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# DEFICIENT VALUES OF ENTIRE FUNCTIONS AND THEIR DERIVATIVES

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ABSTRACT. Let $f(z)$ be entire and of finite order, $f^{(n)}$ be the $n$th derivative, and $\Delta_n(f) = \Sigma\delta(a, f^{(n)})$, the sum of all deficient values of $f^{(n)}$. The authors show that $\Delta_n(f)$ can be strictly increasing.

Let $f(z)$ be entire of order $\rho < \infty$, and for $0 \leq j < \infty$ let

$$\Delta_j(f) = \sum_{|a|<\infty} \delta(a, f^{(j)}),$$

where $f^{(0)} = f$ and $f^{(j)}$ is the $j$th derivative. Using the relation [4, p. 104] $\Sigma\delta(a, f) \leq \delta(0, f')$, it is clear that $\Delta_j(f)$ is nondecreasing in $j$ while $\Delta_j(f) \leq 1$ for all $j$. Professor W. H. J. Fuchs [7, p. 167] recently asked if it is possible that $\Delta_j(f)$ be strictly increasing. In this paper we give an affirmative answer. More precisely, we have the stronger

THEOREM. *Let $c_{jk}$ ($j = 0, 1, 2, \ldots$; $k = 1, 2, \ldots, K_j$; $1 \leq K_j \leq \infty$) be finite complex numbers, with $c_{jk} \neq c_{jk'}$ ($k \neq k'$). Given $\frac{1}{2} < \rho < \infty$, and an increasing sequence $\{n_j\}$ of integers, there exists an entire function $f(z)$ of order $\rho$, mean type, such that*

$$\delta(c_{jk}, f^{(n_j)}) > 0$$

*for all $j$ and $k$.*

Recently, two of us [8] proved that if $\Delta = \lim \Delta_j(f) = 1$, then $\Delta_j(f) \equiv 1$ for $j > j_0(f)$. In the example here $\Delta$ is considerably less than 1.

Our proof is based on N. Arakeylan's method [2] which produces entire functions of finite order having an infinite set of deficient values. Here, we have a set of deficient functions rather than numbers, but Arakelyan's method is sufficiently flexible to adapt to this situation. The restriction $\rho > \frac{1}{2}$ is essential, since if $\rho \leq \frac{1}{2}$, then $\Delta_j(f) \equiv 0$ for all $j$.

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## 1. Preliminary propositions.

PROPOSITION 1 (MERGELYAN [6, p. 125]). *Let* $\mathfrak{L}$: $z = z(t)$, $0 \leq t \leq 1$, *be a simple rectifiable curve of length* $L$, $z(0) = a$, $z(1) = b$. *If* $d > 0$ *and* $0 < \varepsilon < 1$, *there exists a polynomial* $P(z)$ *such that*

$$\left|\frac{1}{z-a} - P\left(\frac{1}{z-b}\right)\right| < \varepsilon \tag{1.1}$$

*holds except in a d-neighborhood of* $\mathfrak{L}$ *and also*[2]

$$\left|P\left(\frac{1}{z-b}\right)\right| < \exp\left\{\left(1 + \log\left(1 + \frac{1}{\varepsilon d}\right)\right)e^{A(L/d)+A}\right\} \qquad (|z-b| > d). \tag{1.2}$$

PROPOSITION 2 (MERGELYAN [5, p. 61]). *Let* $f(z)$ *be analytic in the sector* $|\arg z| \leq \alpha/2$, *let the number* $\rho$ *satisfy the condition* $0 \leq \rho \leq \pi/\alpha$, *and* $\varepsilon > 0$, $\eta > 0$ *be any numbers. Then there exists an entire function* $G(z)$ *with*

$$|f(z) - G(z)| < \varepsilon \exp(-|z|^\rho) \tag{1.3}$$

*in the sector* $|\arg z| < \alpha/2 - \eta$ *and*

$$\log|G(z)| < (1 + r)^{\pi/(2\pi-\alpha)}\left\{K + k \max_{0 \leq t \leq kr+1} \frac{t^\rho + \log^+ M(t,f)}{(1+t)^{\pi/(2\pi-\alpha)}}\right\} \tag{1.4}$$

*in the whole plane; in* (1.4) $k$ *is a constant depending on* $\eta$, $K$ *depends on* $\varepsilon$ *and* $\eta$, *and* $M(t,f) = \max|f(te^{i\theta})|$ $(|\theta| \leq \alpha/2)$.

