(\epsilon_\zeta \in dx, J(\zeta) = i, \zeta < \infty), \quad \lambda \geq 0,$$

where  $\mathbf{b}_i \geq 0$  and  $\zeta = \inf\{s \geq 0 : \epsilon(s) > 0\}$  for the canonical excursion  $\epsilon$ .

Returning to our earlier remarks, recall that each of the local times  $L^{(i)}$ ,  $i \in E$ , can be defined up to an independent multiplicative constant. Suppose that we replace  $L^{(i)}$  by  $cL^{(i)}$  for some  $i \in E$ , where  $c > 0$ . It is a straightforward exercise to show that this change results in the process  $(H^+, L^{-1}, J^+)$  being subject to a multiplicative change of time on the event  $\{J^+ = i\}$ . In turn this has the consequence that the matrix exponent  $\kappa$  is changed by the same multiplicative constant in its  $i$ -th row. More generally, this reasoning shows that the variability in the multiplicative constants of  $L^{(i)}$ ,  $i \in E$  means that  $\kappa$  is unique up to pre-multiplication by a diagonal matrix with strictly positive diagonal elements. This observation does not disturb any of the forthcoming computations however.

For the case of the Lamperti-stable MAP, on account of the fact that the stable processes we consider in this paper do not creep, we can immediately set  $\mathbf{b}_i = 0$  for  $i = 1, 2$ . In the case that  $\alpha \in (0, 1]$ , points are polar and the underlying stable process explores arbitrarily large distances from the origin. When  $\alpha \in (1, 2)$ , the MAP in question represents the stable process until first hitting the origin, which occurs almost surely. We cannot rely on the range of the stable process until this time being unbounded and therefore  $n_i(\zeta = \infty) > 0$ , for  $i = 1, 2$ .

Let us recall the following result, which is a special case of a general Markov additive renewal limit theorem given in the Appendix of [18]. Such limit theorems are classical and can be found in many other contexts; see [29], [20] and [2, 3] to name but a few. First we fix some notation. For each  $a > 0$ , let

$$T_a = \inf\{t > 0 : H^+(t) > a\}.$$

**Lemma 4.1.** *Suppose that the ladder height process  $(H^+, J^+)$  does not experience killing, that is to say,  $L_\infty = \infty$ . Then for  $x > 0$  and  $i, j \in \{1, 2\}$ ,*

$$\begin{aligned} & \lim_{a \rightarrow \infty} \mathbf{P}_{0,i}(H^+(T_a) - a \in dx, J^+(T_a) = j) \\ &= \frac{1}{\mathbf{E}_{0,\pi}(H^+(1))} \left[ \pi_j n_j(\epsilon(\zeta) > x, J(\zeta) = j, \zeta < \infty) + \pi_k \Lambda_{k,j} (1 - F_{k,j}^+(x)) \right] dx, \end{aligned} \quad (4.1)$$

where  $k \in \{1, 2\}$  is such that  $k \neq j$  and  $\int_{[0,\infty)} e^{-\lambda x} F_{k,j}^+(dx) = \mathbf{E}[e^{-\lambda U_{k,j}^+}]$ . As a more refined version of the above statement, we also have that

$$\begin{aligned} & \lim_{a \rightarrow \infty} \mathbf{P}_{0,i}(H^+(T_a) - a \in dx, J^+(T_a) = j, J^+(T_a-) = j) \\ &= \frac{1}{\mathbf{E}_{0,\pi}(H^+(1))} \pi_j n_j(\epsilon(\zeta) > x, J(\zeta) = j, \zeta < \infty) dx, \end{aligned} \quad (4.2)$$

For all limits above, we interpret the right hand side as zero when  $\mathbf{E}_{0,\pi}(H^+(1)) = \infty$ .

**Remark 4.2.** Recall that the local time  $L$  can be constructed up to an arbitrary multiplicative constant. If one follows how this constant permeates through to the definition of  $n_i$ ,  $i = 1, 2$  and  $\Lambda_{i,j}$ ,  $i, j = 1, 2$ , we see that they are also defined up to the same multiplicative constant. For this reason, we shall, without loss of generality assume that this constant is chosen such that  $\mathbf{E}_{0,\pi}(H^+(1)) = 1$ .

As will be explained in the forthcoming computations, the above Lemma provides the key to picking out the individual components that contribute to the matrices  $\kappa(\lambda)$  and  $\hat{\kappa}(\lambda)$  by decomposing the left-hand side of (4.1) and (4.2) in terms of the behaviour of the associated stable process. Ultimately what we shall see is that all computations boil down to identities that come from the so-called two-sided exit problem for the stable process, which is originally due to [32] (see also Exercise 7.7 of [26]).

To elaborate in a little more detail, let us introduce the stopping times for the stable process: for each  $a \in \mathbb{R}$ ,

$$\tau_a^+ = \inf\{t > 0 : X_t > a\} \text{ and } \tau_a^- = \inf\{t > 0 : X_t < a\}.$$

**Theorem 4.3.** *Suppose that  $X$  is a stable process as described in the introduction. Then, for  $\theta \geq 0$  and  $x \in (0, 1)$ ,*

$$\begin{aligned} \mathbb{P}_x(X_{\tau_1^+} - 1 \in d\theta; \tau_1^+ < \tau_0^-) \\ = \frac{\sin(\pi\alpha\rho)}{\pi} (1-x)^{\alpha\rho} x^{\alpha\hat{\rho}} \theta^{-\alpha\rho} (\theta+1)^{-\alpha\hat{\rho}} (\theta+1-x)^{-1} d\theta. \end{aligned}$$

Equivalently, by scaling and translation, for  $\theta \geq 0$  and  $x \in (-1, 1)$ ,

$$\begin{aligned} \mathbb{P}_x(X_{\tau_1^+} - 1 \in d\theta; \tau_1^+ < \tau_{-1}^-) \\ = \frac{\sin(\pi\alpha\rho)}{\pi} (1-x)^{\alpha\rho} (1+x)^{\alpha\hat{\rho}} \theta^{-\alpha\rho} (\theta+2)^{-\alpha\hat{\rho}} (\theta+1-x)^{-1} d\theta. \end{aligned}$$

Note, we have given two forms of the expression in the theorem above purely for convenience as they will both be used in the computations below.

### 5 The ascending MAP exponent for $\alpha \in (0, 1]$

When  $\alpha \in (0, 1]$ , we have that points are polar for the stable process  $X$ . In particular, as a rssMp, the origin is not accessible. Recalling that MAPs respect the same trichotomy as Lévy processes in terms of drifting and oscillating, we thus have that the Lamperti-stable MAP satisfies  $\limsup_{t \rightarrow \infty} \xi(t) = \infty$ . This means that the conditions of Lemma 4.1 are satisfied.

