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DEFECT ZERO p–BLOCKS FOR FINITE SIMPLE GROUPS

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Abstract. We classify those finite simple groups whose Brauer graph (or decomposition matrix) has a p-block with defect 0, completing an investigation of many authors. The only finite simple groups whose defect zero $p-$blocks remained unclassified were the alternating groups A_n. Here we show that these all have a p-block with defect 0 for every prime $p \geq 5$. This follows from proving the same result for every symmetric group S_n, which in turn follows as a consequence of the *t-core partition conjecture*, that every non-negative integer possesses at least one t-core partition, for any $t \geq 4$. For $t \geq 17$, we reduce this problem to Lagrange's Theorem that every non-negative integer can be written as the sum of four squares. The only case with $t < 17$, that was not covered in previous work, was the case $t = 13$. This we prove with a very different argument, by interpreting the generating function for t-core partitions in terms of modular forms, and then controlling the size of the coefficients using Deligne's Theorem (née the *Weil Conjectures*).

We also consider congruences for the number of p-blocks of S_n, proving a conjecture of Garvan, that establishes certain multiplicative congruences when $5 \leq p \leq 23$. By using a result of Serre concerning the divisibility of coefficients of modular forms, we show that for any given prime p and positive integer m, the number of $p-$blocks with defect 0 in S_n is a multiple of m for almost all n. We also establish that any given prime p divides the number of $p-$modularly irreducible representations of S_n, for almost all n.

1. Introduction

An *ordinary* representation of a group G of degree n is a group homomorphism from G to $Gl_n(\mathbb{C})$, the group of invertible $n \times n$ matrices with complex coefficients. Such a representation may be viewed as a homomorphism from G to the group of isomorphisms of an $n-$dimensional complex vector space V to itself. An *irreducible* representation is an ordinary representation which does not have a non-trivial stable subspace; and, in a finite group G, the equivalence classes of such representations are in 1-1 correspondance with the conjugacy classes of G. In the symmetric group S_n, a conjugacy class is the set of permutations with a given cycle structure, and so they are in a natural 1-1 correspondance with the set of partitions of n (a *partition* of n is a non-increasing sequence of positive integers whose sum is n). Thus the number of irreducible representations of S_n equals $p(n)$, the number of partitions of n; which Hardy and Ramanujan showed is $\sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$.

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Young [10,19] described a natural correspondence between partitions of n and irreducible representations of S_n: Given a partition $[\lambda] = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ of n (where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$), Young constructed a set of matrices for the representation of S_n attached to $[\lambda]$ (which we also denote by $[\lambda]$) by examining the action of S_n on *Young tableaux*, combinatorial objects constructed from the *Ferrers-Young diagram* of $[\lambda]$. The Ferrers-Young diagram of a partition $[\lambda]$ of n is an array of nodes with λ_k nodes in the k^{th} row. We assign numbers to the rows and columns, and coordinates to the nodes, just as we do for a matrix. The (i,j) *hook* is the set of nodes directly below, together with the set of nodes directly to the right of, the (i,j) node, as well as the (i,j) node itself (that is, the nodes (i,k) with $k \geq j$ together with the nodes (k,j) with $k \geq i$). The *hook number*, denoted by $H(i,j)$, is the total number of nodes on the (i,j) hook. A *t-core partition of n* is a partition of n in which none of the hook numbers are divisible by t.

Example. The Ferrers-Young diagram of the partition $4+3+1$ of 8 is

$$\begin{array}{ccccc}
 & 1 & 2 & 3 & 4 \\
1 & \bullet_{(1,1)} & \bullet_{(1,2)} & \bullet_{(1,3)} & \bullet_{(1,4)} \\
2 & \bullet_{(2,1)} & \bullet_{(2,2)} & \bullet_{(2,3)} & \\
3 & \bullet_{(3,1)} & & &
\end{array}$$

The hooks at $(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1)$, have hook numbers $6, 4, 3, 1, 4, 2, 1, 1$, respectively. Therefore the partition $4+3+1$ of 8 is a t-core partition for $t = 5$ and for all $t \geq 7$.

The Young tableaux of $[\lambda]$ are given by the $n!$ different ways of assigning the numbers 1 through n to the nodes of the Ferrers-Young diagram, each node getting a different number. A *standard tableau* is one where the numbers are increasing as one goes right or down (that is, the (i,j) entry is less than or equal to the (I,J) entry whenever $i \leq I$ and $j \leq J$). Young showed how to formulate the properties of a given representation $[\lambda]$ of S_n, in terms of certain d-by-d matrices of rational numbers, one for each element of S_n, which he constructed from the combinatorial properties of the set of standard tableaux (here d is the number of standard tableaux for $[\lambda]$). See Appendix I for further details.

It turns out that the hooks of $[\lambda]$ are intimately connected with the structure of the associated representation. In fact the degree d of the representation (which equals the number of standard tableaux) is given by the Frame-Thrall-Robinson hook formula [10,6.1.19]:

$$d = \frac{n!}{\prod_{i,j} H(i,j)}. \tag{1}$$

Although Young's matrices for the representation $[\lambda]$ of S_n have rational entries, there is a change of basis under which all of the matrices have integer entries [19, 12.13]; hence we may reduce these entries modulo a given prime p to obtain the corresponding *p-modular representation* $[\bar{\lambda}]$. Under this reduction, the characteristic 0 representations of S_n form equivalence classes, known as *p-blocks*. Let $\rho_1, \rho_2, \ldots, \rho_{p(n)}$ denote the ordinary irreducible representations of S_n. The *Brauer graph* is constructed by associating a node to each such representation, and then connecting two nodes i and j by an edge if and only if the reductions of ρ_i and

$\rho_j \mod p$ contain a common p-modularly irreducible constituent. The $p-$blocks are the connected components of the Brauer graph (see [9] for $p-$block theory via characters).

Of special interest are those p-blocks which consist of a single characteristic 0 representation (that is, the corresponding vertex in the Brauer graph is isolated); these are the p-blocks with *defect zero*. Brauer's *Problem 19*, one of many conjectures and problems posed in [3], asks for a description of the number of defect zero p-blocks for a finite group in terms of its invariants. In [20], Robinson solved this problem; however it is difficult to determine his invariants for many groups. Since finite simple groups were classified, there has been some interest in classifying all simple groups with defect zero $p-$blocks. For example, using the methods of Deligne and Lustzig, Michler [14] and Willems [23] proved that every finite simple group of Lie type possesses a defect zero $p-$block, for every prime p. However it turns out that some finite simple groups do not have a defect zero p-block for certain primes p: for example almost all alternating groups A_n have neither defect zero 2-blocks nor defect zero $3-$blocks (see corollary 2 below for details).