## 2. Proof of the theorem.

2.1. It is no loss of generality to assume $n_j = j$.

Choose $0 < \alpha \leq \min(\pi/\rho, 2\pi - \pi/\rho)$ and $\gamma^j$ ($j = 0, 1, 2, \ldots$) such that

$$0 < \gamma^0 < \gamma^1 < \cdots < \alpha/2,$$

and then, for each $j$, choose

$$\gamma^j < \gamma_{j1} < \gamma_{j2} < \cdots < \gamma_{jK_j} < \gamma^{j+1}.$$

Then we let

$$\gamma_{j,-k} = -\gamma_{jk}, \qquad (j = 0, 1, 2, \ldots; k = 1, 2, \ldots, K_j),$$

$$\alpha_{jk} = \min\left\{\tfrac{1}{2}(\gamma_{jk+1} - \gamma_{jk}), \tfrac{1}{2}(\gamma_{jk} - \gamma_{jk-1})\right\} = \alpha_{j,-k},$$

where

$$\gamma_{jk_j+1} = \gamma^{j+1}, \qquad \gamma_{j0} = \gamma^j \qquad (j = 0, 1, 2, \ldots).$$

Put

$$E_{jkn} = \left\{re^{i\theta}: 2^n \leq r \leq 2^{n+1}, |\theta - \gamma_{jk}| \leq (1/16)\alpha_{jk}\right\},$$

$$E^1_{jkn} = \left\{re^{i\theta}: (15/16)2^n \leq r \leq (17/16)2^{n+1}, |\theta - \gamma_{jk}| \leq (1/8)\alpha_{jk}\right\},$$

$$E^2_{jkn} = \left\{re^{i\theta}: (7/8)2^n \leq r \leq (9/8)2^{n+1}, |\theta - \gamma_{jk}| \leq (1/4)\alpha_{jk}\right\}.$$

We will construct an entire function $f(z)$ which satisfies

$$|f(z)| < \exp\{A(|z|^\rho + 1)\} \tag{2.1}$$

[2]Here and henceforth $A$ denotes a generic positive absolute constant.

for all $z$ and, for a positive sequence $\varepsilon_{jk}$ to be determined by (2.5),

$$|f(z) - (c_{jk}/j!)z^j| < 2\exp\{-A\varepsilon_{jk}|z|^\rho\} \tag{2.2}$$

for

$$\begin{aligned} z &\in E^1_{jkn} \qquad (n \geqslant n_{jk}, n \text{ even}),\\ z &\in E^1_{j,-k,n} \qquad (n \geqslant n_{jk}, n \text{ odd}). \end{aligned} \tag{2.3}$$

The $n_{jk}$ are chosen precisely in (2.15) below. Let $H$ be the set of $(j, k, n)$ which appear in (2.3) and denote a typical element $(j, k, n)$ of $H$ by $h$.

We see that $f(z)$ is our required function. In fact, when $z \in E_h$, a disk with center at $z$ and radius $10^{-2}\alpha_{jk}2^n$ is contained completely in $E^1_{jkn} = E^1_h$. According to Cauchy's inequality and (2.2),

$$\begin{aligned} |(f(z) - (c_{jk}/j!)z^j)^{(j)}| &< 2\cdot 10^{2j}j!\,\frac{\exp\{-A\varepsilon_{jk}[|z| - (10)^{-2}\alpha_{jk}2^n]^\rho\}}{(\alpha_{jk}2^n)^j}\\ &< 2\cdot 10^{2j}j!\,\frac{\exp\{-A\varepsilon_{jk}|z|^\rho\}}{(\alpha_{jk}2^n)^j} \qquad (z \in E_h), \end{aligned}$$

i.e.

$$\frac{1}{|f^{(j)}(z) - c_{jk}|} > \frac{A}{(2\cdot 10^{2j}j!)}\alpha^j_{jk}2^{nj}\exp\{A\varepsilon_{jk}|z|^\rho\} \qquad (z \in E_h),$$

where $h = (j, k, n) \in H$. On noting from (2.1) that

$$T(r, f^{(j)}) = m(r, f^{(j)}) \leqslant m(r, f) + m(r, f^{(j)}/f) < A(r^\rho + 1),$$

we obtain by integrating over $(|z| = r) \cap E_h$

$$\delta(c_{jk}, f^{(j)}) \geqslant A\alpha_{jk}\varepsilon_{jk} > 0 \qquad (j = 0, 1, 2, \ldots; k = 1, 2, \ldots, K_j).$$

