Tohoku Math. J.
64 (2012), 287–315

DEFICIENCIES OF HOLOMORPHIC CURVES IN ALGEBRAIC VARIETIES

Dedicated to Professor Junjiro Noguchi on his 60th birthday

YOSHIHIRO AIHARA

(Received October 27, 2009, revised September 26, 2011)

Abstract. We study property of Nevanlinna's deficiency as functions on linear systems in smooth complex projective algebraic varieties. We give a structure theorem for the set of deficient divisors. This structure theorem yields that the set of values of deficiency is at most countable. Moreover, we have a correspondence between the deficiencies and the linear systems.

Introduction. Since Cartan [C], Weyl-Weyl [W1] and Ahlfors [A] established the classical theory of holomorphic curves in complex projective spaces, many works have been done for its generalization in various ways. The aim of this paper is to study the properties of the deficiencies of holomorphic curves as functions on linear systems on algebraic varieties. We define the deficiency for the base locus of a linear system by means of the new language in the value distribution theory for coherent ideal sheaves due to Noguchi-Winkelmann-Yamanoi [NWY]. We give a structure theorem for the set of deficient divisors and prove that the set of values of deficiency is at most countable. We also show that the values of deficiency correspond to the families of linear systems with the non-empty base loci.

Let M be a smooth complex projective algebraic variety and $L \to M$ an ample line bundle. For a transcendental holomorphic curve $f : \boldsymbol{C} \to M$, Nevanlinna's deficiency $\delta_f(D)$ of f can be regraded as a function on the complete linear system $|L|$. It is an important problem to study the structure of the set

$$\{D \in |L|\,;\, \delta_f(D) > 0\}$$

of deficient divisors (cf. Stoll [St, p. 54]). When the dimension of M is greater than one, the structure of the above set is very complicated. Shiffman constructed a dominant holomorphic mapping $f : \boldsymbol{C}^2 \to \boldsymbol{P}_2(\boldsymbol{C})$ and linear pencils of lines in $\boldsymbol{P}_2(\boldsymbol{C})$ on which $\delta_f(D)$ is a constant function [Shi1]. It follows from this result that there exist algebraically non-degenerate holomorphic curves in $\boldsymbol{P}_2(\boldsymbol{C})$ with the same property. In his construction, each pencil has the non-empty base locus. He also considered a proximity function for higher codimensional subvarieties of a special type. Shiffman's result is considered to be a prototype of our study

2000 *Mathematics Subject Classification.* Primary 32H30.
Key words and phrases. Nevanlinna theory, holomorphic curve, deficiency, linear system.
This research was partially supported by Grant-in-Aid for Scientific Research, ((C) No. 21540205), Japan Society for the Promotion of Science.

on defect functions. There have been several contributions from this point of view. Among these, we are especially interested in works by Ochiai [O] and Nochka [Nc1], [Nc2]. We now recall their results.

We let $\Gamma(M, L)$ denote the space of all holomorphic sections of $L \to M$ and $|L|$ the complete linear system defined by L. Let $W \subseteq \Gamma(M, L)$ be a linear subspace with $l_0 + 1 = \dim W \geq 2$. Denote by Λ the linear system determined by W, that is, $\Lambda = \boldsymbol{P}(W)$. The linear system Λ may have the non-empty base locus. Let $D_1, \dots, D_q$ be divisors in Λ such that $D_j = (\sigma_j)$ for $\sigma_j \in W$. We first give a definition of *subgeneral position*. Set $Q = \{1, \dots, q\}$ and take a basis $\psi_0, \dots, \psi_{l_0}$ of W. We write

$$\sigma_j = \sum_{k=0}^{l_0} c_{jk}\psi_k$$

for each $j \in Q$. For a subset $R \subseteq Q$, we define a matrix A_R by $A_R = (c_{jk})_{j\in R, 0\leq k\leq l_0}$.

DEFINITION 0.1. Let $N \geq l_0$ and $q \geq N + 1$. We say that $D_1, \dots, D_q$ are in N-subgeneral position in Λ if

$$\operatorname{rank} A_R = l_0 + 1 \quad \text{for every subset} \quad R \subseteq Q \quad \text{with} \quad \sharp R = N + 1\,.$$

If they are in l_0-subgeneral position, we simply say that they are in general position.

The above definition is different from the usual one (cf. [No1, p. 339]). In fact, the divisors $D_1, \dots, D_q$ are usually said to be in N-subgeneral position in Λ provided that

$$\bigcap_{j\in R} D_j = \emptyset \quad \text{for every subset} \quad R \subseteq Q \quad \text{with} \quad \sharp R = N + 1.$$

However, the divisors $D_1, \dots, D_q$ may have a common point when they are in N-subgeneral position in the sense of Definition 0.1. Thus our definition is weaker than the usual one. Throughout this paper we use "N-subgeneral position" in the sense of Definition 0.1.

Let $f : \boldsymbol{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ, namely, the image of f is not contained in the support of any divisor in Λ. Then Ochiai showed that there exists a non-negative constant e_0 such that $\delta_f(D)$ defines a function $\delta_f : \Lambda \to [e_0, 1]$ and the set $\{D \in \Lambda\,; \delta_f(D) > e_0\}$ is a null set for the Lebesgue measure of Λ. The constant e_0 is given by

$$e_0 = \liminf_{r\to+\infty} \frac{m_f(r, \Lambda)}{T_f(r, L)}\,,$$

where $m_f(r, \Lambda)$ is a nondecreasing function in r depending on f and Λ. Now we assume that $D_1, \dots, D_q \in \Lambda$ are in general position. The defect relation

$$\sum_{j=1}^{q} (\delta_f(D_j) - e_0) \leq (1 - e_0)(l_0 + 1)$$

holds under a certain condition on the growth of f. In the classical case where M is a complex projective space $\boldsymbol{P}_n(\boldsymbol{C})$ and Λ is the complete linear system of the hyperplane bundle $\mathcal{O}_{\boldsymbol{P}_n}(1)$, we see $e_0 = 0$. Hence the classical defect relation is contained in Ochiai's one. Until now, the geometric significance of the constant e_0 has not been understood well. Nochka [Nc1] lately pointed out that the existence of a subset $E(f)$ of $\boldsymbol{R}^+$ at which the characteristic function of f has abnormally fast growth gives an influence on deficiencies. In order to avoid this problem, he introduced a notion of modified deficiency $\tilde{\delta}_f(H)$ for hyperplanes in $\boldsymbol{P}_n(\boldsymbol{C})$ from an analytic point of view. He improved Ochiai's theorem in the case where $M = \boldsymbol{P}_n(\boldsymbol{C})$ and $L = \mathcal{O}_{\boldsymbol{P}_n}(1)$. In particular, he obtained some remarkable results on the structure of the set of deficient hyperplanes and gave an example that shows his result is optimal. The works of Ochiai and Nochka are based on some methods in integral geometry as in [A] and [W1].

In this paper, we will give generalizations of the results of Ochiai and Nochka, especially theorems concerning the structure of the set of deficient divisors and the set of values of defect functions as we announced in [Ai]. Our first goal is to give a geometric meaning of the constant e_0. Let $\mathcal{I}_0$ be the coherent ideal sheaf of the structure sheaf $\mathcal{O}_M$ over M that defines the base locus of Λ as a complex analytic space. We notice here that $\mathcal{I}_0$ is a subsheaf of $\mathcal{O}_M$. We denote by B_Λ the base locus as a complex analytic space and by $\operatorname{Bs}\Lambda$ its support, that is,

$$B_\Lambda = (\operatorname{Supp}(\mathcal{O}_M/\mathcal{I}_0),\ \mathcal{O}_M/\mathcal{I}_0) \quad \text{and} \quad \operatorname{Bs}\Lambda = \operatorname{Supp}(\mathcal{O}_M/\mathcal{I}_0).$$

We also let $m_f(r, \mathcal{I}_0)$ denote the proximity function for $\mathcal{I}_0$. Then it is the proximity function for B_Λ. We define a defect of f for B_Λ by

$$\delta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{m_f(r, \mathcal{I}_0)}{T_f(r, L)}.$$

By making use of Crofton type formula, we see

$$e_0 = \delta_f(B_\Lambda).$$

This gives us a geometric meaning of e_0. In particular, $e_0 = 0$ if $\operatorname{Bs}\Lambda = \emptyset$. It is worth noting that there exist f and Λ with the non-empty base locus such that f does not hit $\operatorname{Bs}\Lambda$ but $0 < \delta_f(B_\Lambda) < 1$ (see Section 6). Hence $\delta_f(B_\Lambda)$ has a potential theoretical character. We have then the following inequality of the second main theorem type:

THEOREM 0.2. *Let $f : \boldsymbol{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ and $D_1, \ldots, D_q \in \Lambda$ divisors in N-subgeneral position. Then an inequality*

$$(q - 2N + l_0 - 1)(T_f(r, L) - m_f(r, \mathcal{I}_0)) \le \sum_{j=1}^{q} N(r, f^*D_j) + S_f(r)$$

holds, where $S_f(r) = O(\log T_f(r, L)) + O(\log r)$ as $r \to +\infty$ except on a Borel subset $E \subseteq [1, +\infty)$ with finite measure.

We let $\tilde{\delta}_f(D)$ and $\tilde{\delta}_f(B_\Lambda)$ denote modified deficiencies in the sense of Nochka (for the definition, see Section 4). In general, $\delta_f(D) \le \tilde{\delta}_f(D)$ and $\tilde{\delta}_f(D) = \delta_f(D)$ if f is of finite type. We also see that $\tilde{\delta}_f(D) \ge \tilde{\delta}_f(B_\Lambda)$ for all $D \in \Lambda$. We get the following defect relation without assuming any growth condition on f.

THEOREM 0.3. *Let Λ, f and $D_1, \dots, D_q$ be as in Theorem* 0.2. *Then*

$$\sum_{j=1}^{q}(\tilde{\delta}_f(D_j) - \tilde{\delta}_f(B_\Lambda)) \leq (1 - \tilde{\delta}_f(B_\Lambda))(2N - l_0 + 1)\,.$$

We consider the sets $\mathfrak{D}_f$ and $\widetilde{\mathfrak{D}}_f$ of divisors defined by

$$\mathfrak{D}_f = \{D \in \Lambda\,;\, \delta_f(D) > \delta_f(B_\Lambda)\} \quad \text{and} \quad \widetilde{\mathfrak{D}}_f = \{D \in \Lambda\,;\, \tilde{\delta}_f(D) > \tilde{\delta}_f(B_\Lambda)\}\,.$$

We can show that $\mathfrak{D}_f$ and $\widetilde{\mathfrak{D}}_f$ are $\mathcal{P}$-polar in Λ. In particular, the Hausdorff dimensions of those sets are at most $2l_0 - 2$. By making use of the above defect relation, we have a structure theorem for $\widetilde{\mathfrak{D}}_f$.

THEOREM 0.4. *The set $\widetilde{\mathfrak{D}}_f$ is a union of at most countably many linear systems included in Λ.*

By the above theorem, we have a family $\{\Lambda_j\}$ of at most countably many linear systems in Λ such that $\widetilde{\mathfrak{D}}_f = \bigcup_j \Lambda_j$. We define $\mathfrak{L} = \{\Lambda_j\} \cup \{\Lambda\}$. Let $\tilde{\delta}_f(\Lambda)$ be the set of values of the function $\tilde{\delta}_f : \Lambda \to [0,\ 1]$. Then we can establish the correspondence between the values in $\tilde{\delta}_f(\Lambda)$ and the subfamilies of $\mathfrak{L}$, which is the main result in this paper.

MAIN THEOREM 0.5. *The set $\tilde{\delta}_f(\Lambda)$ is an at most countable subset of* $[0,\ 1]$. *For each $\alpha \in \tilde{\delta}_f(\Lambda)$, there exists a unique finite subfamily $\mathfrak{L}_\alpha = \{\Lambda_j^{(\alpha)}\}$ of $\mathfrak{L}$ such that $\alpha = \tilde{\delta}_f(B_{\Lambda_j^{(\alpha)}})$ for all $\Lambda_j^{(\alpha)} \in \mathfrak{L}_\alpha$ and $\alpha \neq \tilde{\delta}_f(B_{\Lambda_j})$ for all $\Lambda_j \in \mathfrak{L} \setminus \mathfrak{L}_\alpha$.*

It is known that holomorphic curves without defect are dense in the space of holomorphic curves $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ with respect to a certain kind of topology. Moreover, we can eliminate all defects of f by a small deformation of f (see [M1] and [M2]). In [O], Ochiai exhibits an example of $\delta_f(B_\Lambda) = 1$. We can show the existence of holomorphic curves with $0 < \delta_f(B_\Lambda) < 1$ in the case where $M = \boldsymbol{P}_n(\boldsymbol{C})$ and $L = \mathcal{O}_{\boldsymbol{P}_n}(1)$.