In order to apply the aforesaid lemma, write  $X^{(x)}$  to indicate the initial value of the stable process, i.e.  $X_0^{(x)} = x \in \mathbb{R} \setminus \{0\}$ . Thanks to self-similarity, and the Lamperti-Kiu representation, we have, for example, that, on the event  $\{J^+(T_a) = 1\}$ , i.e.  $\{X_{\tau_{e^a}^+ \wedge \tau_{-e^a}^-} > e^a\}$ ,

$$\exp\{H^+(T_a) - a\} = \frac{X_{\tau_{e^a}^+ \wedge \tau_{-e^a}^-}^{(x)}}{e^a} = {}^d X_{\tau_1^+ \wedge \tau_{-1}^-}^{(xe^{-a})} \tag{5.1}$$

Hence, taking limits, we have, for  $i, j \in \{1, 2\}$  and  $u > 0$ ,

$$\lim_{a \uparrow \infty} \mathbf{P}_{0,i}(H^+(T_a) - a > u, J^+(T_a) = 1) = \lim_{x \rightarrow 0} \mathbb{P}_x(X_{\tau_1^+} > e^u, \tau_1^+ < \tau_{-1}^-).$$

Moreover, recalling Lemma 4.1, if we write  $\overline{X}_t = \sup_{s \leq t} X_s$  and  $\underline{X}_t = \inf_{s \leq t} X_s$ ,  $t \geq 0$ ,



then we also have for Lebesgue almost every  $u > 0$ ,

$$\begin{aligned} & \pi_1 n_1(\epsilon(\zeta) > u, J(\zeta) = 1, \zeta < \infty) \\ &= -\frac{d}{du} \lim_{x \rightarrow 0} \mathbb{P}_x \left( X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+ -} > |X_{\tau_1^+ -}|, \tau_1^+ < \tau_{-1}^- \right) \\ &= -\frac{d}{du} \int_0^1 \mathbb{P}(X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+ -} \in dz, \tau_1^+ < \tau_{-z}^-). \end{aligned} \tag{5.2}$$

In order to progress our computations further and reach the goal of producing an identity for  $\kappa(\lambda)$ , we shall first establish some intermediary results, starting with the following.

**Lemma 5.1.** For  $\lambda \geq 0$ ,

$$\Phi_1(\lambda) = \frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho})}{\pi} \kappa_{\alpha\rho+1, \alpha\hat{\rho}}(\lambda).$$

*Proof.* We start by noting that, for  $y \in [0, 1]$ ,

$$\begin{aligned} & \mathbb{P}(X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+ -} \leq y, \tau_1^+ < \tau_{-z}^-) \\ &= \mathbb{P}(X_{\tau_y^+} > e^u, \tau_y^+ < \tau_{-z}^-) \\ &= \mathbb{P}_{z/(z+y)} \left( X_{\tau_1^+} - 1 > \frac{e^u - y}{z + y}, \tau_1^+ < \tau_0^- \right) \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_{\frac{e^u - y}{z + y}}^\infty \left( \frac{y}{y + z} \right)^{\alpha\rho} \left( \frac{z}{z + y} \right)^{\alpha\hat{\rho}} t^{-\alpha\rho} (t + 1)^{-\alpha\hat{\rho}} \left( t + 1 - \frac{z}{z + y} \right)^{-1} dt, \end{aligned}$$

where the penultimate probability follows from scaling and the final equality follows from Theorem 4.3. Hence, it follows that

$$\begin{aligned} & \pi_1 n_1(\epsilon(\zeta) > u, J(\zeta) = 1, \zeta < \infty) \\ &= -\int_0^1 \frac{d}{dy} \frac{d}{du} \mathbb{P}(X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+ -} \leq y, \tau_1^+ < \tau_{-z}^-) \Big|_{y=z} dz \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^1 e^u \frac{d}{dy} y^{\alpha\rho} z^{\alpha\hat{\rho}} (e^u - y)^{-\alpha\rho} (e^u + z)^{-\alpha\hat{\rho}} \Big|_{y=z} dz \\ &= \alpha\rho \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^1 e^u z^{\alpha-1} (e^u - z)^{-\alpha\rho-1} (e^u + z)^{-\alpha\hat{\rho}} dz \end{aligned} \tag{5.3}$$

Finally we can now take Laplace transforms and compute

$$\begin{aligned} \pi_1 \Phi_1(\lambda) &= \lambda \int_0^\infty e^{-\lambda u} \pi_1 n_1(\epsilon(\zeta) > u, J(\zeta) = 1, \zeta < \infty) \\ &= \lambda \alpha\rho \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^1 \int_0^\infty e^{-(\lambda+\alpha)u} z^{\alpha-1} (1 + ze^{-u})^{-\alpha\rho} (1 - ze^{-u})^{-(\alpha\rho+1)} du dz \\ &= \alpha\rho \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^\infty (1 - e^{-\lambda w}) \frac{e^{-\alpha w}}{(1 + e^{-w})^{\alpha\hat{\rho}} (1 - e^{-w})^{\alpha\rho+1}} dw \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \kappa_{\alpha\rho+1, \alpha\hat{\rho}}(\lambda), \end{aligned} \tag{5.4}$$

where, in order to get the third equality, we have first substituted  $\theta = ze^{-u}$  in the second equality, then used Fubini's Theorem and finally substituted  $\theta = e^{-w}$ . The result follows once we recall that  $\pi_1 = \sin(\pi\alpha\rho)/(\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}))$ .  $\square$

We are now in a position to identify  $\kappa(\lambda)$  as presented in Theorem 2.6 when  $\alpha \in (0, 1]$ .

*Proof of Theorem 2.6 (i).* In the spirit of Lemma 4.1, we also note that, again with the help of Theorem 4.3, we have, for  $u > 0$ ,

$$\begin{aligned} & \lim_{a \rightarrow \infty} \mathbf{P}_{0,i}(H^+(T_a) - a \leq u; J^+(T_a) = 1) \\ &= \mathbb{P}(X_{\tau_1^+} \leq e^u; \tau_1^+ < \tau_{-1}^-) \\ &= \mathbb{P}_{\frac{1}{2}}(X_{\tau_1^+} - 1 \leq \frac{1}{2}(e^u - 1); \tau_1^+ < \tau_0^-) \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \left(\frac{1}{2}\right)^\alpha \int_0^{\frac{1}{2}(e^u-1)} t^{-\alpha\rho}(1+t)^{-\alpha\hat{\rho}}(t+1/2)^{-1} dt, \end{aligned} \tag{5.5}$$

where we understand  $\mathbf{P}_{0,i}$  be the law of the MAP with matrix exponent  $\mathbf{F}$  issued from  $(0, i)$ . Moreover, it follows that, for  $\lambda \geq 0$ ,

$$\Theta_1(\lambda) := \lim_{a \rightarrow \infty} \mathbf{P}_{0,i}(e^{-\lambda(H^+(T_a)-a)}; J^+(T_a) = 1) = \frac{\sin(\pi\alpha\rho)}{\pi} \int_0^\infty e^{-\lambda u} \frac{e^{-\alpha u}}{(1 - e^{-u})^{\alpha\rho}(1 + e^{-u})^{\alpha\hat{\rho}}} du.$$