2.2. We now construct a function $Q(\zeta, z)$. When $h = (j, k, n) \in H$, let $C_h$ be the arc of the circle $|z| = (9/8)2^{n+1}$ linking $\partial E^2_h$ to the point $z = -(9/8)2^{n+1}$ which does not meet the positive axis, and let $D_h$ be the $(2^{n-5}\alpha_{jk})$-neighborhood of $C_h \cup \partial E^2_h$. If $\zeta$ is an arbitrary point of $\partial E^2_h$, we may connect $\zeta$ to $z = -(9/8)2^{n+1}$ by a curve contained in $C_h \cup \partial E^2_h$. This curve has length less than $A\cdot 2^n$, so Proposition 1 produces a rational function $Q(\zeta, z)$ with a unique pole at $-(9/8)2^{n+1}$ such that

$$|Q(\zeta, z) - 1/(\zeta - z)| < \eta_h \qquad (\zeta \in \partial E^2_h, z \notin D_h). \tag{2.4}$$

Thus let

$$\varepsilon_{jk} = \exp(-A/\alpha_{jk}) \tag{2.5}$$

so that $\varepsilon_{jk} > 0$ and

$$\sum_{j,k}\varepsilon_{jk} = \sum_{j,k}\exp(-A/\alpha_{jk}) \leqslant \sum_{j,k}\alpha_{jk}/A \leqslant A.$$

Then we choose

$$\eta_h = \eta_{jkn} = \alpha^{-1}_{jk}2^{-(jn+2n+2j)}\exp\{-4^{\rho+1}\varepsilon_{jk}2^{n\rho}\},$$

and observe that

$$\{1 + \log(1 + A/\eta_h 2^n \alpha_{jk})\} \exp(A/\alpha_{jk}) < A 4^\rho \varepsilon_{jk} 2^{n\rho} \exp(A/\alpha_{jk}) < A 4^\rho \varepsilon_{jk}^{1/2} \qquad (n > n_{jk}).$$

Further, recall the choice of $\alpha$ from the beginning of §2.1. Then if $|\arg z| < \alpha/2$, it is clear that $|z - (-(9/8)2^{n+1})| > A 2^{n+1}$. With these choices of $\varepsilon$ and $\eta_h$ in (1.2) and (2.4) we have

$$\left| Q(\zeta, z) - \frac{1}{\zeta - z} \right| < \alpha_{jk}^{-1} 2^{-(jn+2n+2j)} \exp\{-4^{\rho+1} \varepsilon_{jk} \cdot 2^{n\rho}\} \qquad (\zeta \in \partial E_h^2, z \notin D_h), \tag{2.6}$$

$$|Q(\zeta, z)| < \exp(A 4^\rho \varepsilon_{jk}^{1/2} 2^{n\rho}) \qquad (|\arg z| \leqslant \alpha/2, \zeta \in \partial E_h^2). \tag{2.7}$$

2.3. The next proposition is essentially in [1], [2, p. 96].

PROPOSITION 3. *Let* $0 < \alpha \leqslant \min(\pi/\rho, 2\pi - \pi/\rho)$, $\gamma_{jk}$ *and* $\alpha_{jk}$ *be as in* §2.1, *and set*

$$\psi(z) = \exp(-\varepsilon_{jk} z^\rho) \qquad (|\arg z - \gamma_{jk}| < \tfrac{3}{4}\alpha_{jk}). \tag{2.8}$$

*Then there exists a function* $\omega(z)$ *holomorphic in* $|\arg z| < \alpha/2$ *such that*

$$|\omega(z)| < \exp(1 + |z|^\rho) \qquad (|\arg z| < \alpha/2) \tag{2.9}$$

*and for* $h = (j, k, n) \in H$

$$A < |\omega(z)/\psi(z)| < A \qquad \left(z \in \bigcup_H E_h^2\right). \tag{2.10}$$

PROPOSITION 4. *We can choose* $n_{jk}$ *so that if*

$$g(z) = (C_j/j!) z^j \qquad (z \in E_h^2, n > n_{jk}), \tag{2.11}$$

*there there exists a function* $F(z)$ *analytic in* $|\arg z| \leqslant \alpha/2$ *such that*

$$|F(z)| < \exp A(|z|^\rho + 1) \qquad (|\arg z| \leqslant \alpha/2) \tag{2.12}$$

*and*

$$|F(z) - (C_{jk}/j!) z^j| < |\exp(-\varepsilon_{jk} z^\rho)| \qquad (z \in E_h^1, n > n_{jk}). \tag{2.13}$$

PROOF. Let $\omega(z)$ be the function obtained in Proposition 3. Then if $h = (jkn) \in H$, and

$$g_h(z) = \frac{1}{2\pi i} \int_{\partial E_h^2} \frac{g(\zeta)}{\omega(\zeta)} Q(\zeta, z)\, d\zeta,$$