THEOREM 0.6. *Let $\Lambda \subseteq |\mathcal{O}_{\boldsymbol{P}_n}(1)|$ and suppose that* $\mathrm{Bs}\,\Lambda \neq \emptyset$. *Let e_0 be an arbitrary positive number less than one. Then there exists an algebraically non-degenerate transcendental holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ of finite type such that $e_0 = \delta_f(B_\Lambda)$.*

In Sections 1 and 2, we recall some facts in Nevanlinna theory for holomorphic curves and basic results in value distribution theory for coherent ideal sheaves. In Sections 3 and 4, we will proved Theorems 0.2 and 0.3. In the proof of Theorem 0.3, we need to give a precise estimate for the error term in Theorem 0.2. Its proof is somewhat complicated. Hence we will give the proof in Section 7. By making use of results in Section 4, we give the structure theorem for the set of deficient divisors in Section 5. The proofs of Theorems 0.4 and 0.5 are based on Nochka's idea. In Section 6, we prove Theorem 0.6 and discuss the existence of holomorphic curves with deficiencies.

Acknowledgment. The author would like to thank Professors Yoshihisa Kitagawa, Seiki Mori and Junjiro Noguchi for their useful advice and valuable comments. He is also grateful to the referee for his/her valuable comments.

1. Preliminaries. We recall some known facts on Nevanlinna theory for holomorphic curves. For details, see [NO] and [Sh].

Let z be the natural coordinate in $\boldsymbol{C}$ and $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$. Set

$$\Delta(r) = \{z \in \boldsymbol{C}\,;\, |z| < r\} \quad \text{and} \quad C(r) = \{z \in \boldsymbol{C}\,;\, |z| = r\}\,.$$

For a (1,1)-current φ of order zero on $\boldsymbol{C}$, we set

$$N(r,\ \varphi) = \int_1^r \langle \varphi,\ \chi_{\Delta(t)} \rangle \frac{dt}{t}\,,$$

where $\chi_{\Delta(r)}$ denotes the characteristic function of $\Delta(r)$. Let M be a projective algebraic manifold and $L \to M$ a positive line bundle over M. We denote by $\Gamma(M, L)$ the space of all holomorphic sections of $L \to M$. Let $|L| = \boldsymbol{P}(\Gamma(M, L))$ be the complete linear system defined by L. Denote by $\|\cdot\|$ a hermitian fiber metric in L and by ω its Chern form. Let $f : \boldsymbol{C} \to M$ be a meromorphic mapping. We set

$$T_f(r,\ L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^*\omega\,,$$

and call it the characteristic function of f with respect to L. If

$$\liminf_{r \to +\infty} \frac{T_f(r, L)}{\log r} = +\infty\,,$$

then f is said to be *transcendental*. We define the order ρ_f of $f : \boldsymbol{C} \to M$ by

$$\rho_f = \limsup_{r \to +\infty} \frac{\log T_f(r, L)}{\log r}\,.$$

We notice that the definition of ρ_f is independent of the choice of positive line bundle $L \to M$. We call f of finite type if $\rho_f < +\infty$. Let $D = (\sigma) \in |L|$ with $\|\sigma\| \le 1$ on M. Assume that $f(\boldsymbol{C})$ is not contained in Supp$D$. We define the proximity function of D by

$$m_f(r,\ D) = \int_{C(r)} \log\left(\frac{1}{\|\sigma(f(z))\|}\right) \frac{d\theta}{2\pi}\,.$$

Then we have the following first main theorem for holomorphic curves:

THEOREM 1.1. *Let $L \to M$ be a line bundle over M and let $f : \boldsymbol{C} \to M$ be a nonconstant holomorphic curve. Then*

$$T_f(r, L) = N(r, f^*D) + m_f(r, D) + O(1)$$

for $D \in |L|$ with $f(\boldsymbol{C}) \not\subseteq$ Supp$D$, where $O(1)$ stands for a bounded term as $r \to +\infty$.

Let f and D be as in Theorem 1.1. We define Nevanlinna's deficiency $\delta_f(D)$ by

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r,\ D)}{T_f(r,\ L)}\,.$$

We have then the defect functions δ_f defined on $|L|$. If $\delta_f(D) > 0$, then D is called a deficient divisor in the sense of Nevanlinna. Let $E = \sum_j \nu_j p_j$ be an effective divisor on

$\boldsymbol{C}$, where $\nu_j \in \boldsymbol{Z}^+$ and $p_j \in \boldsymbol{C}$ are distinct points. For a positive integer k, we define the truncated counting function of E by

$$N_k(r,\ E) = \sum_j \min\{k,\ \nu_j\}\, N(r,\ p_j)\,.$$

In general, for an effective divisor D on M, we write $\mathcal{O}_M(D)$ for the line bundle determined by D. We now consider the case where $M = \boldsymbol{P}_n(\boldsymbol{C})$. Let $\mathcal{O}_{\boldsymbol{P}_n}(1) \to \boldsymbol{P}_n(\boldsymbol{C})$ be the hyperplane bundle over $\boldsymbol{P}_n(\boldsymbol{C})$ and ω_0 the Fubini-Study form on $\boldsymbol{P}_n(\boldsymbol{C})$. In the case where $M = \boldsymbol{P}_n(\boldsymbol{C})$ and $L = \mathcal{O}_{\boldsymbol{P}_n}(1)$, we always take ω_0 for ω and we simply write $T_f(r)$ for $T_f(r, \mathcal{O}_{\boldsymbol{P}_n}(1))$. The following form of $T_f(r)$ is due to Cartan [C]:

$$T_f(r) = \int_{C(r)} \log \max_{0 \le j \le n} |f_j(z)| \frac{d\theta}{2\pi} + O(1)\,, \tag{1.2}$$

where $f = (f_0, \ldots, f_n)$ is a reduced representation of f. For a meromorphic function f on $\boldsymbol{C}$ and a point $a \in \boldsymbol{P}_1(\boldsymbol{C})$, we write $N(r,\ a,\ f)$ for $N(r,\ f^*a)$. We also write $m(r,\ f)$ for $m_f(r,\ \infty)$. Let $L \to \boldsymbol{P}_n(\boldsymbol{C})$ be a positive line bundle over $\boldsymbol{P}_n(\boldsymbol{C})$. Then $L = \mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes d}$ for some positive integer d and $D \in |L|$ is a hypersurface of degree d in $\boldsymbol{P}_n(\boldsymbol{C})$. It is clear that

$$T_f(r,\ L) = d\, T_f(r) + O(1)\,.$$

We have the following second main theorem for holomorphic curves due to Cartan-Nochka:

THEOREM 1.3. *Let $H_1, \ldots, H_q$ be hyperplanes in N-subgeneral position in $|\mathcal{O}_{\boldsymbol{P}_n}(1)|$. Let $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ be a nonconstant holomorphic curve that is non-degenerate with respect to $\mathcal{O}_{\boldsymbol{P}_n}(1)$. We let W_f denote the Wronskian of f. Then*

$$(q - 2N + n - 1)\, T_f(r) \le \sum_{j=1}^{q} N(r,\ f^*H_j) - N(r,\ (W_f)_0) + S_f(r)\,,$$

where

$$S_f(r) = O(\log T_f(r)) + O(\log r)$$

as $r \to +\infty$ except on a Borel subset $E \subseteq [1, +\infty)$ with finite measure. If f is of finite type, then $E = \emptyset$.

For a simple proof, see [No2].

REMARK 1.4. The above second main theorem can be written in the following form that involves the truncated counting function:

$$(q - 2N + n - 1)\, T_f(r) \le \sum_{j=1}^{q} N_n(r,\ f^*H_j) + S_f(r)\,.$$

2. Value distribution theory for coherent ideal sheaves. In this section we recall some basic facts in value distribution theory for coherent ideal sheaves and give Crofton type formula needed later. For details, see [No1, Chapter 2] and [NWY]. We keep the same notation as in Section 1. Let $f : \boldsymbol{C} \to M$ be a holomorphic curve and $\mathcal{I}$ a coherent ideal sheaf of the structure sheaf $\mathcal{O}_M$ of M. Let $\mathcal{U} = \{U_j\}$ be a finite open covering of M with a partition of unity $\{\eta_j\}$ subordinate to $\mathcal{U}$. We can assume that there exist finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathcal{I})$ such that every stalk $\mathcal{I}_p$over $p \in U_j$ is generated by germs $(\sigma_{j1})_p, \dots, (\sigma_{jl_j})_p$. Set

$$\rho_{\mathcal{I}}(p) = \Bigg(\sum_j \eta_j(p) \sum_{k=1}^{l_j} \big|\sigma_{jk}(p)\big|^2 \Bigg)^{1/2} .$$

We take a positive constant C such that $C\rho_{\mathcal{I}}(p) \le 1$ for all $p \in M$. Set

$$\phi_{\mathcal{I}}(p) = -\log \rho_{\mathcal{I}}(p)$$

and call it the proximity potential for $\mathcal{I}$. It is easy to verify that $\phi_{\mathcal{I}}$ is well defined up to addition of a bounded continuous function on M. We now define the proximity function $m_f(r, \mathcal{I})$ of f for $\mathcal{I}$, or equivalently, for the complex analytic subspace (may be non-reduced)

$$Y = (\mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}), \mathcal{O}_M/\mathcal{I}) ,$$

by

$$m_f(r, \mathcal{I}) = \int_{C(r)} f^*\phi_{\mathcal{I}}(z) \frac{d\theta}{2\pi}$$

provided that $f(\boldsymbol{C})$ is not contained in Supp$\,Y$. For $z_0 \in f^{-1}(\mathrm{Supp}\ Y)$, we can choose an open neighborhood U of z_0 and a positive integer ν such that

$$f^*\mathcal{I} = ((z - z_0)^\nu) \quad \text{on} \ \ U .$$

Then we see

$$\log \rho_{\mathcal{I}}(f(z)) = \nu \log |z - z_0| + h_U(z) \quad \text{for} \ \ z \in U,$$

where h_U is a C^∞-function on U. Thus we have the counting functions $N(r, f^*\mathcal{I})$ and $N_l(r, f^*\mathcal{I})$ as in Section 1. Moreover, we set

$$\omega_{\mathcal{I},f} = -dd^c h_U \quad \text{on} \ \ U ,$$

and thus obtain a well-defined smooth (1, 1)-form on $\boldsymbol{C}$. Define the characteristic function $T_f(r, \mathcal{I})$ of f for $\mathcal{I}$ by

$$T_f(r, \mathcal{I}) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} \omega_{\mathcal{I},f} .$$

We summarize the basic facts on value distribution theory for coherent ideal sheaves due to Noguchi-Winkelmann-Yamanoi as follows [NWY, Theorem 2.9]:

THEOREM 2.1. *Let $f : \boldsymbol{C} \to M$ and $\mathcal{I}$ be as above. Then the following hold*:

(i) (First Main Theorem) $T_f(r, \mathcal{I}) = N(r, f^*\mathcal{I}) + m_f(r, \mathcal{I}) + O(1)$.

(ii) *If* $L \to M$ *be an ample line bundle, then* $T_f(r, \mathcal{I}) = O(T_f(r, L))$.

(iii) *Let* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *be coherent ideal sheaves. Suppose that* $f(\boldsymbol{C})$ *is not contained in* $\mathrm{Supp}(\mathcal{O}_M/\mathcal{I}_1 \otimes \mathcal{I}_2)$. *Then*

$$T_f(r, \mathcal{I}_1 \otimes \mathcal{I}_2) = T_f(r, \mathcal{I}_1) + T_f(r, \mathcal{I}_2) + O(1)$$

and

$$m_f(r, \mathcal{I}_1 \otimes \mathcal{I}_2) = m_f(r, \mathcal{I}_1) + m_f(r, \mathcal{I}_2) + O(1)\,.$$

(iv) *Let* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *and* f *be as in* (iii). *If* $\mathcal{I}_1 \subset \mathcal{I}_2$, *then*

$$m_f(r, \mathcal{I}_2) \le m_f(r, \mathcal{I}_1) + O(1)\,.$$

For a proof, see [NWY, §2].