From the theory of modular representations of finite groups, we know [10,6.1.18] that an ordinary irreducible representation of a finite group G is p-modularly irreducible and has defect zero if and only if the power of p dividing the degree of the representation is equal to the power of p dividing $|G|$. By the Frame-Thrall-Robinson hook formula (1), this can only happen for a representation of S_n if it is associated to a p-core partition of n. Moreover if $[\lambda]$ is a $p-$core partition of n, then the restriction of its associated representation to A_n (which may be a reducible representation) has the property that all of its irreducible components form their own defect zero $p-$blocks [10,6.1.46]. Thus a zero defect p-block for S_n implies the existence of a zero defect p-block for A_n. We have thus explained the following well-known result:

Proposition 1. *Every p-core partition of n corresponds to a different defect zero p-block in S_n, which itself implies the existence of a defect zero p-block in A_n.*

In fact more is known: If $p > 2$ then A_n has a defect zero $p-$block if and only if S_n has one (for $p = 2$ the situation is a little more complicated).

Let $c_t(n)$ be the number of t-core partitions of n. Garvan, Kim and Stanton [6] proved that $c_t(n)$ equals the number of integer representations of n by the quadratic form

$$(2)\quad \frac{t}{2}(x_0^2 + x_1^2 + \cdots + x_{t-1}^2) + \sum_{i=0}^{t-1} i x_i, \qquad \text{where} \quad x_0 + x_1 + x_2 + \cdots + x_{t-1} = 0.$$

For example, for $t = 2$ this corresponds to the number of integer solutions x_1 to $n = 2x_1^2 + x_1$. Thus $c_2(n) = 0$ for almost all integers n, and so S_n has no defect zero 2-blocks for almost all n. By the last proposition we know that A_n has a defect zero 2-block if S_n has one. However if $n = 2m^2 + m + 2$ for some integer m, then A_n does have a defect zero $2-$block even though S_n does not. In these cases there are $2-$blocks for S_n, consisting of two representations each of whose restrictions to A_n form their own defect zero $2-$block for A_n. However if $n \neq 2m^2 + m$ nor $2m^2 + m + 2$, then A_n does not have a defect zero $2-$block. Hence A_n does not have a defect zero $2-$block for almost all n.

The generating function for $c_t(n)$ is given [6, 11] by:

$$\sum_{n=0}^{\infty} c_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}. \tag{3}$$

If $p=2$ then Jacobi's identity gives

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{1-q^n} = \sum_{n=0}^{\infty} c_2(n)q^n = \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}}.$$

This confirms what we had above, replacing n by $2x_1$ or $-2x_1-1$, as n is even or odd.

We shall see in section 3 that $c_3(n) = \sum_{d|3n+1} \left(\frac{d}{3}\right)$ where $\left(\frac{\cdot}{3}\right)$ is the Legendre symbol; and is therefore 0 if and only if there exists a prime $p \equiv 2 \pmod 3$ such the exact power of p dividing $3n+1$ is odd. By elementary sieve theory we thus see that there are $\asymp N/\sqrt{\log N}$ integers $n \le N$ for which $c_3(n)$ is non-zero. Therefore A_n has no defect zero 3-blocks for almost all n.

Garvan, Olsson, Stanton and many others have speculated that $c_p(n) > 0$ for all integers $n \ge 0$, whenever prime $p \ge 5$. From Proposition 1 this then implies that every symmetric group S_n and every alternating group A_n has a $p-$block with defect zero, for each prime $p \ge 5$. In fact, it has even been conjectured that $c_t(n) > 0$ for all integers $t > 3$, the so-called *t-core partition conjecture*. However since $c_{kt}(n) \ge c_t(n)$ whenever k is a positive integer, it follows that the t-core partition conjecture may be deduced by showing that $c_t(n) > 0$ for all integers $n \ge 0$, for all primes $t \ge 5$, as well as for $t = 4, 6$ and 9.

The result was proved for $p=5$ and $p=7$ by Erdmann and Michler [4] using 'abaci'; and from exact formulae for $c_5(n)$ and $c_7(n)$ in [6]. The second author [17,18] proved the result for $4 \le t \le 11$, using the theories of quadratic and modular forms. In this paper we complete the proof of the t-core partition conjecture:

Theorem 1. *Every non-negative integer n has at least one t-core partition, provided $t \ge 4$.*

Corollary 1. *For any positive integer n and any prime $p \ge 5$, the symmetric group S_n and the alternating group A_n have a $p-$block with defect 0.*

We are thus able to complete the classification of defect zero $p-$blocks in finite simple groups (using the main classification theorem, [8] and [14,23]):

Corollary 2. *Every finite simple group G has a p-block of defect 0, for every prime p, except in the following special cases:*

- *G has no 2-block of defect 0 if it is isomorphic to $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, C1, C3, BM$, or A_n where $n \ne 2m^2+m$ nor $2m^2+m+2$ for any integer m.*
- *G has no 3-block of defect 0 if it is isomorphic to $Suz, C3$, or A_n with $3n+1 = m^2 r$ where r is squarefree and divisible by some prime $q \equiv 2 \mod 3$.*

See Appendix II for a brief description of how these groups arise.

In section 4 we investigate congruence properties of $c_p(n)$, the number of p-blocks with defect 0 for S_n. In addition to verifying certain multiplicative congruences for $c_p(n)$ where $5 \le p \le 23$, conjectured in [7], we prove:

Theorem 2. *For any prime p and positive integer m, in almost all symmetric groups S_n, the number of $p-$blocks with defect 0 is a multiple of m. If p is any prime, then the number of $p-$modularly irreducible representations of S_n is almost always a multiple of p.*

By contrast, $p(n) \mod m$ (where $p(n)$ is the number of irreducible representations of S_n) is believed to follow no simple patterns, except in the special arithmetic progressions found by Ramanujan [2].

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2. The $t-$core conjecture for $t \geq 17$

In this section we prove the conjecture for $t \geq 17$. The idea behind the proof is to use the freedom available in choosing so many variables in (2) to reduce the problem to an application of Lagrange's Theorem, that every non-negative integer is representable as a sum of four squares. To begin with, though, we show that all 'small' n can be represented as in (2).

Lemma 1. *Every integer $n \leq t^2/4$ may be represented by* (2) *with each $x_i = -1, 0$ or* 1.