Suppose now we define  $f(x) = e^{-\alpha x}(1 - e^{-x})^{-\alpha\rho}(1 + e^{-x})^{-\alpha\hat{\rho}}$ . A straightforward computation shows that

$$f'(x) = -f(x) \left\{ \frac{\alpha\rho}{(1 - e^{-x})} + \frac{\alpha\hat{\rho}}{(1 + e^{-x})} \right\}.$$

This will be useful in the following computation, which also uses the conclusion of Lemmas 4.1 and 5.1 as well as integration by parts:

$$\begin{aligned} \pi_2 \Lambda_{2,1} \mathbf{K}(\lambda)_{2,1} &= \Theta_1(\lambda) - \pi_1 \frac{\Phi_1(\lambda)}{\lambda} \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \left\{ - \int_0^\infty \frac{(1 - e^{-\lambda x})}{\lambda} f'(x) dx - \int_0^\infty \frac{(1 - e^{-\lambda x})}{\lambda} f(x) \frac{\alpha\rho}{(1 - e^{-x})} dx \right\} \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\kappa_{\alpha\rho, \alpha\hat{\rho}+1}(\lambda)}{\lambda}, \end{aligned}$$

for  $\lambda \geq 0$ .

Summarising the above computations, as well as the statement of Lemma 5.1, we have that, up to the multiplicative constant  $(\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}))/\pi$ ,

$$\Phi_1(\lambda) = \kappa_{\alpha\rho+1, \alpha\hat{\rho}}(\lambda) \text{ and } \Lambda_{2,1} \mathbf{K}(\lambda)_{2,1} = \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})} \frac{\kappa_{\alpha\rho, \alpha\hat{\rho}+1}(\lambda)}{\lambda}, \quad \lambda \geq 0,$$

and hence, again up to the same multiplicative constant,

$$\Lambda_{2,1} = \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})} \kappa'_{\alpha\rho, \alpha\hat{\rho}+1}(0+).$$

By exchanging the roles of  $\rho$  and  $\hat{\rho}$ , we similarly get expressions for  $\Phi_2(\lambda)$ ,  $\Lambda_{1,2} \mathbf{K}(\lambda)_{1,2}$  and  $\Lambda_{1,2}$ . Putting the pieces together into (2.3) we have the required form for  $\kappa(\lambda)$ .  $\square$

## 6 The ascending MAP exponent for $\alpha \in (1, 2)$

When  $\alpha \in (1, 2)$  the computations in the previous section break down as we must take account of the fact that the Lamperti-Kiu transform now only describes the stable process up to the first hitting of the origin. The way we will deal with this is to take advantage of a trick that emerges from the Esscher transform for MAPs.

Revisiting Proposition 3.2 and the proof of the Riesz-Bogdan-Żak transform in Theorem 3.1, let us consider the MAP corresponding to  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ . Recall that

its matrix exponent,  $F^\circ(z)$ , was given by (3.1). As well as being expressed via an Esscher transform of  $F(z)$  (see the proof of Theorem 3.1), it also enjoys a Wiener–Hopf factorisation so that

$$\begin{aligned}
 -F^\circ(i\theta) &= \Delta_{\pi^\circ}^{-1} \hat{\kappa}^\circ(i\theta)^T \Delta_{\pi^\circ} \kappa^\circ(-i\theta) \\
 &= -\Delta_{\mathbf{v}}(\alpha - 1)^{-1} F(i\theta + \alpha - 1) \Delta_{\mathbf{v}}(\alpha - 1),
 \end{aligned}
 \tag{6.1}$$

for  $z \in \mathbb{R}$ , where  $\pi^\circ$  is the stationary distribution associated to the underlying Markov chain of  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ ,  $\kappa^\circ$  is the matrix exponent of the ascending ladder MAP of  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$  and  $\hat{\kappa}^\circ$  is that of its dual. It is easily verified that

$$\pi^\circ \propto \begin{bmatrix} \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) \end{bmatrix}$$

and hence, without loss of generality we may assume that  $\pi^\circ = \mathbf{v}(\alpha - 1)$ . Inserting the Wiener–Hopf factorisation into the expression on the right-hand side of (6.1), in a straightforward fashion, one readily deduces that, for  $\lambda \geq 0$

$$\kappa(\lambda) = \Delta_{\pi^\circ} \kappa^\circ(\lambda + \alpha - 1) \Delta_{\pi^\circ}^{-1}.
 \tag{6.2}$$

This can also be verified by directly performing the Esscher transform to the process  $(H^+, J^+)$  through an application of Proposition 3.2 at the stopping time  $L_t^{-1}$ ,  $t > 0$ . The reader should be careful to note that an additive shift of  $\alpha - 1$  in the argument of  $F$ , the exponent of  $(\xi, J)$ , corresponds to an additive shift of  $-(\alpha - 1)$  in the argument of  $\kappa$  as the exponent of  $(H^+, J^+)$  is given by  $-\kappa(-z)$ .

Thanks to (6.2), it therefore follows that, to know  $\kappa(\lambda)$ , it suffices to compute  $\kappa^\circ(\lambda)$ . This is favourable on account of the fact that the origin is polar for  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ , which implies that the MAP corresponding to  $F^\circ(z)$  does not drift to  $-\infty$  and hence the conditions of Lemma 4.1 are met. As we shall soon see, in dealing with  $\kappa^\circ(\lambda)$  via this lemma, we shall make effective use of the Riesz–Bogdan–Żak transform and a result of Kyprianou et al. [25] concerning the first entry of a stable process into  $(-1, 1)$  when issued from an increasingly large distance from the origin. It is worth remarking that the latter, in particular, should be expected. Indeed, if instead of computing  $\kappa$ , we exploited duality of the stable process, allowing us to relate  $\kappa$  to  $\hat{\kappa}$ , and made it our objective to study  $\hat{\kappa}$ , then using Lemma 4.1 for the ascending ladder height MAP of the dual would be tantamount to studying first entry of the stable process into a finite interval. In fact, as we shall later see in Section 7, we shall reverse this logic and use the identity that we develop here for  $\kappa$  to compute  $\hat{\kappa}$ .

Using obvious notation, let us write

$$\kappa^\circ(\lambda) = \text{diag}(\Phi_1^\circ(\lambda), \Phi_2^\circ(\lambda)) - \Lambda^\circ \circ \mathbf{K}^\circ(\lambda), \quad \lambda \geq 0.$$

The analogue of Lemma 5.1, but now for  $\kappa^\circ$ , reads as follows.