When $\mathcal{I}$ defines an effective divisor D on M, we easily see

$$T_f(r, \mathcal{I}) = T_f(r, \mathcal{O}_M(D)) + O(1) \quad \text{and} \quad m_f(r, \mathcal{I}) = m_f(r, D) + O(1)\,.$$

Let $L \to M$ be an ample line bundle and $W \subseteq \Gamma(M, L)$ a subspace with $\dim W \ge 2$. Let $\Lambda = \boldsymbol{P}(W)$. We define a coherent ideal sheaf $\mathcal{I}_0$ in the following way: For each $p \in M$, the stalk $\mathcal{I}_{0,p}$ is generated by all germs σ_p for $\sigma \in W$. Then $\mathcal{I}_0$ defines the base locus B_Λ of Λ as a complex analytic subspace, that is,

$$B_\Lambda = (\mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}_0),\ \mathcal{O}_M/\mathcal{I}_0)\,.$$

Hence $\mathrm{Bs}\,\Lambda = \mathrm{Supp}(\mathcal{O}_M/\mathcal{I}_0)$. In the case where B_Λ contains non-zero effective divisor on M, the sheaf $\mathcal{I}_0$ can be written as $\mathcal{I}_0 = \mathcal{I}_1 \otimes \mathcal{I}_2$. Here $\mathcal{I}_1$ defines an effective divisor on M and $\mathrm{codim}\,\mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}_2) \ge 2$. Otherwise, put $\mathcal{I}_1 = \mathcal{O}_M$ and we have $\mathcal{I}_0 = \mathcal{I}_2$ with $\mathrm{codim}\,\mathrm{Supp}\,(\mathcal{O}_M/\mathcal{I}_2) \ge 2$. We now give some Crofton type formulas. Let $f : \boldsymbol{C} \to M$ be a nonconstant holomorphic curve that is non-degenerate with respect to Λ. We fix a basis $\{\sigma_0, \dots, \sigma_l\}$ for W. We identify Λ with $\boldsymbol{P}_l(\boldsymbol{C})$ via an isomorphism $T : \Lambda \to \boldsymbol{P}_l(\boldsymbol{C})$ defined by

$$\tilde{T} : \sigma = \sum_{j=1}^{l} c_j \sigma_j \mapsto (c_0, \dots, c_l)\,.$$

We denote by $[\sigma]$ the elements in $\boldsymbol{P}_l(\boldsymbol{C})$ corresponding to $D = (\sigma)$ in Λ. For a function $g(D)$ defined on Λ, we define the integration of g over Λ by

$$\int_{D \in \Lambda} g(D)\,d\mu(D) = \int_{[\sigma] \in \boldsymbol{P}_l(\boldsymbol{C})} g(T^*[\sigma])\,d\mu([\sigma])\,,$$

where μ is the invariant measure on $\boldsymbol{P}_l(\boldsymbol{C})$ normalized as $\mu(\boldsymbol{P}_l(\boldsymbol{C})) = 1$. Then we have well-known Crofton's formula:

THEOREM 2.2. *Suppose that* $\mathrm{Bs}\,\Lambda = \emptyset$. *Then*

$$T_f(r, L) = \int_{D \in \Lambda} N(r, f^*D)\,d\mu(D) + O(1)\,.$$

For a proof, see [Shi1] and [Shi2]. The assumption $\mathrm{Bs}\,\Lambda = \emptyset$ is essential in Theorem 2.2. In the case where $\mathrm{Bs}\,\Lambda \neq \emptyset$, we have the following generalized Crofton's formula due to Ryoichi Kobayashi (see [No1, Theorem 2.3.28]):

THEOREM 2.3. *Suppose that* $\mathrm{Bs}\,\Lambda \neq \emptyset$ *and* $f(\boldsymbol{C}) \not\subseteq \mathrm{Bs}\,\Lambda$. *Then*

$$\int_{D\in\Lambda} m_f(r,\ D)d\mu(D) = m_f(r,\ \mathcal{I}_0) + O(1)\,,$$

and hence

$$T_f(r,\ L) = \int_{D\in\Lambda} N(r,\ f^*D)d\mu(D) + m_f(r,\ \mathcal{I}_0) + O(1)\,.$$

For a reader's convenience, we give a proof here.

PROOF. We first notice that there exists an effective divisor D on M such that $L = \mathcal{O}_M(D)$. We write $\mathcal{I}_0 = \mathcal{I}_1 \otimes \mathcal{I}_2$ as above. Then $\mathcal{I}_1$ defines an effective divisor D_1 on M. Take $\tau_1 \in \Gamma(M,\ \mathcal{O}_M(D_1))$ so that $(\tau_1) = D_1$. We may assume that $\{\sigma_0, \dots, \sigma_l\}$ is a basis for W. For each j, we write

$$\sigma_j = \tau_1 \otimes \tau_{2j} \quad \text{for} \quad \tau_{2j} \in \Gamma(M,\ \mathcal{O}_M(D - D_1))\,.$$

An arbitrary section $\sigma \in \Gamma(M,\ L)$ can be written as

$$\sigma = \tau_1 \otimes \left(\sum_{j=0}^{l} c_j \tau_{2j}\right), \quad \sum_{j=0}^{l} |c_j|^2 = 1\,.$$

Hence

$$-\log\|\sigma(p)\| = -\log\|\tau_1(p)\| + \phi_{\mathcal{I}_2} + \log\frac{\left(\sum|\tau_{2j}(p)|^2\right)^{1/2}}{|\sum c_j\tau_{2j}(p)|} + b(p)\,,$$

where $b(p)$ is a C^∞-function on M. Thus we see

$$m_f(r,\ (\sigma)) = m_f(r,\ D_1) + m_f(r,\ \mathcal{I}_2) + \int_{C(r)} \log\frac{\left(\sum|\tau_{2j}(f(z))|^2\right)^{1/2}}{|\sum c_j\tau_{2j}(f(z))|}\frac{d\theta}{2\pi} + O(1)\,.$$

On the other hand, we have

$$\int_{[\sigma]\in \boldsymbol{P}_l(\boldsymbol{C})} \log\frac{\left(\sum|\tau_{2j}|^2\right)^{1/2}}{|\sum c_j\tau_{2j}|} d\mu([\sigma]) = C(l)\,,$$

where $C(l)$ is a constant depending only on l. In fact, a direct calculation gives

$$C(l) = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{l}\right)$$

(see [W1, p. 519]). Thus we obtain

$$\int_{D\in\Lambda} m_f(r,\ D)d\mu(D) = m_f(r,\ D_1) + m_f(r,\ \mathcal{I}_2) + O(1)\,.$$

This yields our assertion. □

3. Inequality of the second main theorem type. This section is devoted to giving an inequality of the second main theorem type for holomorphic curves that gives an improvement of the results of Ochiai and Nochka.

Let $L \to M$ be an ample line bundle and $W \subseteq \Gamma(M, L)$ a linear subspace with $\dim W = l_0 + 1 \geq 2$. Let $\Lambda = \boldsymbol{P}(W)$ be a linear system included in $|L|$ and $\mathcal{I}_0$ the coherent ideal sheaf of $\mathcal{O}_M$ that defines the base locus of Λ as a complex analytic subspace B_Λ. Let $f : \boldsymbol{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ. Let ν be a positive integer. Here we assume Lemma 4.3 which will be proved in Section 7. We define $S_f(r; \nu)$ by

$$S_f(r\,; \nu) = C_1 \log^+ T_f(r, L) + C_2 \log r + C_3 \log \nu + C_4\,,$$

where $C_1, \dots, C_4$ are the positive constants which satisfy the inequality in Lemma 4.3. We will show the following theorem of the second main theorem type.

THEOREM 3.1. *Let $f : \boldsymbol{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ. Let $D_1, \dots, D_q \in \Lambda$ be divisors in N-subgeneral position. Then*

$$(q - 2N + l_0 - 1)\,(T_f(r, L) - m_f(r, \mathcal{I}_0)) \leq \sum_{j=1}^{q} N(r, f^* D_j) + S_f(r; \nu)$$

as $r \to +\infty$ except on a Borel subset $E \subseteq [1, +\infty)$ with finite measure. The exceptional set E depends on f and ν. If f is of finite type, then $E = \emptyset$.

PROOF. Let $\Phi_\Lambda : M \to \boldsymbol{P}(W^*)$ be a natural meromorphic mapping defined by Λ, where W^* is the dual of W (cf. [NO, p. 68]). Then we have the linearly non-degenerate holomorphic curve

$$F_\Lambda = f \circ \Phi_\Lambda : \boldsymbol{C} \to \boldsymbol{P}(W^*)\,.$$

Let $\{\psi_0, \dots, \psi_{l_0}\}$ be a basis for W. As in Section 2, we identify Λ with $\boldsymbol{P}_{l_0}(\boldsymbol{C})$ via an isomorphism $T : \Lambda \to \boldsymbol{P}_{l_0}(\boldsymbol{C})$ defined by

$$\tilde{T} : \sigma = \sum_{j=0}^{l_0} c_j \psi_j \mapsto (c_0, \dots, c_{l_0})\,.$$

Set $T(D) = [\sigma]$ when $D = (\sigma)$. There exists an entire function f_0 on $\boldsymbol{C}$ such that if we set

$$\Psi(z) = \psi_0(f(z))/f_0(z)\,, \tag{3.2}$$

then $\Psi : \boldsymbol{C} \to L$ is a nonvanishing holomorphic mapping. We define nonzero constant entire functions $f_1, \dots, f_{l_0}$ on $\boldsymbol{C}$ such that

$$\psi_j(f(z)) = f_j(z)\Psi(z) \quad \text{for } j = 1, \dots, l_0\,. \tag{3.3}$$

Then there exists an entire function ψ on $\boldsymbol{C}$ such that $\{f_0/\psi, \dots, f_{l_0}/\psi\}$ has no common zero in $\boldsymbol{C}$. Hence we have a reduced representation

$$\tilde{F}_\Lambda = (f_0/\psi, \dots, f_{l_0}/\psi)$$

of $F_\Lambda : \boldsymbol{C} \to \boldsymbol{P}(W^*) \cong \boldsymbol{P}_{l_0}(\boldsymbol{C})$. Note that F_Λ may be a rational mapping.

Let B be the zero divisor of ψ on $\boldsymbol{C}$ and let $\mathcal{I}_0$ be the defining ideal sheaf of B_Λ. Then it is easy to verify that $f^*\mathcal{I}_0$ is the defining ideal sheaf of B. Hence we have

$$N(r,\ f^*\mathcal{I}_0) = N(r,\ B) + O(1)\,.$$

Let $D = (\sigma) \in \Lambda$ and $[\sigma] = (c_0 : \cdots : c_{l_0})$. We define a hyperplane H_D in $\boldsymbol{P}_{l_0}(\boldsymbol{C})$ by

$$H_D : c_0\zeta_0 + \cdots + c_{l_0}\zeta_{l_0} = 0\,,$$

where $(\zeta_0, \dots, \zeta_{l_0})$ is a homogeneous coordinate system in $\boldsymbol{P}_{l_0}(\boldsymbol{C})$. It is easy to see that

$$N(r, f^*D) = N(r, F_\Lambda^* H_D) + N(r,\ f^*\mathcal{I}_0) + O(1)\,. \tag{3.4}$$

Indeed, we write $\sigma = \sum_{j=0}^{l_0} c_j\psi_j$ and $[\sigma] = (c_0 : \cdots : c_{l_0})$. Thus we see

$$(f^*\sigma)(z) = \psi(z)\Big(\sum_{j=0}^{l_0} c_j(f_j(z)/\psi(z))\Big)\Psi(z) = \psi(z)\Big(\sum_{j=0}^{l_0} c_j f_j(z)\Big)\Psi(z).$$

This shows (3.4). Define

$$m_f(r,\ \Lambda) = \int_{C(r)} \log\Big(\sum_{j=0}^{l_0} \|\psi_j(f(z))\|^2\Big)^{-1/2}\frac{d\theta}{2\pi}\,.$$

We will show

$$T_f(r,\ L) - T_{F_\Lambda}(r) = N(r,\ f^*\mathcal{I}_0) + m_f(r,\ \Lambda) + O(1)\,. \tag{3.5}$$

Let ω (resp. ω_0) be the Chern form of $L \to M$ (resp. $\mathcal{O}_{\boldsymbol{P}_n}(1) \to \boldsymbol{P}_{l_0}(\boldsymbol{C})$). By (3.2) and (3.3), we see

$$\begin{aligned}\log\Big(\sum_{j=0}^{l_0}\|\psi_j(f(z))\|^2\Big)^{1/2} &= |\psi(z)|\Big(\sum_{j=0}^{l_0}|f_j(z)/\psi(z)|^2\Big)^{1/2}\|\Psi(z)\| \\ &= \Big(\sum_{j=0}^{l_0}|f_j(z)/\psi(z)|^2\Big)^{1/2}|\psi(z)||f_0(z)|^{-1}\|\psi_0(f(z))\|\,.\end{aligned}$$

Thus we get

$$\begin{aligned}dd^c\log\Big(\sum_{j=0}^{l_0}\|\psi_j(f(z))\|^2\Big)^{-1/2} &= -dd^c\log\|\psi_0(f(z))\| - dd^c\log\Big(\sum_{j=0}^{l_0}|f_j(z)/\psi(z)|^2\Big)^{1/2} \\ &= f^*\omega - F_\Lambda^*\omega_0\end{aligned}$$

for $z \in \boldsymbol{C} \setminus f^{-1}(\mathrm{Bs}\,\Lambda)$. Now, by means of Jensen's formula, we have (3.5).