Proof. For any fixed integer $k \leq t/2$, we let $x_{t-k} = x_{t-(k-1)} = \cdots = x_{t-1} = -1$, and for a given $I \subset \{0, 1, 2, \ldots, t-k-1\}$ of size k, we let $x_i = 1$ if $i \in I$, and $x_i = 0$ otherwise. Then the integer given by (2) is $\binom{k+1}{2} + \sum_{i \in I} i$.

We claim that, for any given $k \leq m+1$, and integer $r \in \left[\binom{k}{2}, km - \binom{k}{2}\right]$, there is a subset I of $\{0, 1, 2, \ldots, m\}$ with $\sum_{i \in I} i = r$. We prove this by induction on r: it is certainly true for $r = \binom{k}{2}$ by taking $I = \{0, 1, 2, \ldots, k-1\}$. Assume that it is true for $r-1$; that is, we have $\sum_{j \in J} j = r - 1$. Now select the largest $j \in J$ with $j \leq m$ for which $j + 1 \notin J$. Evidently such a j exists (unless $J = \{m+1-k, \ldots, m\}$, in which case $r > km - \binom{k}{2}$), so let I be the set J with j replaced by $j+1$, and then $\sum_{i \in I} i = 1 + \sum_{j \in J} j = r$.

Thus above we see that every integer in

$$\left[\binom{k+1}{2} + \binom{k}{2}, \binom{k+1}{2} + k(t-k-1) - \binom{k}{2}\right] = [k^2, k(t-k)]$$

is so represented by (2). Taking the union of these intervals for $0 \leq k \leq t/2$ gives the result. □

We will prove

Proposition 2. *Any integer $n \geq 3t+9$ may be represented by* (2)*, provided $t \geq 17$.*

Now, since $3t + 9 < t^2/4$ for $t \geq 15$, Lemma 1 combined with Proposition 2 implies:

Theorem 3. *Every non-negative integer has at least one $t-$core partition, provided $t \geq 17$; that is, any integer $n \geq 0$ may be represented by* (2) *once $t \geq 17$.*

Proof of Proposition 2. For any integer $n \geq 3t+9$, let n_0 be the least residue of n (mod 3). If $n_0 = 1$ then let $x_0 = 1$ and $x_{t-1} = -1$; if $n_0 = 2$ then let $x_1 = 1$ and $x_{t-1} = -1$. We also let

$$\begin{aligned} x_2 = x_4 &= a+\alpha, & x_3 &= -2a+\alpha,\\ x_5 = x_7 &= b+\beta, & x_6 &= -2b+\beta,\\ x_8 = x_{10} &= c+\gamma, & x_9 &= -2c+\gamma,\\ x_{11+J} = x_{13+J} &= d+\delta, & x_{12+J} &= -2d+\delta, \end{aligned}$$

for some $0 \leq J \leq t-15$, where a, b, c, d are integers which are yet to be chosen, and $\alpha, \beta, \gamma, \delta \in \{-1, 0, 1\}$ are also yet to be chosen and must satisfy $\alpha+\beta+\gamma+\delta = 0$. We take all other $x_i = 0$. Substituting these numbers into (2) we get

$$n_0 + 3tm + \frac{3t}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2) + 9(\alpha+2\beta+3\gamma+4\delta) + 3J\delta,$$

where $m = a^2+b^2+c^2+d^2$. By Lagrange's Theorem we can, of course, so represent any non-negative integer m by an appropriate choice of a, b, c, d. So in order to represent n by (2) in the manner described above, we must find non-negative integers m and $J \leq t-15$, as well as $\alpha, \beta, \gamma, \delta \in \{-1, 0, 1\}$ satisfying $\alpha+\beta+\gamma+\delta = 0$, for which

$$N = tm + \frac{t}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2) + 3(\alpha+2\beta+3\gamma+4\delta) + J\delta, \tag{4}$$

where $N = [n/3] \geq t+3$.

Now if we select $\alpha = \beta = 0,\ \gamma = -\delta = \pm 1$ in (4), then the right side equals $t(m+1) + \delta(J+3)$. Similarly if we select $\alpha = \gamma = 0,\ \beta = -\delta = \pm 1$ we get $t(m+1)+\delta(J+6)$; and if we select $\beta = \gamma = 0,\ \alpha = -\delta = \pm 1$ we get $t(m+1)+\delta(J+9)$. As J runs through the integers $0 \leq J \leq t-15$, and letting $\delta = \pm 1$, we represent every integer N in $[tm+6, t(m+1)-3]$ as well as in $[t(m+1)+3, t(m+2)-6]$, for $m = 0, 1, 2, \ldots$. This gives respresentations by (4) of all $N \geq t+3$, except those in the intervals $[tm-2, tm+2]$, for $m \geq 2$.

Finally take $\beta = \gamma = -\delta,\ \alpha = \delta = \pm 1$ in (4) so that the right side equals $t(m+2) + J\delta$. As J runs through the integers $0 \leq J \leq t-15$, and letting $\delta = \pm 1$, we represent every integer N in $[t(m+1)+15, t(m+3)-15]$, for $m = 0, 1, 2, \ldots$. In particular this includes the intervals $[tm-2, tm+2]$ for every $m \geq 2$, since $t \geq 17$.

Therefore we can represent every integer $N \geq t+3$ by (4), and thus every integer $n \geq 3t+9$ by (2). □

Remark. In the proof above we picked $x_i = 0, 1$ or -1 for all $i \notin [2, 9] \cup [11+J, 13+J]$. If instead we let these x_i be in the same residue class mod $3t$ as before, but now allow them to be any integer satisfying $|x_i| \leq \frac{\sqrt{n}}{t}$ for $i \geq 1$, with x_0 chosen so that the sum of these integers is 0, then we can pick x_j for $j \in [2, 9] \cup [11+J, 13+J]$ in an analogous way to before. We have thus proved that the number of t-core partitions of an integer n is $> \left(\frac{\sqrt{n}}{2t}\right)^{t-13}$ for $t > 13$.

3. The t–core conjecture for $t \leq 16$

As noted in the introduction, the t-core conjecture holds for any multiple of t once it has been proved for t. Therefore it suffices to prove the t-core conjecture for $t = 4, 6, 9$ and any prime ≥ 5, since any integer ≥ 4 must be divisible by one of these numbers. Given Theorem 3, it only remains to prove the conjecture for $t = 4, 5, 6, 7, 9, 11$ and 13. All these cases have been handled in previous work except for $t = 13$. We briefly summarize the techniques that have been used:

For $t = 4$ one can use (2) to show that $c_4(n)$ is the number of integer solutions x, y, z of $8n + 5 = x^2 + 2y^2 + 2z^2$ (see [17]); and this is always > 0 by the work of Gauss.