**Lemma 6.1.** *We have*

$$\Phi_1^\circ(\lambda) = \frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha\hat{\rho})} c(\alpha) \phi_{\alpha\rho+1, \alpha\hat{\rho}}(\lambda), \quad \lambda \geq 0.$$

where

$$c(\alpha) 2^{\alpha-1} \frac{\Gamma(2 - \alpha)}{\Gamma(1 - \alpha\hat{\rho})\Gamma(1 - \alpha\rho)}.$$

*Proof.* Referring to (4.1) and the discussion leading to (5.2), we have, for  $u > 0$ ,

$$\begin{aligned} & \pi_1^\circ n_1^\circ(\epsilon(\zeta) > u, J(\zeta) = 1, \zeta < \infty) \\ &= -\frac{d}{du} \lim_{x \rightarrow 0} \int_0^1 \mathbb{P}_x^\circ \left( X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+} \in dz, |\underline{X}_{\tau_1^+}| < z, \tau_1^+ < \tau_{-1}^- \right) \\ &= -\frac{d}{du} \int_0^1 \lim_{x \rightarrow 0} \mathbb{P}_x^\circ \left( X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+} \in dz, \tau_1^+ < \tau_{-z}^- \right) \\ &= -\frac{d}{du} \int_0^1 \lim_{x \rightarrow 0} \frac{d}{dy} \mathbb{P}_x^\circ \left( X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+} \leq y, \tau_1^+ < \tau_{-z}^- \right) \Big|_{y=z} dz. \end{aligned} \tag{6.3}$$

Moreover, defining, for  $a < b$ ,

$$\tau^{(a,b)} = \inf\{t > 0 : X_t \in (a, b)\},$$

and writing  $\hat{\mathbb{P}}_x$ ,  $x \in \mathbb{R}$ , for the probabilities of  $-X$  (the negative of a stable process), we have, for  $0 < x < y < 1$  and  $u > 0$ ,

$$\begin{aligned} & -\frac{d}{du} \lim_{x \rightarrow 0} \mathbb{P}_x^\circ \left( X_{\tau_1^+} > e^u, \bar{X}_{\tau_1^+} \leq y, \tau_1^+ < \tau_{-z}^- \right) \\ &= -\frac{d}{du} \lim_{x \rightarrow 0} \mathbb{P}_x^\circ \left( X_{\tau_y^+} > e^u, \tau_y^+ < \tau_{-z}^- \right) \\ &= -\frac{d}{du} \lim_{x \rightarrow 0} \mathbb{P}_{-1/x} \left( X_{\tau^{(-1/y, 1/z)}} \in (-e^{-u}, 0) \right) \\ &= -\frac{d}{du} \lim_{x \rightarrow 0} \hat{\mathbb{P}}_{1/x} \left( X_{\tau^{(-1/z, 1/y)}} \in (0, e^{-u}) \right) \\ &= \hat{p}_{\pm\infty} \left( \frac{2yz e^{-u} - z + y}{y + z} \right) \frac{2yz}{y + z} e^{-u}, \end{aligned} \tag{6.4}$$

where, momentarily, we will assume the limit,

$$\hat{p}_{\pm\infty}(\theta) := \lim_{|z| \rightarrow \infty} \hat{\mathbb{P}}_z(X_{\tau^{(-1,1)}} \in d\theta) / d\theta, \quad \theta \in [-1, 1],$$

exists. In order to continue the process of identifying an explicit expression for (6.4), we appeal to a distributional identity found in [25] which gives the law of the position of a stable process when it first enters a finite interval. We can, in principle, take limits in the expression for  $\hat{\mathbb{P}}_z(X_{\tau^{(-1,1)}} \in dy)$ , in particular, showing that  $\hat{p}_\pm$  is well defined. It turns out to be more convenient to root deeper into the proof of Theorem 1.1. of [25] and fish out an alternative expression. From equation (20) in [25] we have, for  $y \in (-1, 1)$  and  $\alpha \in (1, 2)$ ,

$$\begin{aligned} \hat{p}_{+\infty}(y) &:= \lim_{z \rightarrow \infty} \hat{\mathbb{P}}_z(X_{\tau^{(-1,1)}} \in dy) / dy \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} (1+y)^{-\alpha\hat{\rho}} (1-y)^{-\alpha\rho} \\ &\quad \times \lim_{z \rightarrow \infty} \left[ (y+1)(z-1)^{\alpha\rho} (z+1)^{\alpha\hat{\rho}-1} (z-y)^{-1} \right. \\ &\quad \left. + (1-\alpha\hat{\rho}) 2^{\alpha-1} \int_0^{\frac{z-1}{z+1}} t^{\alpha\rho-1} (1-t)^{1-\alpha} dt \right] \\ &= (1-\alpha\hat{\rho}) 2^{\alpha-1} \frac{\sin(\pi\alpha\rho)}{\pi} (1+y)^{-\alpha\hat{\rho}} (1-y)^{-\alpha\rho} \int_0^1 t^{\alpha\rho-1} (1-t)^{1-\alpha} dt \\ &= 2^{\alpha-1} (1-\alpha\hat{\rho}) \frac{\Gamma(\alpha\rho)\Gamma(2-\alpha)}{\Gamma(2-\alpha\hat{\rho})\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} (1+y)^{-\alpha\hat{\rho}} (1-y)^{-\alpha\rho} \\ &= c(\alpha) (1+y)^{-\alpha\hat{\rho}} (1-y)^{-\alpha\rho}. \end{aligned} \tag{6.5}$$

By appealing to duality, one also easily verifies in a similar fashion that

$$\hat{p}_{-\infty}(y) := \lim_{z \rightarrow -\infty} \hat{\mathbb{P}}_z(X_{\tau(-1,1)} \in dy) / dy = p_{\infty}(-y) = \hat{p}_{+\infty}(y),$$

where the function  $p_{\infty}$  is the same as  $\hat{p}_{\infty}$  albeit the roles of  $\rho$  and  $\hat{\rho}$  are interchanged. Hence we may accordingly refer to the function

$$\hat{p}_{\pm\infty}(y) := \hat{p}_{+\infty}(y) = \hat{p}_{-\infty}(y) = c(\alpha)(1+y)^{-\alpha\hat{\rho}}(1-y)^{-\alpha\rho}. \tag{6.6}$$

Returning to (6.3) and (6.4), we therefore have

$$\pi_1^{\circ} n_1^{\circ}(\epsilon(\zeta) > u, J(\zeta) = 1, \zeta < \infty) = c(\alpha) \int_0^1 \hat{\beta}(ze^{-u})e^{-u} dz,$$

where  $\hat{\beta}(\theta) = \{\hat{p}'_{\pm\infty}(\theta)(\theta + 1) + \hat{p}_{\pm\infty}(\theta)\} / 2c(\alpha)$ , which can easily be verified to satisfy

$$\hat{\beta}(\theta) = \frac{\alpha\rho}{(1-\theta)^{\alpha\rho+1}(1+\theta)^{\alpha\hat{\rho}}} - \frac{(\alpha-1)/2}{(1-\theta)^{\alpha\rho}(1+\theta)^{\alpha\hat{\rho}}},$$

for  $\theta \in [-1, 1]$ . We can now compute

$$\begin{aligned} \pi_1^{\circ} \Phi_1^{\circ}(\lambda) &= c(\alpha)\lambda \int_0^{\infty} \int_0^1 e^{-(\lambda+1)u} \hat{\beta}(ze^{-u}) dz du \\ &= c(\alpha) \int_0^{\infty} (1 - e^{-\lambda w}) \hat{\beta}(e^{-w}) e^{-w} dw, \quad \lambda \geq 0, \end{aligned}$$

as required. Note that the second equality condenses two changes of variable and an application of Fubini's theorem, similar in spirit to earlier computations, into one step. The details are straightforward and left to the reader.  $\square$