We next show

$$m_f(r,\ \mathcal{I}_0) = m_f(r,\ \Lambda) + O(1)\,. \tag{3.6}$$

By (3.4), (3.5) and Theorem 2.2, we have

$$\begin{aligned}\int_{D\in\Lambda} N(r,\ f^*D)d\mu(D) &= \int_{H_D\in \boldsymbol{P}_{l_0}(\boldsymbol{C})} N(r,\ F_\Lambda^* H_D)d\mu(H_D) + N(r,\ f^*\mathcal{I}_0) + O(1)\\ &= T_{F_\Lambda}(r) + N(r,\ f^*\mathcal{I}_0) + O(1)\\ &= T_f(r,\ L) - m_f(r,\ \Lambda) + O(1)\,.\end{aligned}$$

On the other hand, by Theorem 2.3, we have

$$\int_{D\in\Lambda} N(r,\ f^*D)d\mu(D) = T_f(r,\ L) - m_f(r,\ \mathcal{I}_0) + O(1)\,.$$

Hence we get (3.6). By (3.5) and (3.6), we conclude that

$$\begin{aligned}T_f(r,\ L) - T_{F_\Lambda}(r) &= N(r,\ f^*\mathcal{I}_0) + m_f(r,\ \mathcal{I}_0)\\ &= T_f(r,\ \mathcal{I}_0) + O(1)\,.\end{aligned}\tag{3.7}$$

By Theorem 1.3, we obtain

$$(q-2N+l_0-1)T_{F_\Lambda}(r) \le \sum_{j=1}^{q} N(r,\ F_\Lambda^* H_{D_j}) - N(r,\ (W_{F_\Lambda})_0) + S_{F_\Lambda}(r)\,.$$

It follows from (3.7) that

$$(q-2N+l_0-1)(T_f(r,\ L) - T_f(r,\ \mathcal{I}_0)) \le \sum_{j=1}^{q} N(r,\ F_\Lambda^* H_{D_j}) - N(r,\ (W_{F_\Lambda})_0) + S_{F_\Lambda}(r)\,.$$

By (3.4) and (3.7), we have the following inequality:

$$\begin{aligned}(q-2N+l_0-1)\,(T_f(r, L) - m_f(r,\ \mathcal{I}_0)) &\le \sum_{j=1}^{q} N(r, f^*D_j) - N(r,\ (W_{\psi F_\Lambda})_0) + S_{F_\Lambda}(r)\\ &\le \sum_{j=1}^{q} N(r, f^*D_j) + S_{F_\Lambda}(r)\,.\end{aligned}$$

By Lemma 4.3 below, we have that $S_{F_\Lambda}(r) \le S_f(r;\nu)$ for $r \in [1,+\infty)\setminus E$, where $E := E(f;\nu)$. For the definition of $E(f;\nu)$, see Section 4. Therefore, we have our assertion. □

REMARK 3.8. The above inequality of the second main theorem type does not involve the truncated counting function. If $N(r,\ f^*\mathcal{I}_0) = o(T_f(r,\ L))$, by the above proof, we see

$$(q-2N+l_0-1)\,(T_f(r, L) - m_f(r,\ \mathcal{I}_0)) \le \sum_{j=1}^{q} N_{l_0}(r, f^*D_j) + o(T_f(r,\ L))\,.$$

In general, it is difficult to give the truncation level for counting functions.

4. A defect relation. In this section, we deduce the defect relation from Theorem 3.1. We first give the definition of modified deficiency. This modification ensures that a defect relation is valid without any condition on the growth of f. In what follows, we assume that f, L and Λ satisfy the same conditions as in Section 3. We let $E(f;\nu)$ denote the set of all $r \in [1, +\infty)$ with

$$T_f(r, L) + \nu \le T_f\left(r + \frac{1}{(T_f(r, L)+\nu)^2}, L\right),$$

where ν is a positive integer. Then the Lebesgue measure $|E(f;\nu)|$ is finite (see Lemma 7.4), and $E(f;\nu_2) \subseteq E(f;\nu_1)$ if $\nu_1 < \nu_2$. Set

$$E(f) = \bigcap_{\nu \in \mathbf{Z}^+} E(f;\nu).$$

We call $E(f)$ the exceptional growth set for f. The existence of non-empty $E(f)$ affects on the deficiency. If $E(f;\nu) = \emptyset$ for some ν, then $E(f) = \emptyset$. In the case where f is of finite type, we set $E(f) = \emptyset$. We now define a modified deficiency following Nochka [Nc1].

We define νth Nevanlinna's deficiency $\delta_f(D;\nu)$ by

$$\delta_f(D;\nu) = \liminf_{\substack{r\to+\infty \\ r \notin E(f;\nu)}} \frac{m_f(r, D)}{T_f(r, L)}.$$

It is clear that $\delta_f(D;\nu_2) \le \delta_f(D;\nu_1)$ if $\nu_1 < \nu_2$. We define the modified deficiency of f in the sense of Nochka by

$$\tilde{\delta}_f(D) = \lim_{\nu\to+\infty} \delta_f(D;\nu).$$

It is clear that $\delta_f(D) \le \tilde{\delta}_f(D)$ and $\delta_f(D) = \tilde{\delta}_f(D)$ if f is of finite type. We define a deficiency $\delta_f(B_\Lambda)$ for B_Λ by

$$\delta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{m_f(r, \mathcal{I}_0)}{T_f(r, L)}.$$

We define $\tilde{\delta}_f(B_\Lambda;\nu)$ and $\tilde{\delta}_f(B_\Lambda)$ by the same way. We consider the set of divisors

$$\mathfrak{D}_f = \{D \in \Lambda\,;\, \delta_f(D) > \delta_f(B_\Lambda)\} \quad \text{and} \quad \widetilde{\mathfrak{D}}_f = \{D \in \Lambda\,;\, \tilde{\delta}_f(D) > \tilde{\delta}_f(B_\Lambda)\}.$$

The following proposition plays an important role in what follows.

PROPOSITION 4.1. *The deficiencies $\delta_f(D)$(resp.$\tilde{\delta}_f(D)$) are not less than $\delta_f(B_\Lambda)$ (resp. $\tilde{\delta}_f(B_\Lambda)$) for all $D \in \Lambda$. The sets $\mathfrak{D}_f$ and $\widetilde{\mathfrak{D}}_f$ are null sets in the sense of Lebesgue measure in Λ.*

PROOF. The assertion for $\delta_f(D)$ is proved in [O]. By Theorem 2.2, we have

$$\int_{D\in\Lambda} \frac{m_f(r, D)}{T_f(r, L)} d\mu(D) = (1+o(1))\frac{m_f(r, \mathcal{I}_0)}{T_f(r, L)}.$$

Hence

$$\liminf_{\substack{r\to+\infty \\ r\notin E(f;\nu)}} \int_{D\in\Lambda} \frac{m_f(r,\ D)}{T_f(r,\ L)} d\mu(D) \leq \liminf_{\substack{r\to+\infty \\ r\notin E(f;\nu)}} \frac{m_f(r,\ \mathcal{I}_0)}{T_f(r,\ L)} .$$

This implies

$$\int_{D\in\Lambda} \tilde{\delta}_f(D;\ \nu) d\mu(D) \leq \tilde{\delta}_f(B_\Lambda;\ \nu) .$$

Hence, by letting $\nu \to +\infty$, we see

$$\int_{D\in\Lambda} \tilde{\delta}_f(D) d\mu(D) \leq \tilde{\delta}_f(B_\Lambda) .$$

On the other hand, by (3.2) and (3.3), we see

$$T_f(r,\ L) - N(r,\ f^*D) \geq m_f(r,\ \mathcal{I}_0) + O(1) ,$$

and hence

$$\frac{m_f(r,\ D)}{T_f(r,\ L)} \geq (1 + o(1)) \frac{m_f(r,\ \mathcal{I}_0)}{T_f(r,\ L)} .$$

Thus we have $\tilde{\delta}_f(D) \geq \tilde{\delta}_f(B_\Lambda)$ for all $D \in \Lambda$. This shows

$$\int_{D\in\Lambda} \tilde{\delta}_f(D) d\mu(D) = \tilde{\delta}_f(B_\Lambda) .$$

This yields that $\widetilde{\mathfrak{D}}_f$ is a null set in the sense of Lebesgue measure in Λ. □

We have then the defect function $\tilde{\delta}_f : \Lambda \to [\tilde{\delta}_f(B_\Lambda),\ 1]$. In the next section we will give more precise estimate for the size of the sets $\mathfrak{D}_f$ and $\widetilde{\mathfrak{D}}_f$. We now show the following defect relation:

THEOREM 4.2. *Let Λ, f and $D_1, \dots, D_q$ be as in Theorem* 3.1. *Then*

$$\sum_{j=1}^{q} (\tilde{\delta}_f(D_j) - \tilde{\delta}_f(B_\Lambda)) \leq (2N - l_0 + 1)(1 - \tilde{\delta}_f(B_\Lambda)) .$$

Let $S_{F_\Lambda}(r)$ be as in the proof of Theorem 3.1. For the proof of Theorem 4.2, we need the following estimate for $S_{F_\Lambda}(r)$:

LEMMA 4.3. *There exist positive constants $C_1, \dots, C_4$ independent of ν such that the estimate*

$$S_{F_\Lambda}(r) \leq C_1 \log^+ T_f(r,\ L) + C_2 \log r + C_3 \log \nu + C_4$$

is valid for $r \in [1,\ +\infty) \setminus E(f;\ \nu)$.

Note that the exceptional set $E(f;\ \nu)$ for $S_{F_\Lambda}(r)$ is independent of a choice of divisors D_j. The definition of the error term $S_{F_\Lambda}(r)$ and the proof of Lemma 4.3 are complicated. Hence we will give the definition of $S_{F_\Lambda}(r)$ and will prove Lemma 4.3 in Section 7.

PROOF OF THEOREM 4.2. By Theorem 3.1 and Lemma 4.3, we see that

$$\sum_{j=1}^{q}\frac{m_f(r, D_j)}{T_f(r,\ L)} \le \frac{qm_f(r,\mathcal{I}_0)}{T_f(r,\ L)} + (2N-l_0+1)\left(1-\frac{m_f(r,\mathcal{I}_0)}{T_f(r,\ L)}\right) + \frac{S_f(r;\nu)}{T_f(r,\ L)}$$

for $r \in [1, +\infty) \setminus E(f;\nu)$. Now, it follows from the definition of limit inferior that there exists a sequence $\{r_i\}$ with $r_i \notin E(f;\nu)$ such that

$$\lim_{i\to+\infty}\frac{m_f(r_i,\ \mathcal{I}_0)}{T_f(r_i,\ L)} = \liminf_{\substack{r\to+\infty\\ r\notin E(f;\nu)}}\frac{m_f(r,\ \mathcal{I}_0)}{T_f(r,\ L)}\,.$$

We can take a subsequence $\{r_{i_k}\}$ of $\{r_i\}$ such that the limit

$$\lim_{k\to+\infty}\sum_{j=1}^{q}\frac{m_f(r_{i_k},\ D_j)}{T_f(r_{i_k},\ L)}$$

exists. Then we see

$$\begin{aligned}\sum_{j=1}^{q}\tilde{\delta}_f(D_j;\nu) &\le \liminf_{\substack{r\to+\infty\\ r\notin E(f;\nu)}}\sum_{j=1}^{q}\frac{m_f(r,\ D_j)}{T_f(r,\ L)}\\ &\le \lim_{k\to+\infty}\sum_{j=1}^{q}\frac{m_f(r_{i_k},\ D_j)}{T_f(r_{i_k},\ L)}\\ &\le \lim_{i\to+\infty}\left\{\frac{qm_f(r_i,\mathcal{I}_0)}{T_f(r_i,\ L)} + (2N-l_0+1)\left(1-\frac{m_f(r_i,\mathcal{I}_0)}{T_f(r_i,\ L)}\right)\right\}\\ &\quad + \lim_{i\to+\infty}\frac{S_f(r_i;\nu)}{T_f(r_i,\ L)}\\ &= q\tilde{\delta}_f(B_\Lambda;\nu) + (2N-l_0+1)(1-\tilde{\delta}_f(\Lambda;\nu))\,.\end{aligned}$$

This yields that

$$\sum_{j=1}^{q}(\tilde{\delta}_f(D_j;\nu) - \tilde{\delta}_f(B_\Lambda;\nu)) \le (2N-l_0+1)(1-\tilde{\delta}_f(\Lambda;\nu))\,.$$

Thus, by letting $\nu \to +\infty$, we have the desired conclusion. □

5. The set of deficient divisors. In this section we consider the sets of deficient divisors. We give theorems concerning structures of those sets that are our main theorems. Let $L \to M$ be an ample line bundle and $W \subseteq \Gamma(M,\ L)$ a subspace with $\dim W = l_0+1 \ge 2$. Set $\Lambda = \boldsymbol{P}(W)$. Let $f : \boldsymbol{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ. We define the sets $\mathfrak{D}_f$ and $\widetilde{\mathfrak{D}}_f$ as in Section 4. In Section 4 we see that those sets are null sets in the sense of Lebesgue measure. By making use of Sadullaev's method [S1], we give an improvement of Proposition 4.1.