For $t = 6$ one can factor the generating function (3) into a product of two formal power series: The first is the generating function for the number of representations of an integer as the sum of three triangular numbers, and the second has all non-negative coefficients with leading term 1. Thus $c_6(n) > 0$ since Gauss's *Eureka* theorem asserts that every non-negative integer can be represented as the sum of three triangular numbers.

Similarly for $t = 9$ we factor the generating function (3) into a product of two formal power series, the first of which is a power of an Eisenstein series, that we prove has all positive coeffcients, and the second of which has all non-negative coefficients with leading term 1. This forces all the coefficients of the resulting product to be positive.

For prime values of t we can prove the result by explicitly computing the generating function (3) as the sum of an Eisenstein series and a cusp form. This was done for $t = 11$ in [17], and we shall do it for $t = 13$ here. We begin by recalling various definitions and facts from the theory of modular forms:

Let $\mathfrak{H}$ be the upper half of the complex plane and let $SL_2(\mathbb{Z})$ act on it by linear fractional transformations. If N is a positive integer, then let $\Gamma_0(N)$ denote the subgroup of $SL_2(\mathbb{Z})$ defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \text{ and } c \equiv 0 \mod N \right\}.$$

Given a positive integer k and a Dirichlet character $\chi \mod N$, we say that a meromorphic function on $\mathfrak{H}$ is a *modular form of weight k with character* χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $z \in \mathfrak{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

If $f(z)$ is holomorphic on $\mathfrak{H}$ as well as at the cusps (that is, at the rational z), then we say that $f(z)$ is a *holomorphic modular form of type* (k, χ) *and level* N. The set of all such forms is denoted $M_k(N, \chi)$ and is a finite dimensional $\mathbb{C}-$vector space. The subspace of $M_k(N, \chi)$ consisting of those modular forms which also vanish at the cusps, *the cusp forms*, is denoted by $S_k(N, \chi)$.

Every modular form $f(z)$ in $M_k(N, \chi)$ admits a Fourier expansion at infinity of the form

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

where $q := e^{2\pi i z}$. The *Hecke operators* T_p are natural linear transformations which preserve $M_k(N,\chi)$ and $S_k(N,\chi)$. For every prime p the image of the modular form $f(z)$ is defined by

$$f(z) \mid T_p := \sum_{n=0}^{\infty}(a(pn)+\chi(p)p^{k-1}a(n/p))q^n. \tag{5}$$

We say that $f(z)$ is an *eigenform* with respect to the Hecke operator T_p if there exists a complex number λ_p satisfying

$$f(z) \mid T_p = \lambda_p f(z).$$

One can construct modular forms out of forms of a lower level, since if $f(z) \in S_k(N,\chi)$ then both $f(z)$ and $f(mz)$ belong to $S_k(mN,\chi)$, for any positive integer m. For any given N we can form the vector space generated by all modular forms in $S_k(N,\chi)$ obtained in this way from all $S_k(M,\chi)$, where M and the conductor of χ both divide N. This is the subspace of $S_k(N,\chi)$ of *oldforms*, denoted by $S_k^{\mathrm{old}}(N,\chi)$. The orthogonal complement of $S_k^{\mathrm{old}}(N,\chi)$ in $S_k(N,\chi)$ is $S_k^{\mathrm{new}}(N,\chi)$.

It turns out that $S_k^{\mathrm{new}}(N,\chi)$ has a basis of *newforms*, which are defined to be elements of this space which are also eigenforms of all of the Hecke operators T_p. Throughout we shall assume that a newform $f(z)$ is normalized so that its Fourier expansion is of the form

$$f(z) = q + \sum_{n=2}^{\infty} a(n)q^n$$

(that is, we divide out to make the leading coefficient 1). With this normalization we obtain, for every prime p,

$$f(z) \mid T_p = a(p)f(z). \tag{6}$$

Since newforms are eigenforms of the Hecke operators, the Fourier coefficients $a(n)$ possess nice multiplicative properties. Specifically the coefficients satisfy

$$a(mn) = a(m)a(n) \quad \text{if} \quad \gcd(m,n) = 1,$$

and

$$a(p^r) = a(p)a(p^{r-1}) - \chi(p)p^{k-1}a(p^{r-2}). \tag{7}$$

For more details on the theory of modular forms see [13,15].

Deligne's Theorem implies that if $f(z) = q + \sum_{n=2}^{\infty} a(n)q^n$ is a newform of type (k,χ) of level N, then for every prime p which does not divide N we have

$$\mid a(p) \mid \leq 2p^{\frac{k-1}{2}}.$$

In [15, 4.6.17] Miyake shows that we get the better upper bound $\mid a(p) \mid \leq p^{\frac{k-1}{2}}$ when p does divide N. This allows us to obtain the following upper bound for $|a(n)|$:

Lemma 2. *If $f(z) = q + \sum_{n=2}^{\infty} a(n)q^n$ is a newform of type (k, χ) of level N, then*

$$|a(n)| < n^{\frac{k-1}{2}}(1+\sqrt{2})^{\Omega(n)},$$

where $\Omega(n)$ denotes the total number of prime divisors of n, counting multiplicity.

Proof. Since $a(n)$ is a multiplicative function we need only prove the result for $n = p^r$, the powers of a fixed prime p. Now define $u_r = |a(p^r)/p^{\frac{r(k-1)}{2}}|$, so that (7) implies $u_r \le 2u_{r-1} + u_{r-2}$ for any $r \ge 2$, with $u_0 = 1$ and $u_1 \le 2$ (by the comments above). By an induction hypothesis we immediately deduce that $u_r \le (1+\sqrt{2})^r$ for all r, and the proposition follows. $\square$

Dedekind's $\eta-$function, defined by the infinite product

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

is a modular form of weight $1/2$. Evidently the eta-quotient

$$\frac{\eta^t(tz)}{\eta(z)} = q^{\frac{t^2-1}{24}} \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)} = \sum_{n=0}^{\infty} c_t(n) q^{n+\frac{t^2-1}{24}}, \tag{8}$$

and so we can try to interpret the generating function for $c_p(n)$ as a modular form. In fact, it is shown in [11] that if $p \ge 5$ is prime then $\eta^p(pz)/\eta(z) \in M_{\frac{p-1}{2}}(p, \chi)$ where $\chi(d) = \left(\frac{d}{p}\right)$ is the usual Legendre symbol. In particular it is established that

$$\frac{\eta^p(pz)}{\eta(z)} = \alpha_p E_p(z) + f(z) \tag{9}$$

where α_p is a positive constant, $E_p(z)$ is the Eisenstein series with weight $\frac{p-1}{2}$ centered at 0, and $f(z) \in S_{\frac{p-1}{2}}(p, \chi)$. However there are no lower levels dividing p which have non-trivial character χ, and so $f(z) \in S^{\text{new}}_{\frac{p-1}{2}}(p, \chi)$. Thus we may write $f(z) = \alpha_p \sum_i c_i f_i(z)$, a linear combination of newforms $f_i(z)$ of level p. Hecke showed that the Fourier expansion of $E_p(z)$ is

$$E_p(z) := \sum_{n=1}^{\infty} \sigma_p(n) q^n$$

where $\sigma_p(n) := \sum_{d|n} \chi(n/d) d^{\frac{p-3}{2}}$.