*Proof of Theorem 2.7 (i).* Let  $\mathbf{P}_{0,i}^{\circ}$  be the law of the MAP whose matrix exponent is  $\mathbf{F}^{\circ}$ , issued from  $(0, i)$ . Following previous reasoning, we compute,

$$\begin{aligned} \lim_{a \rightarrow \infty} \mathbf{P}_{0,i}^{\circ}(H^+(T_a) - a \leq u; J^+(T_a) = 1) &= \lim_{x \rightarrow 0} \mathbb{P}_x^{\circ}(X_{\tau_1^+} \leq e^u; \tau_1^+ < \tau_{-1}^-) \\ &= \lim_{x \rightarrow 0} \hat{\mathbb{P}}_{1/x}(X_{\tau(-1,1)} \in (e^{-u}, 1)) \\ &= \int_{e^{-u}}^1 \hat{p}_{\pm\infty}(y) dy. \end{aligned} \tag{6.7}$$

Hence, for  $\lambda \geq 0$ ,

$$\Theta_1^{\circ}(\lambda) := \lim_{a \rightarrow \infty} \mathbf{P}_{0,i}^{\circ}(e^{-\lambda(H^+(T_a)-a)}; J^+(T_a) = 1) = \int_0^{\infty} e^{-(\lambda+1)u} \hat{p}_{\pm\infty}(e^{-u}) du,$$

and so, again appealing to obvious notation, from Lemma 4.1,

$$\begin{aligned} \pi_2^{\circ} \Lambda_{2,1}^{\circ} \mathbf{K}^{\circ}(\lambda)_{2,1} &= \Theta_1^{\circ}(\lambda) - \pi_1^{\circ} \frac{\Phi_1^{\circ}(\lambda)}{\lambda} \\ &= \int_0^{\infty} \frac{(1 - e^{-\lambda u})}{\lambda} \{\hat{p}'_{\pm\infty}(e^{-u})e^{-u} + \hat{p}_{\pm\infty}(e^{-u})\} e^{-u} du \\ &\quad - c(\alpha) \int_0^{\infty} \frac{(1 - e^{-\lambda u})}{\lambda} \hat{\beta}(e^{-u}) e^{-u} du \\ &= \int_0^{\infty} \frac{(1 - e^{-\lambda u})}{\lambda} \frac{1}{2} \{\hat{p}'_{\pm\infty}(e^{-u})(e^{-u} - 1) + \hat{p}_{\pm\infty}(e^{-u})\} e^{-u} du \\ &= c(\alpha) \frac{\phi_{\alpha\rho, \alpha\hat{\rho}+1}(\lambda)}{\lambda}, \quad \lambda \geq 0. \end{aligned}$$

This tells us that up to the multiplicative constant  $c(\alpha)(\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}))/\sin(\pi\alpha\rho)\sin(\pi\alpha\hat{\rho})$ , for  $\lambda \geq 0$ ,

$$\Lambda_{2,1}^\circ \mathbf{K}^\circ(\lambda)_{2,1} = \sin(\pi\alpha\hat{\rho}) \frac{\phi_{\alpha\rho, \alpha\hat{\rho}+1}(\lambda)}{\lambda} \quad \text{and} \quad \Lambda_{2,1}^\circ = \sin(\pi\alpha\hat{\rho}) \phi'_{\alpha\rho, \alpha\hat{\rho}+1}(0+).$$

By exchanging the roles of  $\rho$  and  $\hat{\rho}$ , we now have enough identities to complete fill out the entries of  $\kappa(\lambda)$ . □

### 7 The ascending ladder MAP for the dual process

Using (2.4) and (3.5) a straightforward computation gives us

$$\hat{\mathbf{F}}(z) = \begin{bmatrix} -\frac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\hat{\rho}+z)\Gamma(1-\alpha\hat{\rho}-z)} & \frac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \frac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\frac{\Gamma(\alpha+z)\Gamma(1-z)}{\Gamma(\alpha\rho+z)\Gamma(1-\alpha\rho-z)} \end{bmatrix},$$

which is well defined for  $\text{Re}(z) \in (-\alpha, 1)$ . Glancing back, one quickly realises that  $\hat{\mathbf{F}}(z)$  takes the form of  $\mathbf{F}_{\alpha-1}(z)$  (from the Proof of Theorem 3.1), but with the roles of  $\rho$  and  $\hat{\rho}$  interchanged; otherwise written

$$\hat{\mathbf{F}}(z) = \mathbf{F}_{\alpha-1}(z)|_{\rho \leftrightarrow \hat{\rho}} = \mathbf{F}^\circ(z)|_{\rho \leftrightarrow \hat{\rho}} \tag{7.1}$$

for  $\text{Re}(z) \in (-\alpha, 1)$ . This tells us that the dual of  $(\xi, J)$  is the MAP which underlies the  $\text{rsmMp}(X, \hat{\mathbb{P}}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$  through the Lamperti-Kiu transform.

We may now proceed to calculate  $\hat{\kappa}(\lambda)$  by again taking advantage of the Esscher transform to remove the effects of killing. Indeed, referring to (7.1) and noting that, without loss of generality, we may take

$$\mathbf{v}(\alpha-1)|_{\rho \leftrightarrow \hat{\rho}} = \boldsymbol{\pi},$$

it is immediately clear that

$$\hat{\kappa}(\lambda) = \boldsymbol{\Delta}_\pi^{-1} \boldsymbol{\kappa}(\lambda + 1 - \alpha)|_{\rho \leftrightarrow \hat{\rho}} \boldsymbol{\Delta}_\pi, \quad \lambda \geq 0.$$

Referring to (6.2), this means, in particular that

$$\hat{\kappa}(\lambda) = \boldsymbol{\kappa}^\circ(\lambda)|_{\rho \leftrightarrow \hat{\rho}}, \quad \lambda \geq 0. \tag{7.2}$$

That is to say  $\hat{\kappa}(\lambda)$  is nothing more than  $\boldsymbol{\kappa}^\circ(\lambda)$  albeit with the roles of  $\rho$  and  $\hat{\rho}$  interchanged.

**Remark 7.1.** Whist (7.2) is helpful for computing  $\hat{\kappa}$  in the case that  $\alpha \in (1, 2)$ , it also tells us something about  $\boldsymbol{\kappa}^\circ$  in the case that  $\alpha \in (0, 1)$ , where  $\boldsymbol{\kappa}^\circ$  has not already been evaluated. For the latter regime of  $\alpha$ , appealing to Theorem 2.6 (ii), we note that  $\hat{\kappa}(0)_{1,1} > -\hat{\kappa}(0)_{1,2}$  and, similarly,  $\hat{\kappa}(0)_{2,2} > -\hat{\kappa}(0)_{2,1}$ . This implies that the ascending ladder MAP associated to  $(X, \hat{\mathbb{P}}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ , and hence  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ , is subject to killing. In turn, this means that, when  $\alpha \in (0, 1)$ , the MAP associated to  $\mathbf{F}^\circ(z)$  drifts to  $-\infty$  and, accordingly,  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ , is a real-valued self-similar Markov process which experiences absorption at the origin.