DEFINITION 5.1. A subset S of Λ is said to be $\mathcal{P}$-polar in Λ provided that, for any coordinate chart $(U,\ \phi)$ in Λ, there exists a plurisubharmonic function v on $\phi(U) \subseteq \boldsymbol{C}^{l_0}$ with $v \not\equiv -\infty$ and $\phi(S\cap U) = \{v = -\infty\}$.

THEOREM 5.2. *The sets $\mathfrak{D}_f$ and $\widetilde{\mathfrak{D}}_f$ are $\mathcal{P}$-polar sets in Λ. In particular, the Hausdorff dimensions of those sets are at most $2l_0 - 2$.*

PROOF. Take a basis $\{\psi_0, \dots, \psi_{l_0}\}$ of W. We identify Λ with $\boldsymbol{P}_{l_0}(\boldsymbol{C})$ via an isomorphism $T : \Lambda \to \boldsymbol{P}_{l_0}(\boldsymbol{C})$ defined by

$$\tilde{T} : \sigma = \sum_{j=1}^{l_0} c_j \sigma_j \mapsto (c_0, \dots, c_{l_0}) .$$

Let $(\zeta_0 : \cdots : \zeta_{l_0})$ be a homogeneous coordinate system in $\boldsymbol{P}_{l_0}(\boldsymbol{C})$. We take the coordinate open set U_k by

$$U_k = \{(\zeta_0 : \cdots : \zeta_{l_0}) \in \Lambda \,;\, \zeta_k \neq 0\} \quad \text{for } k = 0, \dots, l_0$$

and the local coordinate system

$$\xi^{(k)} = (\xi_1^{(k)}, \dots, \xi_{l_0}^{(k)}) = \left(\frac{\zeta_0}{\zeta_k}, \dots, \frac{\zeta_{k-1}}{\zeta_k}, \frac{\zeta_{k+1}}{\zeta_k}, \dots, \frac{\zeta_{l_0}}{\zeta_k}\right) \quad \text{for each } k .$$

Let $D \in T^{-1}(U_k)$ defined by $\sum_{j=0}^{l_0} \zeta_j \psi_j = 0$. Then there exists a plurisubharmonic function $v_k(r, \xi^{(k)})$ in U_k such that

$$N(r, f^*D) = v_k(r, \xi^{(k)}) + s_k(\xi^{(k)}), \tag{5.3}$$

where $s_k(\xi^{(k)})$ is an L^1-function in U_k. Indeed, by Jensen's formula, we see

$$N(r, f^*D) = \int_{C(r)} \log \left| \sum_{j=0}^{l_0} \zeta_j \psi_j(f(z)) \right| \frac{d\theta}{2\pi} + O(1) .$$

We define $v_k(r, \xi^{(k)})$ by

$$v_k(r, \xi^{(k)}) = \int_{C(r)} \log \left| \sum_{j=0}^{l_0} \frac{\zeta_j}{\zeta_k} \psi_j(f(z)) \right| \frac{d\theta}{2\pi} .$$

It is clear that $v_k(r, \xi^{(k)})$ is a plurisubharmonic function in U_k. If we set

$$s_k(\xi^{(k)}) = N(r, f^*D) - v_k(r, \xi^{(k)}) ,$$

we have (5.3). We now write $v_k(r, D)$ for $v_k(r, \xi^{(k)})$ and $s_k(D)$ for $s_k(\xi^{(k)})$. It follows from Theorem 2.3 and (5.3) that

$$T_f(r, L) = \int_{D \in \Lambda} v_k(r, D) d\mu(D) + m_f(r, \mathcal{I}_0) + O(1) . \tag{5.4}$$

By Proposition 4.1 and (5.3), we see

$$\begin{aligned} \limsup_{r \to +\infty} \frac{N(r, f^*D)}{T_f(r, L)} &= \limsup_{r \to +\infty} \frac{v_k(r, D)}{T_f(r, L)} \\ &\leq 1 - \delta_f(B_\Lambda) \end{aligned}$$

for $D \in T^{-1}(U_k)$. Because of (5.4), there exists at least one $D \in T^{-1}(U_k)$ such that

$$\begin{aligned} v_k(D) &:= \limsup_{r\to+\infty} \frac{v_k(r,\ D)}{T_f(r,\ L)} \\ &= 1 - \delta_f(B_\Lambda)\,. \end{aligned}$$

By the maximal principle for plurisubharmonic functions, the regularization $v_k^*(D)$ of $v_k(D)$ is identically equal to $1 - \delta_f(B_\Lambda)$. It follows from Lelong's theorem that the set

$$T(\mathfrak{D}_f) \cap U_k = \{T(D) \in U_k;\ v_k(D) < v_k^*(D)\}$$

is a $\mathcal{P}$-polar set in U_k (cf. e.g., [S2, §12]). This yields that $\mathfrak{D}_f$ is a $\mathcal{P}$-polar set in Λ, and hence the Newton capacity of $\mathfrak{D}_f$ is zero. Thus we see that the Hausdorff dimension of $\mathfrak{D}_f$ does not exceed $2l_0 - 2$ (see [HK, Theorem 5.13]). We also have the assertion for $\widetilde{\mathfrak{D}}_f$ by the method similar to the above. □

We next give a structure theorem for $\widetilde{\mathfrak{D}}_f$. In what follows, we consider points in Λ as zero-dimensional linear systems included in Λ. For a sufficiently small positive number ε, set

$$\widetilde{\mathfrak{D}}_\varepsilon = \{D \in \Lambda\,;\, \tilde{\delta}_f(D) \geq \tilde{\delta}_f(B_\Lambda) + \varepsilon\}\,.$$

Then it is clear that

$$\widetilde{\mathfrak{D}}_f = \bigcup_{\varepsilon>0} \widetilde{\mathfrak{D}}_\varepsilon\,.$$

THEOREM 5.5. *The set* $\widetilde{\mathfrak{D}}_\varepsilon$ *is a union of finitely many linear systems included in* Λ. *In particular, the set* $\widetilde{\mathfrak{D}}_f$ *is a union of at most countably many linear systems in* Λ.

For a proof, we first show a lemma. Let V be an $(l+1)$-dimensional complex vector space and Σ a subset of V. We say that Σ is *of maximal rank* if $\sharp\Sigma \geq l+1$ and any distinct $l+1$ vectors $a_1, \dots, a_{l+1}$ in Σ are linearly independent.

LEMMA 5.6. *Let S be an infinite subset of V that generates V over* $\boldsymbol{C}$. *Suppose that there exists a positive integer d_0 such that $\sharp\Sigma \leq d_0$ if $\Sigma \subseteq S$ is of maximal rank. Then there exist finitely many proper linear subspaces $V_1, \dots, V_s$ of V such that*

$$S \subseteq V_1 \cup \cdots \cup V_s\,.$$

PROOF OF LEMMA 5.6. Let $\mathfrak{S} = \{S_\alpha\}$ be the family of all sets of maximal rank contained in S. Since $\mathfrak{S} \neq \emptyset$ and $\sharp\Sigma_\alpha \leq d_0$ for $\Sigma_\alpha \in \mathfrak{S}$, there exists an element Σ in $\mathfrak{S}$ such that $\sharp\Sigma_\alpha \leq \sharp\Sigma$ for all $\Sigma_\alpha \in \mathfrak{S}$. Now, we let $U(\Sigma)$ denote the union of all distinct linear subspaces of codimension one in V spanned by distinct l vectors in Σ. We claim that $S \subseteq U(\Sigma)$. Assume the contrary. Then there exists a vector $a \in S$ that is not contained in $U(\Sigma)$. Since the vector a cannot be written as a linear combination of any l linearly independent vectors in $U(\Sigma)$, we see that $\Sigma \cup \{a\}$ is of maximal rank. Hence $\Sigma \cup \{a\} \in \mathfrak{S}$. This contradicts the choice of Σ. Hence $S \subseteq U(\Sigma)$. Since $U(\Sigma)$ is the union of finitely many proper linear subspaces of V, we have our assertion. □

PROOF OF THEOREM 5.5. In the case where $\sharp\widetilde{\mathfrak{D}}_\varepsilon$ is finite, our assertion trivially holds, that is, $\widetilde{\mathfrak{D}}_\varepsilon$ is a union of finitely many zero dimensional linear systems in Λ. We consider the case where $\widetilde{\mathfrak{D}}_\varepsilon$ is an infinite set. Let $S(\varepsilon) := \{\sigma \in W\,;\,(\sigma) \in \widetilde{\mathfrak{D}}_\varepsilon\}$ and denote by $l_1 + 1$ the dimension of the linear space W' generated by $S(\varepsilon)$ over $\boldsymbol{C}$. Now, we consider the linear system $\Lambda' := \boldsymbol{P}(W')$. By Theorem 2.1 and Proposition 4.1, we see that

$$\tilde{\delta}_f(B_\Lambda) \leq \tilde{\delta}_f(B_{\Lambda'}) \leq \tilde{\delta}_f(D)$$

for all $D \in \Lambda'$. If $\tilde{\delta}_f(B_\Lambda) + \varepsilon \leq \tilde{\delta}_f(B_{\Lambda'})$, then $\Lambda' \subseteq \widetilde{\mathfrak{D}}_\varepsilon$. Hence we have $\widetilde{\mathfrak{D}}_\varepsilon = \Lambda'$. By Proposition 4.1, we see $\Lambda' \subsetneq \Lambda$. We suppose that

$$\tilde{\delta}_f(B_{\Lambda'}) < \tilde{\delta}_f(B_\Lambda) + \varepsilon\,.$$

Let Σ be an arbitrary set of maximal rank contained in $S(\varepsilon)$ and let $D_1, \dots, D_q$ be divisors such that $D_j = (\sigma_j)$ with $\sigma_j \in \Sigma$. We now apply Theorem 4.2 for f and Λ'. Then we have

$$\sum_{j=1}^{q}(\tilde{\delta}_f(D_j) - \tilde{\delta}_f(B_{\Lambda'})) \leq (l_1 + 1)(1 - \tilde{\delta}_f(B_{\Lambda'}))\,.$$

Thus we see that $q\varepsilon \leq (l_1 + 1)(1 - \tilde{\delta}_f(B_{\Lambda'}))$, and hence

$$q \leq (l_1 + 1)(1 - \tilde{\delta}_f(B_{\Lambda'}))/\varepsilon\,.$$

If we denote by d_0 the largest integer that does not exceed

$$(l_1 + 1)(1 - \tilde{\delta}_f(B_{\Lambda'}))/\varepsilon\,,$$

then the cardinality of any set of maximal rank contained in $S(\varepsilon)$ is less than d_0. Hence by Lemma 5.6, there exist proper linear subspaces $W_1, \dots, W_s$ of W' such that

$$S(\varepsilon) \subseteq W_1 \cup \cdots \cup W_s\,.$$

If $W_j \subseteq S(\varepsilon)$ for all j, we have

$$S(\varepsilon) = W_1 \cup \cdots \cup W_s\,.$$

We consider the case where $W_j \setminus S(\varepsilon)$ is non-empty for some j. In this case, we will show that we can eliminate sections $\sigma \in W_j \setminus S(\varepsilon)$ if we replace the linear subspace W_j by a finite sets of proper linear subspaces of W_j. By a suitable change of indices, we assume that $W_j \setminus S(\varepsilon) \neq \emptyset$ for $j = 1, \dots, s_1$ and $W_j \subseteq S(\varepsilon)$ for $j = s_1 + 1, \dots, s$. For $j = 1, \dots, s_1$, let W'_j be the linear subspace of W' generated by $S(\varepsilon) \cap W_j$ over $\boldsymbol{C}$. Then it is clear that

$$S(\varepsilon) \subseteq W'_1 \cup \cdots \cup W'_{s_1} \cup W_{s_1+1} \cup \cdots \cup W_s\,. \tag{5.7}$$