Very recently Almkvist [1] has managed to evaluate the constant α_p, getting

$$1/\alpha_p = \frac{\left(\frac{p-3}{2}\right)! p^{p/2}}{(2\pi)^{\frac{p-1}{2}}} \; L\left(\frac{p-1}{2}, \left(\frac{\cdot}{p}\right)\right),$$

where $L(s, \left(\frac{\cdot}{p}\right))$ is the Dirichlet L-function for character $\left(\frac{\cdot}{p}\right)$. He proves that $1/\alpha_p$ is always an integer[1], using a result of Dokshitzer on the denominators of values of Bernoulli polynomials.

[1]In fact $1/\alpha_5 = 1$, $1/\alpha_7 = 8$, $1/\alpha_{11} = 1275$, $1/\alpha_{13} = 33463$. $1/\alpha_{17} = 59901794$, $1/\alpha_{19} = 3708443635$, $1/\alpha_{23} = 27533989805352$, $\ldots$.

In the special case $p = 3$ we replace z by $3z$ so that the quotient of eta-functions is an Eisenstein series; in fact a modular form belonging to $M_1(9, \left(\frac{-3}{n}\right))$, with Fourier expansion

$$\sum_{n=0}^{\infty} c_3(n) q^{3n+1} = \frac{\eta^3(9z)}{\eta(3z)} = \sum_{n=0}^{\infty} \sigma(n) q^n$$

$$\text{where} \quad \sigma(n) = \begin{cases} 0 & \text{if } n \equiv 0 \mod 3, \\ \sum_{d|n} \left(\frac{d}{3}\right) & \text{if } n \equiv 1, 2 \mod 3. \end{cases}$$

Thus $c_3(n) = \sigma(3n+1)$, as noted in the introduction.

We will need a lower bound for $\sigma_p(n)$ for $p \geq 5$:

Lemma 3. *If p is a prime ≥ 5, and $\sigma_p(n)$ is the divisor function defined by*

$$\sigma_p(n) = \sum_{d|n} \chi(n/d) d^{\frac{p-3}{2}},$$

where $\chi(d) = \left(\frac{d}{p}\right)$ is the usual Legendre symbol, then

$$\sigma_p(n) \geq n^{\frac{p-3}{2}} \prod_{q|n,\ q \text{ prime}} \left(1 - \frac{1}{q^{\frac{p-3}{2}}}\right).$$

Proof. Since $\sigma_p(n)$ is a multiplicative function, it suffices to prove the result for $n = q^k$, a prime power. Writing $Q = q^{\frac{p-3}{2}}$ we either have $\sigma_p(q^k) = Q^k$ (if $\chi(p) = 0$) or else

$$\frac{1}{Q^k}\sigma_p(q^k) = 1 \pm \frac{1}{Q} + \frac{1}{Q^2} \pm \frac{1}{Q^3} + \frac{1}{Q^4} + \cdots + \frac{(\pm 1)^k}{Q^k} \geq 1 \pm \frac{1}{Q} \geq 1 - \frac{1}{Q}$$

and the result follows. $\square$

We restrict our attention here to lower bounds for $c_p(n)$ once $p \geq 11$:

Theorem 4. *There are more than $\frac{2\alpha_p}{5} n^{\frac{p-3}{2}}$ p-blocks with defect zero, once n is sufficiently large, provided prime $p \geq 11$.*

Proof. By (9) we see that $c_p(n) = \alpha_p(\sigma_p(N) + \sum_i c_i f_i(N))$ where $N = n + \frac{p^2-1}{24}$ and $f_i(N)$ is the Fourier coefficient of q^N in the Fourier expansion of $f_i(z)$. By Lemma 2, we know that

$$\left| \sum_i c_i f_i(N) \right| \leq \left(\sum_i |c_i| \right) N^{\frac{p-3}{4}} (1 + \sqrt{2})^{\Omega(N)},$$

since each $f_i(z)$ has weight $k = (p-1)/2$. Now suppose τ satisfies $1 + \sqrt{2} = 2^\tau$. Then $(1+\sqrt{2})^{\Omega(N)} = 2^{\tau \Omega(N)} \leq N^\tau$ and so $|\sum_i c_i f_i(N)| \leq (\sum_i |c_i|) N^{\frac{p-3}{4} + \tau}$. On the other hand, since $\frac{p-3}{2} \geq 4$ when $p \geq 11$,

$$\sigma_p(N) \geq N^{\frac{p-3}{2}} \prod_{q|N,\ q \text{ prime}} \left(1 - \frac{1}{q^{\frac{p-3}{2}}}\right) \geq N^{\frac{p-3}{2}} \prod_{q \text{ prime}} \left(1 - \frac{1}{q^4}\right) = \frac{90}{\pi^4} N^{\frac{p-3}{2}}.$$

Therefore, since $N \geq n$ and $90/\pi^4 - 1/2 > 2/5$, we have

$$c_p(n) \geq \alpha_p N^{\frac{p-3}{2}} \left(\frac{90}{\pi^4} - \frac{\sum_i |c_i|}{N^{\frac{p-3}{4}-\tau}} \right) \geq \frac{2\alpha_p}{5} n^{\frac{p-3}{2}},$$

once $N^{\frac{p-3}{4}-\tau} \geq 2\sum_i |c_i|$. □

As mentioned above, Almkvist [1] has now determined the value of α_p explicitly for all primes p. This gives us some hope of finding an explicit upper bound for each c_i in the proof above, which would lead to an explicit version of Theorem 4. However we do not yet know how to do this, and have to work hard to even completely solve the case $p = 13$. What we will do is to fill out the steps of the above proof explicitly, using Maple, so as to determine the actual values of the c_i.