In this sense, when  $\alpha \in (0, 1)$ ,  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$  may be reasonably named the stable process conditioned to be continuously absorbed at the origin. Indeed, in further work following ideas of [16], we hope to give more mathematical substance to this remark.

### 8 Cramér-type results for Lamperti-stable MAPs

A classical computation that emerges from the Wiener–Hopf factorisation in the setting of a Lévy process that drifts to  $-\infty$ , which also has a non-zero root of its characteristic exponent, is Cramér’s asymptotic estimate for the probability of first passage above a threshold, as well as the asymptotic conditional overshoot distribution conditional of first passage. See for example [8]. In the current setting we have noted that when  $\alpha \in (1, 2)$  and when  $\alpha \in (0, 1)$  the ascending ladder processes of the Lamperti-stable MAP and the ascending ladder processes of the dual Lamperti-stable MAP, respectively, undergo killing. Moreover, we have also noted the existence of roots to the leading eigenvalue of the associated matrix exponent. This means that we can expect to see Cramér-type results in each of these regimes. In this respect, we have two main theorems in this section.

**Theorem 8.1.** *When  $\alpha \in (0, 1]$ ,  $\mathbf{P}_{0,i}(T_a < \infty) = 1$ , for  $i = 1, 2$ , where as, when  $\alpha \in (1, 2)$ ,*

$$\lim_{a \rightarrow \infty} e^{(\alpha-1)a} \mathbf{P}_{0,i}(T_a < \infty) = 2^{\alpha-1} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)} \left\{ \frac{\pi}{\sin(\pi\alpha\rho)} \mathbf{1}_{(i=1)} + \frac{\pi}{\sin(\pi\alpha\hat{\rho})} \mathbf{1}_{(i=2)} \right\}.$$

Moreover, when  $\alpha \in (0, 2)$ , for  $i = 1, 2$  and  $u > 0$ ,

$$\begin{aligned} & \lim_{a \rightarrow \infty} \mathbf{P}_{0,i}(H^+(T_a) - a \in du; J^+(T_a) = j | T_a < \infty) \\ &= \begin{cases} \frac{\sin(\pi\alpha\rho)}{\pi} e^{-\alpha u} (1 + e^{-u})^{-\alpha\hat{\rho}} (1 - e^{-u})^{-\alpha\rho} du & \text{if } j = 1, \\ \frac{\sin(\pi\alpha\hat{\rho})}{\pi} e^{-\alpha u} (1 + e^{-u})^{-\alpha\rho} (1 - e^{-u})^{-\alpha\hat{\rho}} du & \text{if } j = 2. \end{cases} \end{aligned}$$

*Proof.* The first claim follows by virtue of the fact that, as noted in Section 5, the ascending ladder MAP  $(H^+, J^+)$  experiences no killing when  $\alpha \in (0, 1]$ . Recall that

$$\tau^{(-1,1)} := \inf\{t > 0 : X_t \in (-1, 1)\}.$$

Thanks to (5.1) and Theorem 3.1,

$$\begin{aligned} & \mathbf{P}_{0,1}(T_a < \infty) \\ &= \mathbf{P}_{e^{-a}}(\tau_1^+ \wedge \tau_{-1}^- < \tau^{\{0\}}) \\ &= \mathbf{P}_{-e^a}^\circ(\tau^{(-1,1)} < \infty) \\ &= \hat{\mathbf{P}}_{e^a}^\circ(\tau^{(-1,1)} < \infty) \tag{8.1} \\ &= \frac{e^{-(\alpha-1)a}}{2 \sin(\pi\alpha\rho)} \hat{\mathbf{E}}_{e^a} \left( \left( 2 \sin(\pi\alpha\rho) \mathbf{1}_{(X_{\tau^{(-1,1)}}) > 0} + 2 \sin(\pi\alpha\hat{\rho}) \mathbf{1}_{(X_{\tau^{(-1,1)}}) < 0} \right) |X_{\tau^{(-1,1)}}|^{\alpha-1} \right), \end{aligned}$$

where, for each  $w \in \mathbb{R} \setminus \{0\}$ ,  $\hat{\mathbf{P}}_w^\circ$  plays the role of  $\mathbf{P}_w^\circ$  with  $\rho$  and  $\hat{\rho}$  interchanged (i.e. it plays the role of  $\mathbf{P}_w^\circ$  for  $-X$ ). Recalling the definition of the limiting distribution  $\hat{p}_{\pm\infty}$

given in (6.6), we thus have

$$\begin{aligned}
 & \lim_{a \rightarrow \infty} \sin(\pi\alpha\rho)e^{(\alpha-1)a} \mathbf{P}_{0,1}(T_a < \infty) \\
 &= c(\alpha) \sin(\pi\alpha\rho) \int_0^1 y^{\alpha-1}(1+y)^{-\alpha\hat{\rho}}(1-y)^{-\alpha\rho} dy \\
 &\quad + c(\alpha) \sin(\pi\alpha\hat{\rho}) \int_0^1 y^{\alpha-1}(1-y)^{-\alpha\hat{\rho}}(1+y)^{-\alpha\rho} dy \\
 & c(\alpha)\pi \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha)\Gamma(1-\alpha\rho)}{\Gamma(1+\alpha\hat{\rho})} {}_2\mathcal{F}_1(\alpha\hat{\rho}, \alpha, \alpha\hat{\rho}+1; -1) \\
 &\quad + c(\alpha)\pi \frac{\sin(\pi\alpha\hat{\rho})}{\pi} \frac{\Gamma(\alpha)\Gamma(1-\alpha\hat{\rho})}{\Gamma(1+\alpha\rho)} {}_2\mathcal{F}_1(\alpha\rho, \alpha, \alpha\rho+1; -1) \\
 &= c(\alpha)\pi, \tag{8.2}
 \end{aligned}$$

where  ${}_2\mathcal{F}_1(a, b, c; z)$  is the usual hypergeometric function and the final equality is a remarkable simplification which follows from one of the many identities for the aforesaid functions. See for example the first formula at the `functions.wolfram.com` webpages [1]. If, on the left-hand side of (8.2), we replace  $\mathbf{P}_{0,1}$  by  $\mathbf{P}_{0,2}$ , the only thing that changes in the statement is that we must replace  $\sin(\pi\alpha\rho)$  by  $\sin(\pi\alpha\hat{\rho})$  on the left-hand side. This completes the proof of the first part of the theorem.