If $W'_j \subseteq S(\varepsilon)$ for $j = 1, \dots, s_1$, then we have

$$S(\varepsilon) = W'_1 \cup \cdots \cup W'_{s_1} \cup W_{s_1+1} \cup \cdots \cup W_s\,.$$

We consider the case where $W'_j \setminus S(\varepsilon)$ is non-empty for some j. We may assume that $W'_j \setminus S(\varepsilon) \neq \emptyset$ for $j = 1, \dots, s_2$ and $W'_j \subseteq S(\varepsilon)$ for $j = s_2 + 1, \dots, s_1$. Since $S(\varepsilon) \cap W_j$ generates

W'_j over $\boldsymbol{C}$, by using the above argument, we obtain finitely many proper linear subspaces $W_1^{(j)}, \dots, W_{t_j}^{(j)}$ of W'_j such that

$$S(\varepsilon) \cap W'_j \subseteq W_1^{(j)} \cup \cdots \cup W_{t_j}^{(j)} .$$

We replace W'_j in (5.7) by $W_1^{(j)} \cup \cdots \cup W_{t_j}^{(j)}$ for $j = 1, \dots, s_2$ and obtain

$$S(\varepsilon) \subseteq \bigcup_{j=1}^{s_2} (W_1^{(j)} \cup \cdots \cup W_{t_j}^{(j)}) \cup (W'_{s_2+1} \cup \cdots \cup W'_{s_1}) \cup (W_{s_1+1} \cup \cdots \cup W_s) .$$

Note that

$$\dim W_k^{(j)} < \dim W'_j$$

for all $j = 1, \dots, s_2$ and $k = 1, \dots, t_j$. If $W_k^{(j)} \setminus S(\varepsilon)$ is non-empty for some j and k, we repeat this argument. Since $\dim W'_j$ is finite and each replacement of the linear subspace by a finite set of linear subspaces decreases the dimension of the initial subspace, we finally obtain finitely many proper linear subspaces $X_1, \dots, X_{t_1}, Y_1, \dots, Y_{t_2}$ of W such that

$$S(\varepsilon) = X_1 \cup \cdots \cup X_{t_1} \cup Y_1 \cup \cdots \cup Y_{t_2} ,$$

where $\dim X_j = 1$ for $j = 1, \dots, t_1$ and $\dim Y_k \geq 2$ for $k = 1, \dots, t_2$. This yields that there exist finitely many linear systems $\Lambda_1, \dots, \Lambda_m$ in Λ such that

$$\widetilde{\mathfrak{D}}_\varepsilon = \bigcup_{j=1}^{m} \Lambda_j .$$

Since $\widetilde{\mathfrak{D}}_f = \bigcup_{n \in \boldsymbol{Z}^+} \widetilde{\mathfrak{D}}_{1/n}$, we see that $\widetilde{\mathfrak{D}}_f$ is a union of at most countably many linear systems in Λ. □

We next consider the set of values of the defect function $\tilde{\delta}_f : \Lambda \to [0,\ 1]$. We will show that the set of values of $\tilde{\delta}_f$ is a countable set.

THEOREM 5.8. *Let $f : \boldsymbol{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ. Then the set of values of modified deficiency of f is at most a countable subset of* $[0,\ 1]$.

PROOF. We consider the set

$$\widetilde{\mathfrak{D}}_{1/n} = \{D \in \Lambda\,;\, \tilde{\delta}_f(D) \geq \tilde{\delta}_f(B_\Lambda) + 1/n\} .$$

For the proof, we first show that $\{\tilde{\delta}_f(D)\,;\, D \in \widetilde{\mathfrak{D}}_{1/n}\}$ is a finite set for each integer $n \geq 2$. We will give the proof by induction on the dimension of Λ. We first consider the case where $\dim \Lambda = 1$. In this case we see $\Lambda \cong \boldsymbol{P}_1(\boldsymbol{C})$. Now, by Theorem 4.2, we see that $\widetilde{\mathfrak{D}}_{1/n}$ is a finite set of points in Λ. Hence the cardinality of the set $\{\tilde{\delta}_f(D)\,;\, D \in \widetilde{\mathfrak{D}}_{1/n}\}$ is finite. Suppose that the above assertion is true for $\dim \Lambda < l_0$. Let us consider the case $\dim \Lambda = l_0$. It follows from Theorem 5.5 that the set $\widetilde{\mathfrak{D}}_{1/n}$ is a union of finitely many linear systems $\Lambda_1, \dots, \Lambda_t$ in Λ. Since $\dim \Lambda_j < l_0$ for all j, by applying the hypothesis of induction for Λ_j, we easily see that $\{\tilde{\delta}_f(D)\,;\, D \in \widetilde{\mathfrak{D}}_{1/n}\}$ is a finite set. Thus we conclude that the cardinality of the set $\{\tilde{\delta}_f(D)\,;\, D \in \widetilde{\mathfrak{D}}_{1/n}\}$ is always finite. This yields that the set of values of $\tilde{\delta}_f$ is at most countable. □

We next show that the set of values of $\tilde{\delta}_f$ corresponds to the family of linear systems in Λ. Let f be as in Theorem 5.8. By Theorem 5.5, there exist at most countably many linear systems $\{\Lambda_j\}$ in Λ such that $\widetilde{\mathfrak{D}}_f = \bigcup_j \Lambda_j$. Define $\mathfrak{L} = \{\Lambda_j\} \cup \{\Lambda\}$. We call $\mathfrak{L}$ *the fundamental family of linear systems for* f. For $\Lambda' \in \mathfrak{L}$, we denote by $\tilde{\delta}_f(\Lambda')$ the set of values of $\tilde{\delta}_f$ on Λ', that is, $\tilde{\delta}_f(\Lambda') = \{\tilde{\delta}_f(D)\,;\, D \in \Lambda'\}$. Then we have the following theorem:

THEOREM 5.9. *Let f be as in Theorem* 5.8 *and $\mathfrak{L}$ the fundamental family of linear systems for f. Then, for each $\alpha \in \tilde{\delta}_f(\Lambda)$, there exists a unique finite subfamily $\mathfrak{L}_\alpha = \{\Lambda_j^{(\alpha)}\}$ of $\mathfrak{L}$ such that*

$$\alpha = \tilde{\delta}_f(B_{\Lambda_j^{(\alpha)}}) \quad \textit{for all} \ \ \Lambda_j^{(\alpha)} \in \mathfrak{L}_\alpha$$

and

$$\alpha \neq \tilde{\delta}_f(B_{\Lambda_j}) \quad \textit{for all} \ \ \Lambda_j \in \mathfrak{L} \setminus \mathfrak{L}_\alpha .$$

Furthermore, there exists a union $\mathfrak{E}_j$ of at most countably many linear systems in $\mathfrak{L}$ such that

$$\tilde{\delta}_f(D) = \tilde{\delta}_f(B_{\Lambda_j^{(\alpha)}}) \quad \textit{for all} \ \ D \in \Lambda_j \setminus \mathfrak{E}_j .$$

In particular, the closure of the inverse image $\tilde{\delta}_f^{-1}(\alpha)$ is a union of finitely many linear systems in $\mathfrak{L}$.

PROOF. Let $\alpha \in \tilde{\delta}_f(\Lambda)$. Then there exists a positive integer n such that that $\tilde{\delta}_f^{-1}(\alpha)$ is included in $\widetilde{\mathfrak{D}}_{1/n}$. Since $\widetilde{\mathfrak{D}}_{1/n}$ is a union of finitely many linear systems in $\mathfrak{L}$, we can take finitely many linear systems $\Lambda_1^{(\alpha)}, \dots, \Lambda_t^{(\alpha)}$ with $\alpha = \tilde{\delta}_f(B_{\Lambda_j^{(\alpha)}})$ for $j = 1, \dots, t$. Hence if $\Lambda_i \in \mathfrak{L}$ and $\Lambda_i \subsetneq \Lambda_j^{(\alpha)}$ for all j, then α is not in $\tilde{\delta}_f(\Lambda_i)$. We apply Theorem 5.5 for the linear system $\Lambda_j^{(\alpha)}$ and obtain at most countable linear systems $\{\Lambda'_k\}$ included in $\Lambda_j^{(\alpha)}$ such that

$$\alpha = \tilde{\delta}_f(D) \quad \text{for all} \ \ D \in \Lambda_j^{(\alpha)} \setminus \Big(\bigcup_k \Lambda'_k\Big) .$$

On the other hand, by Proposition 4.1, we see that $\tilde{\delta}_f(B_{\Lambda_j^{(\alpha)}}) = \tilde{\delta}_f(D)$ for almost all $D \in \Lambda_j^{(\alpha)}$. This yields that $\alpha = \tilde{\delta}_f(B_{\Lambda_j^{(\alpha)}})$. Note that $\Lambda'_k = \Lambda_j^{(\alpha)} \cap \Lambda_k$ for $\Lambda_k \in \mathfrak{L}$. Therefore we have our assertion. □

Note that $\tilde{\delta}_f(D) > \tilde{\delta}_f(B_{\Lambda_j}^{(\alpha)})$ for all $D \in \mathfrak{E}_j$.

COROLLARY 5.10. *If $\tilde{\delta}_f(D) > \tilde{\delta}_f(B_\Lambda)$ for a divisor D in Λ, then there exists a linear system Λ_D included in Λ such that $\tilde{\delta}_f(D) = \tilde{\delta}_f(B_{\Lambda_D})$.*

REMARK 5.11. Let $\mathfrak{L}'$ be an arbitrary at most countable family of linear systems in $|\mathcal{O}_{\boldsymbol{P}_n}(1)|$. Nochka [Nc2] give a condition under which $\mathfrak{L}'$ is the fundamental family of linear systems for a holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$. He also give a method for constructing f and the fundamental family $\mathfrak{L}$ for f from an arbitrary given $\mathfrak{L}'$.

6. The existence of holomorphic curves with deficiencies for base loci. The existence of holomorphic curves with positive deficiencies is a nontrivial matter. In this section, we discuss the existence of holomorphic curves with deficiencies. Throughout this section, we consider holomorphic curves in complex projective spaces. We first give an example of linear system Λ and holomorphic curve f with $0 < \delta_f(B_\Lambda) < 1$. The following example is due to Noguchi [No1, Example 2.3.29].

EXAMPLE 6.1. Let $(\zeta_0, \zeta_1, \zeta_2)$ be a homogeneous coordinate system in $\boldsymbol{P}_2(\boldsymbol{C})$ and W a subspace of $\Gamma(\boldsymbol{P}_2(\boldsymbol{C}), \mathcal{O}_{\boldsymbol{P}_n}(1))$ generated by ζ_1 and ζ_2. Then $\mathrm{Bs}\,\Lambda = \{(1,0,0)\}$. We define an algebraically non-degenerate holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_2(\boldsymbol{C})$ by

$$f(z) = (1, e^z, e^{cz})\,,$$

where c is a positive number greater than one. In this case, we have

$$\phi_{\mathcal{I}_0} = \frac{1}{2}\log\left(\frac{|\zeta_0|^2 + |\zeta_1|^2 + |\zeta_2|^2}{|\zeta_1|^2 + |\zeta_2|^2}\right).$$

Then, a direct calculation gives us the following:

$$T_f(r) = \frac{c}{\pi} r + O(1) \quad \text{and} \quad m_f(r, \mathcal{I}_0) = \frac{1}{\pi} r + O(1)\,.$$

Hence we have $\delta_f(B_\Lambda) = 1/c$. We notice that f does not hit the base locus.