Theorem 5. *Every non-negative integer has at least one* 13*-core partition. Actually n has more than $(n/10)^5$ such partitions.*

Proof. By (9) we have

$$\sum_{n=0}^{\infty} c_{13}(n) q^{n+7} = \frac{\eta^{13}(13z)}{\eta(z)} = \alpha_{13} E_{13}(z) + f(z) \tag{10}$$

where $f(z) \in S_6^{\text{new}}(13, (\frac{\cdot}{13}))$. This space of cusp forms has dimension 6, and Garvan [6] proved that it has the basis

$$b_i(z) := \frac{\eta^{13}(13z)}{\eta(z)} \left(\frac{\eta^2(z)}{\eta^2(13z)} \right)^{7-i} \quad \text{for } 1 \leq i \leq 6.$$

Now it is well known that Eisenstein series lie in the orthogonal complement to the cusp forms, in the space of modular forms, and so $E_{13}(z)$ and the $b_i(z)$ are linearly independent. Therefore in order to write $f(z) = \alpha_{13} \sum_i \gamma_i b_i(z)$ above, and to determine α_{13}, we equate the first seven terms of the Fourier expansions of both sides of (10) and solve the resulting linear equations. First note that

$$b_i(z) = q^i \prod_{n \geq 1} (1-q^n)^{13-2i} (1-q^{13n})^{2i-1} \equiv q^i \prod_{n=1}^{7-i} (1-q^n)^{13-2i} \mod q^8,$$

and so the first few Fourier coefficients are easily determined. We can compute $E_{13}(z)$ from the definition given:

$$E_{13}(z) = q + 31q^2 + 244q^3 + 993q^4 + 3124q^5 + 7564q^6 + 16806q^7 + \dots.$$

So we solve the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 31 & -11 & 1 & 0 & 0 & 0 & 0 \\ 244 & 44 & -9 & 1 & 0 & 0 & 0 \\ 993 & -55 & 27 & -7 & 1 & 0 & 0 \\ 3124 & -110 & -12 & 14 & -5 & 1 & 0 \\ 7564 & 374 & -90 & 7 & 5 & -3 & 1 \\ 16806 & -143 & 135 & -49 & 10 & 0 & -1 \end{pmatrix} \alpha_{13} \begin{pmatrix} 1 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

to get $\alpha_{13} = 1/33463$ and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) = -(1, 42, 578, 3960, 15446, 33462)$, and so

$$33463 \frac{\eta^{13}(13z)}{\eta(z)} = E_{13} - (b_1 + 42b_2 + 578b_3 + 3960b_4 + 15446b_5 + 33462b_6). \tag{11}$$

It is known that $S_6^{\text{new}}(13, \left(\frac{\cdot}{13}\right))$ has a basis of newforms, which we can determine by computing the action of the Hecke operator T_2 on our chosen basis vectors. Specifically we know that $b_i(z)|T_2 = \sum_{n \geq 1}(b_i(2n) - 32b_i(n/2))q^n$ by (5) (where $b_i(n)$ is the nth Fourier coefficient of $b_i(z)$ if n is an integer, 0 otherwise). We can obtain this in terms of the $b_j(z)$ by considering the first six Fourier coefficients of each $b_i(z)|T_2$, and finding the only possible linear combination of the $b_j(z)$ that could give this. Thus

$$\begin{pmatrix} b_1(z) \mid T_2 \\ b_2(z) \mid T_2 \\ b_3(z) \mid T_2 \\ b_4(z) \mid T_2 \\ b_5(z) \mid T_2 \\ b_6(z) \mid T_2 \end{pmatrix} = \begin{pmatrix} -11 & -87 & 374 & -110 & 495 & -3477 \\ 1 & 27 & -90 & 22 & -189 & 945 \\ 0 & -7 & 7 & 21 & 41 & -165 \\ 0 & 1 & 5 & -15 & -5 & 15 \\ 0 & 0 & -3 & 5 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$
$$\bullet \begin{pmatrix} 1 & -11 & 44 & -55 & -110 & 374 \\ 0 & 1 & -9 & 27 & -12 & -90 \\ 0 & 0 & 1 & -7 & 14 & 7 \\ 0 & 0 & 0 & 1 & -5 & 5 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} b_1(z) \\ b_2(z) \\ b_3(z) \\ b_4(z) \\ b_5(z) \\ b_6(z) \end{pmatrix}$$
$$= \begin{pmatrix} -11 & -208 & -1014 & -2197 & 0 & 0 \\ 1 & 38 & 208 & 507 & 0 & 0 \\ 0 & -7 & -56 & -182 & -169 & 0 \\ 0 & 1 & 14 & 56 & 91 & 0 \\ 0 & 0 & -3 & -16 & -38 & -13 \\ 0 & 0 & 1 & 6 & 16 & 11 \end{pmatrix} \begin{pmatrix} b_1(z) \\ b_2(z) \\ b_3(z) \\ b_4(z) \\ b_5(z) \\ b_6(z) \end{pmatrix}$$

For convenience we will write this matrix equation as $B' = TB$. Let us express the $b_i(z)$ as linear combinations of the newforms $f_j(z)$, written as the matrix equation $B = RF$. Note that $F' = DF$ for some diagonal matrix D, since the f_i's are eigenforms of the Hecke operators. Now $TRF = TB = B' = RF' = RDF$, so that $TR = RD$; that is, R is the matrix of eigenvectors of T. The columns of such a matrix are only defined up to a scalar multiple. However, since the f_i are 'normalized' we can determine the correct scalar multiple by comparing the coefficient of q^1 in $B = RF$; that is, we must have $(1, 0, 0, 0, 0, 0)^t = R(1, 1, 1, 1, 1, 1)^t$. Let $S = (R^{-1})^t$. Taking the transpose of $TR = RD$, and then multiplying on the left and right by S, we get $T^tS = SD$; so S is the matrix of eigenvectors of T^t. Moreover, since $(1, 0, 0, 0, 0, 0) = (1, 1, 1, 1, 1, 1)R^t$, thus $(1, 0, 0, 0, 0, 0)S = (1, 1, 1, 1, 1, 1)$; that is the top row of S must be $(1, 1, 1, 1, 1, 1)$. Suppose that Maple gives the eigenvectors $\mathbf{r}_\rho$ and $\mathbf{s}_\rho$ of T and T^t, respectively, corresponding to the eigenvalue ρ. The corresponding column of S is $S_\rho = \mathbf{s}_\rho/\sigma_\rho$ where σ_ρ is the first element of $\mathbf{s}_\rho$. If $R_\rho = m_\rho \mathbf{r}_\rho$ is the corresponding column vector of R then, since $S^tR = I$, we must have $1 = S_\rho^t \cdot R_\rho = \frac{m_\rho}{\sigma_\rho}\mathbf{s}_\rho^t \cdot \mathbf{r}_\rho$, so that $m_\rho = \sigma_\rho/(\mathbf{s}_\rho^t \cdot \mathbf{r}_\rho)$. Using Maple we obtain $3548220835392 R_\rho$ equals