For the next part, we split the proof into the cases that  $\alpha \in (0, 1]$  and  $\alpha \in (1, 2)$ . In the former case the result was already established in (5.5). For the latter case, appealing again to the Riesz-Bogdan-Żak transform, (8.1), (8.2) and (6.5), we have, for  $u > 0$ ,

$$\begin{aligned}
 & \lim_{a \rightarrow \infty} \mathbf{P}_{0,1}(H^+(T_a) - a \leq u; J^+(T_a) = 1 | T_a < \infty) \\
 &= \lim_{a \rightarrow \infty} \frac{\mathbb{P}_{e^{-a}}(X_{\tau_1^+} \leq e^u; \tau_1^+ < \tau_{-1}^- \wedge \tau^{\{0\}})}{\mathbb{P}_{e^{-a}}(\tau_1^+ \wedge \tau_{-1}^- < \tau^{\{0\}})} \\
 &= \lim_{a \rightarrow \infty} \frac{\hat{\mathbb{P}}_{e^a}^\circ(X_{\tau^{(-1,1)}} \in (e^{-u}, 1); \tau^{(-1,1)} < \infty)}{\hat{\mathbb{P}}_{e^a}^\circ(\tau^{(-1,1)} < \infty)} \\
 &= \frac{\sin(\pi\alpha\rho) \int_{e^u}^1 \theta^{\alpha-1} \hat{\rho}_{\pm\infty}(\theta) d\theta}{c(\alpha)\pi} \\
 &= \frac{\sin(\pi\alpha\rho)}{\pi} \int_{e^{-u}}^1 \theta^{\alpha-1} (1+\theta)^{-\alpha\hat{\rho}} (1-\theta)^{-\alpha\rho} d\theta, \quad u \geq 0, \tag{8.3}
 \end{aligned}$$

which is equivalent to the statement in the second part of the theorem when  $i = 1$ . It turns out that the asymptotic is unaffected when  $i = 2$ , however the details are left to the reader to verify. We also leave it as an exercise for the reader to check that when the event  $\{J^+(T_a) = 1\}$  is replaced by  $\{J^+(T_a) = 2\}$  on the left-hand side of (8.3), the resulting asymptotic is the same but with the roles of  $\rho$  and  $\hat{\rho}$  interchanged.  $\square$

It is worth noting from the proof of this theorem that the methodology allows us access to new identities for stable processes with  $\alpha \in (1, 2)$ . For example, the following polynomial asymptotic decay for the probability that the stable processes escapes  $(-1, 1)$  on before hitting the origin.

**Corollary 1.** For  $\alpha \in (1, 2)$ ,

$$\lim_{x \rightarrow 0} s(x)x^{1-\alpha} \mathbb{P}_x(\tau_1^+ \wedge \tau_{-1}^- < \tau^{\{0\}}) = 2^{\alpha-1} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)},$$

where

$$s(x) := \frac{\sin(\pi\alpha\rho)}{\pi} \mathbf{1}_{(x>0)} + \frac{\sin(\pi\alpha\hat{\rho})}{\pi} \mathbf{1}_{(x<0)}.$$



Another example of a new fluctuation result for stable processes is captured in the corollary immediately below.

**Corollary 2.** For  $\alpha \in (1, 2)$ ,  $\theta > 0$  and  $x \in (0, 1)$ ,

$$\begin{aligned} \mathbb{P}_x(X_{\tau_1^+} - 1 \in d\theta, \tau_1^+ < \tau_{-1}^- \wedge \tau^{\{0\}}) &= \frac{\sin(\pi\alpha\rho)}{\pi} (1+x)^{\alpha\hat{\rho}}(1-x)^{\alpha\rho}(2+\theta)^{-\alpha\hat{\rho}}\theta^{-\alpha\rho}(1+\theta-x)^{-1} \\ &\quad - (\alpha-1) \frac{\sin(\pi\alpha\rho)}{\pi} (2+\theta)^{-\alpha\hat{\rho}}\theta^{-\alpha\rho}(1+\theta)^{-1}x^{\alpha-1} \int_1^{1/x} (t-1)^{\alpha\rho-1}(t+1)^{\alpha\hat{\rho}-1} dt. \end{aligned}$$

*Proof.* According to Theorem 3.1, we thus have

$$\begin{aligned} \mathbb{P}_x(X_{\tau_1^+} - 1 > \theta, \tau_1^+ < \tau_{-1}^- \wedge \tau^{\{0\}}) &= \mathbb{P}_x(-1/X_{\tau_1^+} \in (-(1+\theta)^{-1}, 0), \tau_1^+ < \tau_{-1}^- \wedge \tau^{\{0\}}) \\ &= \mathbb{P}_{-1/x}^\circ(X_{\tau^{(-1,1)}} \in (-(1+\theta)^{-1}, 0), \tau^{(-1,1)} < \infty) \\ &= x^{\alpha-1} \hat{\mathbb{E}}_{1/x}(|X_{\tau^{(-1,1)}}|^{\alpha-1}; X_{\tau^{(-1,1)}} \in (0, (1+\theta)^{-1})), \end{aligned}$$

for  $\theta > 0$ . It follows that

$$\mathbb{P}_x(X_{\tau_1^+} - 1 \in d\theta, \tau_1^+ < \tau_{-1}^- \wedge \tau^{\{0\}}) = x^{\alpha-1} \left(\frac{1}{1+\theta}\right)^{\alpha+1} \hat{p}_{1/x} \left(\frac{1}{1+\theta}\right) d\theta,$$

where, for  $x > 1$  and  $y \in [-1, 1]$ ,  $\hat{p}_x(y) = \hat{\mathbb{P}}_x(X_{\tau^{(-1,1)}} \in dy)/dy$ . The latter can be found in Theorem 1.1 of [25] and is given by

$$\begin{aligned} \hat{\mathbb{P}}_x(X_{\tau_{-1}^-} \in dy)/dy &= \frac{\sin(\pi\alpha\rho)}{\pi} (x+1)^{\alpha\hat{\rho}}(x-1)^{\alpha\rho}(1+y)^{-\alpha\hat{\rho}}(1-y)^{-\alpha\rho}(x-y)^{-1} \\ &\quad - (\alpha-1) \frac{\sin(\pi\alpha\rho)}{\pi} (1+y)^{-\alpha\hat{\rho}}(1-y)^{-\alpha\rho} \int_1^x (t-1)^{\alpha\rho-1}(t+1)^{\alpha\hat{\rho}-1} dt. \end{aligned}$$

The result now follows by straightforward algebra. □

Let us now turn our attention to the dual of the Lamperti-stable MAP when  $\alpha \in (0, 1)$ . We denote its law by  $\hat{\mathbb{P}}_{x,i}$ , for  $x \in \mathbb{R}$  and  $i = 1, 2$ . Recall from Section 7 that this MAP corresponds to the rssMp  $(X, \hat{\mathbb{P}}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$ . The analogue of Theorem 8.1 takes the following form.