In this example, f does not hit the non-empty base locus of Λ but $0 < \delta_f(B_\Lambda) < 1$. In [AM], we proved a theorem on the existence of holomorphic curves with deficiencies in the case where $L = \mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes d}$ for an arbitrary positive integer d. In fact, we have the following theorem [AM, Theorem 3.2]:

THEOREM 6.2. *Let $D \in |\mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes d}|$. There exists a positive constant $\lambda(D)$ with $\lambda(D) \le d$ depending only on D that satisfies the following property: Let α be a positive real number such that $\alpha \le \lambda(D)/d$. Then there exists an algebraically non-degenerate holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ such that $\delta_f(D) = \alpha$.*

The proof of the above theorem is based on the classical theory of entire functions, especially Valiron's theorem on entire function of order zero. The resulting holomorphic curves f in Theorem 6.2 is of order zero. Hence $\delta_f(D) = \tilde{\delta}_f(D)$. By making use of Theorem 6.2, we have the following proposition:

PROPOSITION 6.3. *Let e_0 be a positive constant less than one. Then there exist an algebraically non-degenerate transcendental holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ and a linear system Λ in $|\mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes d}|$ such that $\delta_f(B_\Lambda) = e_0$.*

PROOF. We can take $D \in |\mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes d}|$ such that $\lambda(D) = d$(cf. [AM, Remark 3.7]). By Theorem 6.2, we have a transcendental holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ of order zero such that $\delta_f(D) = e_0$. Hence, by Theorem 5.9, there exists a linear system Λ included in $|\mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes d}|$ such that $\delta_f(B_\Lambda) = e_0$. □

Let $\Lambda \subseteq |\mathcal{O}_{\boldsymbol{P}_n}(1)^{\otimes d}|$ be a linear system with the non-empty base locus. We will show the existence of holomorphic curves with $0 < \delta_f(B_\Lambda) < 1$. We will give a proof by constructing a holomorphic curve with the desired property. We recall some known facts on exponential curves. Let $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ be a nonconstant holomorphic curve defined by

$$f(z) = (\exp a_0 z, \dots, \exp a_n z)\,, \tag{6.4}$$

where $a_0, \dots, a_n$ are complex numbers. We denote by $\mathcal{C}_f$ the circumference of the convex polygon spanned by the set $\{a_0, \dots, a_n\}$. If the convex polygon reduces to the segment with the end points with a_j and a_k, then we see $\mathcal{C}_f = 2|a_j - a_k|$. Let H be a hyperplane in $\boldsymbol{P}_n(\boldsymbol{C})$ defined by

$$H : L(z) = \sum_{j=0}^{n} \alpha_j \zeta_j = 0 \quad (\alpha_0, \dots, \alpha_n \in \boldsymbol{C})\,,$$

where $\zeta = (\zeta_0 : \cdots : \zeta_n)$ is a homogeneous coordinate system in $\boldsymbol{P}_n(\boldsymbol{C})$. We define the set J_H of index by $J_H = \{j\,;\alpha_j \neq 0\}$. Let $\mathcal{C}_{f,H}$ be the circumference of the convex polygon spanned by the set $\{a_j\,;\, j \in J_H\}$. According to [A] and [W2], we have

$$T_f(r) = \frac{\mathcal{C}_f}{2\pi} r + O(1)\,. \tag{6.5}$$

We have a simple proof of (6.5) by using Crofton's formula (see [Shi3, p. 630]). The following lemma is also due to Weyl [W2, pp. 95–98]:

LEMMA 6.6. *Let f and H be as above. Then the deficiency of f for H is given by*

$$\delta_f(H) = 1 - \frac{\mathcal{C}_{f,H}}{\mathcal{C}_f}\,.$$

Furthermore, the constant $\mathcal{C}_{f,H}$ depends only on f and J_H.

For a reader's convenience, we introduce the proof of this lemma by Toda [T].

PROOF. We first assume that all α_j's are non-zero. Take hyperplanes $H_0, \dots, H_n$ so that $H_j = \{\zeta_j = 0\}$. Then $H_0, \dots, H_n, H$ are in general position. By Theorem 1.3,

$$\sum_{j=0}^{n} \delta_f(H_j) + \delta_f(H) \le n + 1\,.$$

Since $\delta_f(H_j) = 1$ for $j = 0, \dots, n$, we have $\delta_f(H) = 0$. Next, we assume that $\alpha_{j_k} \neq 0$ for $0 \le j_0 < \cdots < j_m \le n$ and $\alpha_j = 0$ for $j \neq j_0, \dots, j_m$. We define $g : \boldsymbol{C} \to \boldsymbol{P}_m(\boldsymbol{C})$ by $g = (\exp a_{j_0} z, \dots, \exp a_{j_m} z)$. By (6.5), we have

$$T_g(r) = \frac{\mathcal{C}_g}{2\pi} r + O(1)\,.$$

This shows $\mathcal{C}_g = \mathcal{C}_{f,H}$. Hence $\mathcal{C}_{f,H}$ depends only on the index set $\{j\,;\alpha_j \neq 0\}$. We now regard H as a hyperplane in $\boldsymbol{P}_m(\boldsymbol{C})$ by the natural way. Then we have $\delta_g(H) = 0$. Hence we see

$$\begin{aligned}\delta_f(H) &= 1 - \limsup_{r\to+\infty} \frac{N(r,\ f^*H)}{T_f(r)} \\ &= 1 - \limsup_{r\to+\infty} \frac{N(r,\ g^*H)}{T_g(r)} \frac{T_g(r)}{T_f(r)} \\ &= 1 - \frac{\mathcal{C}_{f,H}}{\mathcal{C}_f}\,.\end{aligned}$$

This completes the proof. □

For brevity, we consider the case where $d = 1$.

THEOREM 6.7. *Let $\Lambda \subseteq |\mathcal{O}_{\boldsymbol{P}_n}(1)|$ and e_0 an arbitrary positive number less than one. Suppose* Bs $\Lambda \neq \emptyset$. *Then there exists an algebraically non-degenerate transcendental holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ of finite type such that $e_0 = \delta_f(B_\Lambda)$.*

PROOF. Let W be a subspace of $\Gamma(\boldsymbol{P}_n(\boldsymbol{C}), \mathcal{O}_{\boldsymbol{P}_n}(1))$ such that $\Lambda = \boldsymbol{P}(W)$. Since Bs $\Lambda \neq \emptyset$, we may assume that W is generated by $\zeta_0, \dots, \zeta_l$ with $1 \leq l < n$. Take an exponential curve f defined by (6.4). Since Bs $\Lambda \neq \emptyset$, each $H \in \Lambda$ is defined by

$$\sum_{j=0}^{l} \alpha_j \zeta_j = 0 \quad \text{with} \quad 1 \leq l \leq n-1\,.$$

For an arbitrary positive number e_0 less than one, we can choose $\{\alpha_j\}_{j=0}^n$ such that $\alpha_j \neq 0$ for all j and $1 - (\mathcal{C}_{f,H}/\mathcal{C}_f) = e_0$. Hence, we have $\delta_f(H) = e_0$. Since we can take $\{\alpha_j\}_{j=0}^n$ such that $\alpha_i \neq \alpha_j$ for $i \neq j$, we have the algebraically non-degenerate holomorphic curve f. By Lemma 6.6, if $\alpha_j \neq 0$ for all j, then we see $\mathcal{C}_{f,H}$ is a constant independent of H. This implies

$$e_0 = \delta_f(H) \quad \text{for all}\ \ H \in \Lambda \setminus \left(\bigcup_{i=1}^{t} \Lambda_i\right),$$

where $\Lambda_1, \dots, \Lambda_t$ are finitely many proper linear systems included in Λ. On the other hand, by Proposition 4.1, we see $\delta_f(H) = \delta_f(B_\Lambda)$ for almost all $H \in \Lambda$. This yields that $e_0 = \delta_f(B_\Lambda)$. □

By the above proof, we have the following corollary:

COROLLARY 6.8. *Let Λ be as above. Then there exists a transcendental holomorphic curve $f : \boldsymbol{C} \to \boldsymbol{P}_n(\boldsymbol{C})$ non-degenerate with respect to Λ such that the set of values of δ_f is a finite set $\{e_j\}$ with $0 < e_j < 1$. Furthermore, there exist finitely many linear systems $\{\Lambda_j\}$ included in Λ such that*

$$\delta_f(H) = e_j \quad \textit{for all}\ \ H \in \Lambda_j \setminus \left(\bigcup_k \Lambda_{jk}\right),$$

where $\{\Lambda_{j_k}\}$ *are linear systems included in* Λ_j.

7. Appendix. The proof of Lemma 4.3. In this section we give the proof of Lemma 4.3. We keep the same notation as in the proof of Theorem 3.1. We first give the definition of $S_{F_\Lambda}(r)$ (cf. [No2, p. 344] and [Shi2, p. 558]). We take a basis $\{\phi_0, \dots, \phi_{l_0}\}$ for W and have the isomorphism $\boldsymbol{P}(W) \cong \boldsymbol{P}_{l_0}(\boldsymbol{C})$ through the basis. Take a reduced representation $(f_0, \dots, f_{l_0})$ of F_Λ. Let $L_j(\zeta)$ be linear forms on $\boldsymbol{P}(W)$ such that

$$H_{D_j} = \{L_j(\zeta) = 0\}$$

for $j = 1, \dots, q$. We may assume that $f_0 \not\equiv 0$. Set

$$u_j(z) = L_j(f(z))/f_0(z)$$

for $j = 1, \dots, q$. Let $Q = \{1, \dots, q\}$ and $J \subseteq Q$ with $\sharp J = l_0 + 1$. We define the logarithmic Wronskian $\Delta(u_j;\, j \in J)$ by

$$\Delta(u_j;\, j \in J) = \begin{vmatrix} 1 & \cdots & 1 \\ \dfrac{u'_{j_1}}{u_{j_1}} & \cdots & \dfrac{u'_{j_{l_0+1}}}{u_{j_{l_0+1}}} \\ \vdots & \vdots & \vdots \\ \dfrac{u^{(l_0)}_{j_1}}{u_{j_1}} & \cdots & \dfrac{u^{(l_0)}_{j_{l_0+1}}}{u_{j_{l_0+1}}} \end{vmatrix},$$

where $J = \{j_1, \dots, j_{l_0+1}\}$. Then we define the error term $S_{F_\Lambda}(r)$ by

$$S_{F_\Lambda}(r) = \int_0^{2\pi} \log\left(\sum_{J \subseteq Q,\, \sharp J = l_0+1} |\Delta(u_j;\, j \in J)(z)| \right) \frac{d\theta}{2\pi}.$$

If F_Λ is rational, then it is easy to see that

$$S_{F_\Lambda}(r) = O(1)$$

as $r \to +\infty$. From now on, we assume that F_Λ is transcendental.

LEMMA 7.1. *Let* u_j *be as above. Then*

$$T(r, u_j) + O(1) \le T_{F_\Lambda}(r) \le T_f(r, L) + O(1)$$

for $j = 1, \dots, q$.

PROOF. By (1.2), we have

$$T_{F_\Lambda}(r) = \int_{C(r)} \log \max_{0 \le j \le l_0} |f_j(z)| \frac{d\theta}{2\pi} + O(1).$$

By making use of this representation and (3.5), we easily see our assertion. □

We now recall the Lemma on logarithmic derivative due to Nevanlinna (cf. [N, p. 61]):

LEMMA 7.2. *Let u be a meromorphic function on $\boldsymbol{C}$ and $1 \leq r < R$. Then*

$$m\left(r, \frac{u'}{u}\right) \leq 4\log^{+} T_u(R) + 3\log^{+}\frac{1}{R-r} + 4\log R + 24 + c_u\,,$$

where c_u is a positive constant that depends only on u.

When u is of finite type, as in [N, p. 61], we have that

$$m\left(r, \frac{u'}{u}\right) \leq (4+4\rho)\log r + 8\rho + 32 + c_u$$

for $r \in [1, +\infty)$, where

$$\rho = \limsup_{r\to+\infty}\frac{\log T_u(r)}{\log r}\,.$$

Hence there exist positive constants c_1, c_2, c_3 such that

$$m\left(r, \frac{u'}{u}\right) \leq c_1\log^{+} T_u(r) + c_2\log r + c_3 \tag{7.3}$$

for all $r \in [1, +\infty)$. Now, we assume that u is not of finite type. In this case, we cannot give the estimate (7.3) for all $r \in [1, +\infty)$. Indeed, there exists a meromorphic function for which the estimate (7.3) does not hold for all $r \in [1, +\infty)$. For example, see [GO, p. 92]. To get the estimate of type (7.3), we use the following Borel-Nevanlinna type lemma due to Gol'dberg-Ostrovskii [GO, p. 90]:

LEMMA 7.4. *Let ν be a positive integer. Then there exists a Borel subset $E(f;\nu)$ of $[1, +\infty)$ with the finite Lebesgue measure such that*

$$T_f\left(r + \frac{1}{(T_f(r,L)+\nu)^2}, L\right) \leq T_f(r,L) + \nu$$

for all $r \in [1, +\infty) \setminus E(f;\nu)$.