$$\begin{pmatrix} 1326473734176 + 870096491748\rho + 94767875376\rho^2 + 7911872133\rho^3 + \dots \\ \dots + 918595056\rho^4 - 5815887\rho^5 \\ 317901315600 + 321298585860\rho - 3005304012\rho^2 + 13222526309\rho^3 - \dots \\ \dots - 101172348\rho^4 + 95885041\rho^5 \\ -132164787024 - 199061787876\rho - 3238095204\rho^2 - 6279291529\rho^3 - \dots \\ \dots - 8785908\rho^4 - 40796693\rho^5 \\ 24210604656 + 56495681340\rho + 893004468\rho^2 + 1563549235\rho^3 + \dots \\ \dots + 5006820\rho^4 + 9362375\rho^5 \\ -7406385780\rho - 131628545\rho^3 - 483709\rho^5 \\ -1946375244\rho - 90540651\rho^3 - 694383\rho^5 \end{pmatrix}$$

Substituting this into (11), we find that $33463\frac{\eta^{13}(13z)}{\eta(z)} = E_{13}(z) - \sum_\rho c_\rho f_\rho(z)$, where

$$\begin{aligned} 1182740278464 c_\rho = {} & 11387025509088 - 18832940453556\rho + 544407924080\rho^2 \\ & - 645774441961\rho^3 + 3806036272\rho^4 - 4396980293\rho^5. \end{aligned}$$

Using a floating point routine to evaluate c_ρ for each ρ, we get the values

$$\approx -2.373 \pm 5.33i, \qquad -5.156 \pm 12.901i, \qquad 8.029 \pm 26.472i,$$

so that $2\sum_\rho |c_\rho| < 190$.

From the proof of Theorem 4, we need $(n+7)^{5/2 - \ln(1+\sqrt{2})/\ln 2} \geq 190$, which is true for $n \geq 65$. The result may be verified for $n \leq 64$ by explicit computation. □

By combining Theorems 3 and 5, we obtain Theorem 1.

4. Congruence properties for the number of defect zero $p-$blocks

There has been much interest in congruence properties of $p(n)$ since Ramanujan first conjectured the residue of $p(n)$ modulo powers of 5,7, and 11, for certain values of n. It is now believed that, besides these very special congruences, there are no other moduli m for which $p(n)$ behaves in a predictable way modulo m. However for $c_p(n)$ we will find many congruence properties which follow from the theory of modular forms as developed by Deligne, Serre and Sturm.

Considering $c_t(n)$ where t is a power of 5, 7, or 11, we obtain congruences modulo powers of 5,7, and 11 which are equivalent to the Ramanujan congruences for $p(n)$.

Proposition 3. *If $c_t(n)$ is the number of $t-$core partitions of n, then for all $k \geq 1$ and every integer n we have*

$$\begin{aligned} c_{5^k}(5^k n + \delta_{5,k}) &\equiv 0 \mod 5^k, \\ c_{7^k}(7^k n + \delta_{7,k}) &\equiv 0 \mod 7^{[k/2]+1}, \\ c_{11^k}(11^k n + \delta_{11,k}) &\equiv 0 \mod 11^k, \end{aligned}$$

where $\delta_{p,k} := 1/24 \mod p^k$.

Proof. By Euler's generating function for $p(n)$ and (3), we find that

$$\sum_{n=0}^{\infty} c_{\ell^k}(n) q^n = \left(\sum_{n=0}^{\infty} p(n) q^n\right)\left(\prod_{n=1}^{\infty}(1 - q^{\ell^k n})^{\ell^k}\right).$$

If we let $\prod_{n=1}^{\infty}(1-q^{l^k n})^{l^k} = 1+\sum_{n=1}^{\infty} a_{\ell,k}(n)q^{\ell^k n}$, then we obtain

$$c_{\ell^k}(\ell^k n+\delta_{\ell,k}) = p(\ell^k n+\delta_{l,k}) + \sum_{i=1}^{n} p(\ell^k n-\ell^k i+\delta_{\ell,k})a_{\ell,k}(i).$$

The result then follows immediately from Ramanujan's partition congruences (see [2]), which state that $p(\ell^k m+\delta_{\ell,k}) \equiv 0 \mod \ell^K$ for any integer m (where $K=k$ if $p=5$ or 11, and $K=[k/2]+1$ if $p=7$). □

In [7] Garvan proves various congruences for $c_p(n)$ and conjectures [7,5.5] that for $5\le p\le 23$ we have

$$c_p(prn-\delta_p)+\delta_p c_p(pr-\delta_p)c_p(pn-\delta_p)+r^{p-2}c_p((pn/r)-\delta_p)\equiv 0 \mod p \tag{12}$$

for all primes r and any non-negative integer n, where $\delta_p=\frac{p^2-1}{24}$. This was proved in [7] for $p=5,7$, and 11, and also follows easily from Proposition 3 for these p. To prove Garvan's conjecture for primes $13\le p\le 23$, we show that the modular form $\frac{\eta^p(pz)}{\eta(z)} \mid T_p$ is congruent, modulo p, to a scalar multiple of the unique weight $p-1$ normalized eigenform with respect to $SL_2(\mathbb{Z})$. These are defined as follows: The canonical Eisenstein series of weights 4 and 6 have Fourier expansions

$$E_4(z)=1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n \quad \text{and} \quad E_6(z)=1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n,$$

respectively, where $\sigma_k(n):=\sum_{d\mid n}d^k$. The only normalized cusp form of weight 12, with respect to $SL_2(\mathbb{Z})$, is $f_{12}(z):=\Delta(z)=\eta^{24}(z)$. The only normalized cusp form of weight k (for $k=16,18$ or 22) with respect to $SL_2(\mathbb{Z})$, that is also an eigenform of the Hecke operators, is $f_{16}(z)=E_4(z)\Delta(z), f_{18}(z)=E_6(z)\Delta(z)$ or $f_{22}(z)=E_4(6)E_6(z)\Delta(z)$, respectively.

Proposition 4. *If $13\le p\le 23$ is prime, then*

$$\frac{\eta^p(pz)}{\eta(z)} \mid T_p \equiv 24 f_{p-1}(z) \mod p.$$

Proof. Sturm [22] proved that two modular forms with integer coefficients, both of weight k with respect to $\Gamma_0(N)$, are congruent modulo an integer m if the alleged congruence holds for the first $\frac{k}{12}\prod_{p\mid N}\left(1-\frac{1}{p^2}\right)+1$ terms.