**Theorem 8.2.** If  $\alpha \in (0, 1)$ , then, for  $i = 1, 2$ ,

$$\lim_{a \rightarrow \infty} e^{(1-\alpha)a} \hat{\mathbb{P}}_{0,i}(T_a < \infty) = \frac{2^{1-\alpha}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(2-\alpha)} \left\{ \frac{\pi}{\sin(\pi\alpha\rho)} \mathbf{1}_{(i=1)} + \frac{\pi}{\sin(\pi\alpha\hat{\rho})} \mathbf{1}_{(i=2)} \right\}.$$

If  $\alpha \in [1, 2)$ , then  $\hat{\mathbb{P}}_{0,i}(T_a < \infty) = 1$ , for  $i = 1, 2$ . Moreover, for  $\alpha \in (0, 2)$ ,  $i = 1, 2$  and  $u > 0$ ,

$$\begin{aligned} \lim_{a \rightarrow \infty} \hat{\mathbb{P}}_{0,i}(H^+(T_a) - a \in du; J^+(T_a) = j | T_a < \infty) &= \begin{cases} 2^{\alpha-1} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)} e^{-u}(1+e^{-u})^{-\alpha\rho}(1-e^{-u})^{-\alpha\hat{\rho}} du & \text{if } j = 1 \\ 2^{\alpha-1} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)} e^{-u}(1+e^{-u})^{-\alpha\hat{\rho}}(1-e^{-u})^{-\alpha\rho} du & \text{if } j = 2. \end{cases} \end{aligned}$$

*Proof.* Appealing to (5.1), we start by noting that

$$\hat{\mathbf{P}}_{0,1}(T_a < \infty) = \hat{\mathbf{P}}_{e^{-a}}(\tau_1^+ \wedge \tau_{-1}^- < \infty),$$

and hence, by again making use of Theorem 4.3 and Theorem 3.1, we also have that, when  $\alpha \in (0, 1)$ ,

$$\begin{aligned} & \lim_{a \rightarrow \infty} \sin(\pi\alpha\rho)e^{(1-\alpha)a}\hat{\mathbf{P}}_{0,1}(T_a < \infty) \\ &= \lim_{a \rightarrow \infty} \hat{\mathbb{E}}_{e^{-a}} \left( \left( \sin(\pi\alpha\rho)\mathbf{1}_{(X_{\tau_1^+ \wedge \tau_{-1}^-} > 0)} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{(X_{\tau_1^+ \wedge \tau_{-1}^-} < 0)} \right) |X_{\tau_1^+ \wedge \tau_{-1}^-}|^{\alpha-1} \right) \\ &= \frac{\sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)}{\pi} \int_0^\infty e^{(\alpha-1)u}(e^u - 1)^{-\alpha\hat{\rho}}(e^u + 1)^{-\alpha\rho} \mathbf{d}u \\ &\quad + \frac{\sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)}{\pi} \int_0^\infty e^{(\alpha-1)u}(e^u - 1)^{-\alpha\rho}(e^u + 1)^{-\alpha\hat{\rho}} \mathbf{d}u \\ &= \frac{\sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)}{\pi(1-\alpha\hat{\rho})(1-\alpha\rho)} (1-\alpha\rho) {}_2F_1(1, \alpha\rho, 2-\alpha\hat{\rho}; -1) \\ &\quad + \frac{\sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)}{\pi(1-\alpha\hat{\rho})(1-\alpha\rho)} (1-\alpha\hat{\rho}) {}_2F_1(1, \alpha\hat{\rho}, 2-\alpha\rho; -1) \\ &= \frac{\sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)}{\pi(1-\alpha\hat{\rho})(1-\alpha\rho)} \times 2^{1-\alpha} \frac{\Gamma(2-\alpha\rho)\Gamma(2-\alpha\hat{\rho})}{\Gamma(2-\alpha)} \\ &= \frac{2^{1-\alpha}\pi}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(2-\alpha)}. \end{aligned} \tag{8.4}$$

where the penultimate equality is again remarkably due to a very particular identity for hypergeometric functions; see the second formula in the `functions.wolfram.com` webpage [1]. If we repeat the computation with  $\hat{\mathbf{P}}_{0,1}$  replaced by  $\hat{\mathbf{P}}_{0,2}$ , then the only other thing that changes in (8.4) is that  $\sin(\pi\alpha\rho)$  is replaced by  $\sin(\pi\alpha\hat{\rho})$  on the left-hand side. This completes the proof of the first part of the theorem.

When  $\alpha \in [1, 2)$ , the ascending ladder height MAP of the dual is not killed (see the discussion in Section 7) and hence  $\hat{\mathbf{P}}_{0,i}(T_a < \infty) = 1$ , for  $i = 1, 2$ .

For the next part set  $\alpha \in (0, 1]$ . Starting as we did in the proof of Theorem 2.6 (i), we note from Lemma 4.1, (8.4) and Theorem 4.3 that

$$\begin{aligned} & \lim_{a \rightarrow \infty} \hat{\mathbf{P}}_{0,1}(H^+(T_a) - a \leq u; J^+(T_a) = 1 | T_a < \infty) \\ &= \lim_{a \rightarrow \infty} \hat{\mathbf{P}}_{e^{-a}}(X_{\tau_1^+} \leq e^u; \tau_1^+ < \tau_{-1}^- | \tau_1^+ \wedge \tau_{-1}^- < \infty) \\ &= \lim_{a \rightarrow \infty} \frac{\hat{\mathbf{P}}_{e^{-a}}(X_{\tau_1^+} - 1 \leq e^u - 1, \tau_1^+ < \tau_{-1}^-)}{\hat{\mathbf{P}}_{e^{-a}}(\tau_1^+ \wedge \tau_{-1}^- < \infty)} \\ &= \frac{\Gamma(\alpha\hat{\rho})\Gamma(2-\alpha)}{\Gamma(1-\alpha\rho)} \hat{\mathbb{E}}(X_{\tau_1^+}^{\alpha-1}; X_{\tau_1^+} - 1 \leq e^u - 1, \tau_1^+ < \tau_{-1}^-) \\ &= 2^{\alpha-1} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)} \int_0^{e^u-1} (\theta+1)^{\alpha-2} \theta^{-\alpha\hat{\rho}} (\theta+2)^{-\alpha\rho} \mathbf{d}\theta. \end{aligned}$$

This is equivalent to the second statement of the theorem for  $i = 1$ . The computation when  $i = 2$  can be performed similarly. Replacing the event  $\{J^+(T_a) = 1\}$  by  $\{J^+(T_a) = 2\}$  in the probability above, affects the final equality only exchanging the roles of  $\rho$  and  $\hat{\rho}$ . The details are left to the reader.

Finally, when  $\alpha \in (1, 2)$ , the desired asymptotic can already be found in (6.7), as soon as one notes that  $\hat{\mathbf{P}}_{0,i}$  agrees with  $\mathbf{P}_{0,i}^\circ$  when the roles of  $\rho$  and  $\hat{\rho}$  are exchanged.  $\square$

Similarly to before, one can proceed to extract further identities for the Doob  $h$ -transformed process  $(X, P_x^o)$ ,  $x \in \mathbb{R} \setminus \{0\}$ , however, we leave this for the reader to amuse themselves with.

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