We can give a proof of Lemma 7.4 by using a standard argument in Nevanlinna theory (see [GO, Chapter 3]). Hence we omit the proof here. Now we consider an estimate for the logarithmic derivative of u_j that is not of finite type. By making use of Lemmas 7.2 and 7.4, we have the following estimate:

LEMMA 7.5. *Suppose that u_j is not of finite type. Then there exist positive constants $c_1, \ldots, c_4$ such that*

$$m\left(r, \frac{u_j'}{u_j}\right) \leq c_1\log^{+} T_f(r,L) + c_2\log r + c_3\log\nu + c_4$$

as $r \to +\infty$ except on $E(f;\nu)$. Furthermore, $c_1, \ldots, c_4$ are independent of ν.

PROOF. Let $u = u_j$ and $R = r + 1/(T_f(r,L)+\nu)^2$ in Lemma 7.2. Suppose that $r \notin E(f;\nu)$. Then by Lemmas 7.1 and 7.4, we have

$$m\left(r, \frac{u_j'}{u_j}\right) \leq 4\log^{+} T_{u_j}\left(r + \frac{1}{(T_f(r,L)+\nu)^2}\right)$$

$$
\begin{aligned}
&\quad +3\log^+(T_f(r,L)+\nu)^2+4\log^+\left(r+\frac{1}{(T_f(r,L)+\nu)^2}\right)+c_{u_j} \\
&\le 4\log^+ T_f\left(r+\frac{1}{(T_f(r,L)+\nu)^2}, L\right) \\
&\quad +3\log^+(T_f(r,L)+\nu)^2+4\log^+\left(r+\frac{1}{(T_f(r,L)+\nu)^2}\right)+c_{u_j} \\
&\le 4\log^+(T_f(r,L)+\nu)+3\log^+(T_f(r,L)+\nu)^2+4\log^+ 2r+c_{u_j} \\
&\le 10\log^+ T_f(r,L)+4\log r+10\log\nu+10\log 2+c_{u_j}\,.
\end{aligned}
$$

Hence, if we put $c_1 = 10$, $c_2 = 4$, $c_3 = 10$ and $c_4 = 10\log 2 + c_{u_j}$, then we have our assertion. □

By making use of Lemma 7.5, we have the following estimate (cf. [NO, p. 227]):

LEMMA 7.6. *Let k be a positive integer. Then*

$$
m\left(r, \frac{u_j^{(k)}}{u_j}\right) \le c_1^{(k)}\log^+ T_f(r,\ L) + c_2^{(k)}\log r + c_3^{(k)}\log\nu + c_4^{(k)} \tag{7.7}
$$

for $r \in [1, +\infty) \setminus E(f;\nu)$, where $c_1^{(k)}, \dots, c_4^{(k)}$ are positive constants independent of ν.

PROOF. We will show (7.7) and

$$
T_{u_j^{(k)}}(r) \le (k+1)T_f(r,\ L) + c_1^{(k)}\log^+ T_f(r,\ L) + c_2^{(k)}\log r + c_3^{(k)}\log\nu + c_4^{(k)} \tag{7.8}
$$

by induction on k. Let $r \in [1, +\infty) \setminus E(r;\nu)$. We consider the case where $k = 1$. Lemma 7.5 implies (7.7) for $k = 1$. By Lemma 7.5, we see that

$$
\begin{aligned}
T_{u'_j}(r) &\le (1+1)N(r,\infty,u_j)+m(r,u'_j)+O(1) \\
&\le 2N(r,\infty,u_j)+m(r,u_j)+m\left(r,\frac{u'_j}{u_j}\right)+O(1) \\
&\le 2T_{u_j}(r)+c_1\log^+ T_f(r,L)+c_2\log r+c_3\log\nu+c_4 \\
&\le 2T_f(r,L)+c_1\log^+ T_f(r,L)+c_2\log r+c_3\log\nu+c_4\,.
\end{aligned}
$$

Hence we have

$$
T_{u'_j}(r) \le 2T_f(r,L)+c_1\log^+ T_f(r,L)+c_2\log r+c_3\log\nu+c_4\,.
$$

Thus, we have (7.8) for $k = 1$. □

Next, we assume that (7.7) and (7.8) hold for k with $k \ge 1$. We consider the case where $k+1$. We will show (7.8) for $k+1$. By the hypothesis of induction,

$$
\begin{aligned}
m(r,u_j^{(k+1)}) &\le m(r,u_j)+m\left(r,\frac{u_j^{(k)}}{u_j}\right) \\
&\le m(r,u_j)+c_1^{(k)}\log^+ T_f(r,\ L)+c_2^{(k)}\log r+c_3^{(k)}\log\nu+c_4^{(k)}\,.
\end{aligned}
$$

Hence we have

$$\begin{aligned} T_{u_j^{(k+1)}}(r) &\le (k+2)N(r,\infty,u_j) + m(r,u_j) + c_1^{(k)}\log^+ T_f(r,\ L) + c_2^{(k)}\log r + c_3^{(k)}\log\nu + c_4^{(k)} \\ &\le (k+2)T_f(r,L) + c_1^{(k)}\log^+ T_f(r,\ L) + c_2^{(k)}\log r + c_3^{(k)}\log\nu + c_4^{(k)}\,. \end{aligned}$$

Thus (7.8) is proved for $k+1$. Next, we notice that

$$m\left(r,\ \frac{u_j^{(k+1)}}{u_j}\right) \le m\left(r,\ \frac{u_j^{(k+1)}}{u_j^{(k)}}\right) + m\left(r,\ \frac{u_j^{(k)}}{u_j}\right).$$

By Lemma 7.2 and (7.8), as in the proof of Lemma 7.5, we see that

$$m\left(r,\ \frac{u_j^{(k+1)}}{u_j^{(k)}}\right) \le c_1'\log^+ T_f(r,\ L) + c_2'\log r + c_3'\log\nu + c_4'\,,$$

where $c_1', \dots, c_4'$ are positive constants independent of ν. Hence we have

$$\begin{aligned} &m\left(r,\ \frac{u_j^{(k+1)}}{u_j}\right) \\ &\quad\le (c_1'+c_1^{(k)})\log^+ T_f(r,\ L) + (c_2'+c_2^{(k)})\log r + (c_3'+c_3^{(k)})\log\nu + (c_4'+c_4^{(k)})\,. \end{aligned}$$

This gives us (7.7) for $k+1$. Therefore, we have the desired conclusion. □

Now, we can give an estimate for $S_{F_\Lambda}(r)$ as follows:

LEMMA 7.9. *There exist positive constants* $C_1, \dots, C_4$ *such that*

$$S_{F_\Lambda}(r) \le C_1\log^+ T_f(r,\ L) + C_2\log r + C_3\log\nu + C_4$$

for $r \in [1,\ +\infty) \setminus E(f;\nu)$. *The constants* $C_1, \dots, C_4$ *are independent of* ν.

PROOF. Let $r \in [1,\ +\infty) \setminus E(f;\nu)$. Then, we have

$$\begin{aligned} S_{F_\Lambda}(r) &\le \sum_{J\subseteq Q,\ \sharp J = l_0+1} \int_0^{2\pi} \log|\Delta(u_j;\ j\in J)(z)|\frac{d\theta}{2\pi} \\ &\le \sum_{J\subseteq Q, \sharp J = l_0+1}\ \sum_{j\in J}\sum_{k=1}^{l_0} m\left(r,\ \frac{u_j^{(k)}}{u_j}\right) + O(1) \\ &\le C_1\log^+ T_f(r,L) + C_2\log r + C_3\log\nu + C_4\,. \end{aligned}$$

This gives us the desired conclusion. □

Therefore, we have completed the proof of Lemma 4.3.

REFERENCES

[A] L. V. AHLFORS, The theory of meromorphic curves, Acta. Soc. Sci. Fennicae Nova Ser. A 3 (1941), 3–31.

[Ai] Y. AIHARA, Deficiencies of holomorphic curves in complex projective spaces, Complex analysis and its applications (eds. Y. Imayoshi et al.), 119–125, OCAMI Stud. Vol.2, Osaka Munic. Univ. Press, Osaka, 2007.

[AM] Y. AIHARA AND S. MORI, Deficiencies of meromorphic mappings for hypersurfaces, J. Math. Soc. Japan 97 (2005), 233–258.

[C] H. CARTAN, Sur les zéros des combinaisons linéaires de p fonctions holomorphes donnés, Mathematica (Cluj) 7 (1933), 5–31.

[GO] A. A. GOL'DBERG AND I. V. OSTROVSKII, Value distribution of meromorphic functions, Transl. Math. Monogr., Vol. 236, Amer. Math. Soc., Providence, RI, 2008.

[HK] W. K. HAYMAN AND P. B. KENNEDY, Subharmonic functions Vol. I, Academic Press, London-New York, 1976.

[M1] S. MORI, Elimination of defects of meromorphic mappings of $\boldsymbol{C}^m$ into $\boldsymbol{P}^n(\boldsymbol{C})$, Ann. Acad. Sci. Fennicae Math. 24 (1999), 89–104.

[M2] S. MORI, Defects of holomorphic curves into $\boldsymbol{P}^n(\boldsymbol{C})$ for rational moving targets and a space of meromorphic mappings, Complex Variables Theory and Apl. 43 (2001), 363–379.

[N] R. NEVANLINNA, Le théorème de Picard-Borel et la théorie des fonction méromorphes, Gauthier-Villars, Paris, 1929.

[Nc1] E. I. NOCHKA, On a simple defect relation in Weyl-Ahlfors theory, preprint, 2006.

[Nc2] E. I. NOCHKA, On general forms of a simple defect function in Weyl-Ahlfors theory, preprint, 2006.

[No1] J. NOGUCHI, Nevanlinna theory in several complex variables and Diophantine approximation (in Japanese), Kyoritsu Shuppan, 2003, Tokyo.

[No2] J. NOGUCHI, A note on entire pseudo-holomorphic curves and the proof of Cartan-Nochka's theorem, Kodai Math. J. 28 (2005), 336–346.

[NO] J. NOGUCHI AND T. OCHIAI, Geometric function theory in several complex variables, Transl. of Math. Monogr., Vol. 80, Amer. Math. Soc., Providence, RI, 1990.

[NWY] J. NOGUCHI, J. WINKELMANN AND K. YAMANOI, The second main theorem for holomorphic curves into semi-abelian varieties II, Forumn Math. 20 (2008), 469–503.

[O] T. OCHIAI, Remark on the defect relation of holomorphic curves, Osaka J. Math. 11 (1974), 483–501.

[S1] A. SADULLAEV, Deficient divisors in the Valiron sense, Mat. Sb. 108 (150) (1979), 567–580; English transl. in Math. USSR Sb. 36 (1980), 535–547.

[S2] A. SADULLAEV, Purisubharmonic measures and capacities on complex manifolds, Uspekhi Mat. Nauk 36 (1981), 53–105, 247; English transl. in Russian Math. Surveys 36 (1981), 61–119.

[Sh] B. V. SHABAT, Distribution of values of holomorphic mappings, Transl. of Math. Monogr., Vol. 61, Amer. Math. Soc., Providence, RI, 1985.

[Shi1] B. SHIFFMAN, Applications of geometric measure theory to value distribution theory for meromorphic maps, Value Distribution Theory, Part A, 63–95, Marcel Dekker, New York, 1974.

[Shi2] B. SHIFFMAN, Holomorphic curves in algebraic manifolds, Bull. Amer. Math. Soc. 83 (1977), 553–568.

[Shi3] B. SHIFFMAN, Holomorphic curves and meromorphic maps in complex projective spaces, Indiana Univ. Math. J. 28 (1979), 627–641.

[St] W. STOLL, Value distribution theory for meromorphic maps, Aspects of Math. E7, Vieweg, Braunschweig-Wiesbaden, 1985.

[T] N. TODA, On the defect relation for exponential curves, Bull. Nagoya Inst. Tech. 53 (2001), 135–146.

[W1] H. WEYL AND J. WEYL, Meromorphic curves, Ann. Math. 39 (1938), 516–538.

[W2] H. WEYL, Meromorphic functions and analytic curves, Princeton Univ. Press, Princeton, 1943.

FACULTY OF HUMAN DEVELOPMENT AND CULTURE
FUKUSHIMA UNIVERSITY
FUKUSHIMA 960–1296
JAPAN

E-mail address: aihara@educ.fukushima-u.ac.jp