Now, both $\frac{\eta^p(pz)}{\eta(z)} \mid T_p$ and $\frac{\eta^p(z)}{\eta(pz)}$ are weight $\frac{p-1}{2}$ modular forms on $\Gamma_0(p)$, and so their product has weight $p-1$. We checked (in Maple) that the coefficients of q^m of this modular form and of $24f_{p-1}(z)$ are congruent modulo p, for each $m\le \frac{p^3-p^2-p+1}{12p}$. Thus, by Sturm's theorem the two modular forms are congruent modulo p. But this implies the result since the Fourier expansion of $\frac{\eta^p(z)}{\eta(pz)}$ satisfies the congruence

$$\frac{\eta^p(z)}{\eta(pz)}=\prod_{n=1}^{\infty}\frac{(1-q^n)^p}{(1-q^{pn})}\equiv 1 \mod p,$$

because $(1-Q)^p\equiv 1-Q^p \mod p$. □

As a corollary we deduce Garvan's conjecture [7]:

Corollary 3. *The congruence* (12) *holds for all non-negative integers n, and for all primes p in the range $5 \le p \le 23$.*

Proof. $f_{p-1}(z) := \sum_{n=1}^{\infty} a_p(n)q^n$ is an eigenform of the Hecke operator T_r, and so, by (5) and (6),

$$a_p(r)a_p(n) - a_p(rn) - r^{p-2}a_p(n/r) = 0$$

for all non-negative n. By Proposition 4 this then leads to a congruence modulo p for the coefficients of $(\eta^p(pz)/\eta(z)) \mid T_p$. Since $\delta_p = \frac{p^2-1}{24} \equiv -1 \mod p$, the result then follows immediately from the expansion (8). ☐

It is of interest to note that for $p = 13$ this Proposition implies that

$$c_{13}(13n-7) \equiv 11\tau(n) \mod 13$$

where $\tau(n)$ is Ramanujan's tau-function, the coefficient of q^n in $\Delta(z)$.

In [21], Serre proved that if $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is an integer weight modular form where the coefficients $a(n)$ are algebraic integers all from some fixed number field, then they are almost all divisible by any given non-zero integer. Applying this result to the decomposition (9), we deduce that any given non-zero integer divides $c_p(n)$ for almost all n, for any given prime $p \ge 5$. For $p = 2$ and 3 we noted, in the introduction, that $c_p(n) = 0$ for almost all n, and so is divisible by m.

The generating function for $b_t(n)$, the number of partitions of n all of whose parts are not divisible by t, is

$$\sum_{n=0}^{\infty} b_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})}{1-q^n}.$$

For p prime, $b_p(n)$ is the number of $p-$modularly irreducible representations of S_n [10,6.1.2]. In terms of eta-quotients, we find that the generating function for $b_p(n)$ may be interpreted as the modular function

$$\sum_{n=0}^{\infty} b_p(n)q^{24n+p-1} = \frac{\eta(24pz)}{\eta(24z)} \equiv \eta^{p-1}(24z) \mod p,$$

since $(1-Q)^p \equiv 1 - Q^p \mod p$. For $p = 2$, this implies that $b_2(n)$ is even for almost all n, since, by Euler's identity,

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3n^2+n}{2}}.$$

Every odd prime p divides $b_p(n)$ for almost all n, by Serre's Theorem, since $\eta^{p-1}(24z)$ is a cusp form with integer weight $\frac{p-1}{2}$. We have therefore proved Theorem 2.

Appendix I. Young's matrices

The general construction of Young's matrices, corresponding to a given representation, is rather complicated to describe (see [19,2.1] for details). However, every permutation σ in S_n can be expressed as a product of transpositions of the form $\sigma_r := (r, r+1)$ (where $1 \leq r \leq n-1$), say $\sigma = \sigma_{r_1}\sigma_{r_2}\dots\sigma_{r_m}$. But then the matrix M corresponding to σ is given by $M = M_{r_1}M_{r_2}\dots M_{r_m}$, where M_r is the matrix corresponding to σ_r. Therefore we only need to construct the matrices corresponding to transpositions $\sigma_r := (r, r+1)$.

This matrix has dimension d, the number of standard tableaux for partition $[\lambda]$. We order the standard tableaux $T_1, T_2, \dots, T_d$ in (essentially) lexicographic order: If row i is the first row for which the standard tableaux T' and T'' differ, and j is the first such entry in the ith row, then we write $T' < T''$ if the (i,j) entry in T' is smaller than the (i,j) entry in T''. We then construct M_r, the matrix representing the transposition $(r, r+1)$, as follows:

• If r and $r+1$ are on the same row of T_i then we place the number 1 in the (i,i) entry of M_r.

• If r and $r+1$ are on the same column of T_i then we place the number -1 in the (i,i) entry of M_r.

• Otherwise r and $r+1$ are on neither the same row nor the same column of T_i, and if we swap them around then we will get another standard tableau, call it T_j. If $i < j$, and r and $r+1$ are entries (u,v) and (U,V) of T_i, respectively, then we must have $u < U$ and $v > V$ because of the lexicographic ordering of the tableaux. We then place the numbers $-\rho, 1-\rho^2, 1$, and ρ in the $(i,i), (i,j), (j,i)$, and (j,j) entries of M_r, respectively, where $1/\rho := (U-u) + (v-V)$.

• Zeros are placed everywhere else in M_r.

Appendix II. Simple groups

Here we briefly describe the finite simple groups which are mentioned in Corollary 2. For a long time, the only known sporadic simple groups were the Mathieu groups: $M_{11}, M_{12}, M_{22}, M_{23}$, and M_{24}. These groups are highly transitive permutation groups where the subscript denotes the number of letters in the defining permutations.

Many other sporadic simple groups are obtained by examining the Leech lattice, a 24-dimensional lattice which is defined in terms of the Mathieu group M_{24}. One can obtain $J_2, HS, Suz, C1$ and $C3$ by examining the automorphism group of the Leech lattice. In some cases these groups are realized as automorphism groups of the Leech lattice which stabilize certain low dimensional sublattices, and in other cases they are realized as the full automorphism group of the Leech lattice with an enlarged ring of definition.

The monster group M is the largest of the sporadic simple groups. Several of the sporadic groups are non-abelian composition factors for the centralizer of an element in M. The *Baby Monster* BM is constructed in this manner.

The only remaining sporadic simple group occuring in Corollary 2 is Ru, the Rudvalis group. This group is realized as a 28 dimensional matrix group over the finite field with 2 elements. For more on the sporadic simple groups see [8].

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