Proposition 3 shows that $g_h$ is analytic in $|\arg z| \leqslant \alpha/2$. Using (2.6), (2.8), (2.10) and (2.11), we can obtain

$$\left| g_h(z) - \frac{1}{2\pi i} \int_{\partial E_h^2} \frac{g(\zeta) d\zeta}{\omega(\zeta)(\zeta - z)} \right|$$

$$< A \int_{\partial E_h^2} |c_{jk}|\, |\zeta|^j \alpha_{jk}^{-1} \exp\{-4^{\rho+1} \varepsilon_{jk} 2^{n\rho} + \varepsilon_{jk} |\zeta|^\rho\} \frac{|d\zeta|}{2^{(nj+2n+2j)}}$$

$$< A |c_{jk}| \alpha_{jk}^{-1} \exp(-\varepsilon_{jk} \cdot 2^{n\rho}) \cdot 2^{-n} \qquad (z \notin D_h^1). \tag{2.14}$$

Choose $n_{jk}$ so that

$$\alpha_{jk}^{-1}\exp(-\varepsilon_{jk}\cdot 2^{n\rho}) < \exp\{-(\varepsilon_{jk}/2)\cdot 2^{n\rho}\} \qquad (n > n_{jk}) \tag{2.15}$$

and that

$$A\sum_{H}|c_{jk}|2^{-n} < 1. \tag{2.16}$$

The integral in the left-hand side of (2.14) is given by

$$\frac{1}{2\pi i}\int_{\partial E_h^2}\frac{g(\zeta)d\zeta}{\omega(\zeta)(\zeta - z)} = \begin{cases} 0 & \text{if } z \notin E_h^2, \\ g(z)/\omega(z) & \text{if } z \in E_h^2. \end{cases} \tag{2.17}$$

When $z \in E_h^2 \cup D_h^1$, we have $2^{n-1} < |z| < 2^{n+2}$, so that

$$\begin{aligned} |g_h(z)| &< A|c_{jk}|2^{(n+2)j}\exp\left(A4^{\rho}\varepsilon_{jk}^{1/2}2^{n\rho}\right)\cdot\exp(\varepsilon_{jk}2^{n\rho})2^n \\ &< |c_{jk}|2^{-n}\exp\left(A4^{\rho}\varepsilon_{jk}^{1/2}2^{n\rho}\right) \end{aligned} \tag{2.18}$$

by (2.7) and Proposition 3.

Consequently, from (2.14), (2.16), (2.17) and (2.18),

$$G(z) = \sum_{H} g_h(z)$$

is absolutely convergent for every compact region in the angle $|\arg z| < \alpha/2$ and $G(z)$ is analytic in this angle. Moreover, from (2.14), (2.17) and (2.18), we have

$$|G(z)| < \sum_{H}|g_{jkn}(z)| < \exp(A(1 + |z|^{\rho}))$$

and

$$\left|G(z) - \frac{c_{jk}}{j!}\frac{z^j}{\omega(z)}\right| < 1 \qquad (z \in E_h^1).$$

Thus if $F(z) = G(z)\omega(z)$, then $F(z)$ satisfies the conditions of Proposition 4.

2.4. In order to complete the proof of the theorem, we apply Proposition 2. There exists an entire function $f(z)$ such that

$$|f(z) - F(z)| < \exp(-|z|^{\rho}) \tag{2.19}$$

in the angle $|\arg z| < \alpha/2 - \eta$ and

$$\log|f(z)| < (1 + r)^{\pi/(2\pi-\alpha)}\left\{K + k\max_{0<t<kr+1}\frac{t^{\rho} + \log^{+}M(t, F)}{(1 + t)^{\pi/(2\pi-\alpha)}}\right\}$$

in the whole plane.

On noting (2.12) and

$$\max_{0<t<kr+1}\frac{t^{\rho}}{(1 + t)^{\pi/(2\pi-\alpha)}} < Ak^{\rho}\frac{r^{\rho}}{(1 + r)^{\pi/(2\pi-\alpha)}}$$

we have (2.1). In every $E_h^1$, we obtain from (2.13) and (2.19)

$$\begin{aligned} |f(z) - (c_{jk}/j!)z^j| &< e^{-r^{\rho}} + |F(z) - (c_{jk}/j!)z^j| \\ &< e^{-r^{\rho}} + \exp(-A\varepsilon_{jk}r^{\rho}) < 2\exp(-A\varepsilon_{jk}r^{\rho}). \end{aligned}$$

Thus (2.2) is satisfied and so $f(z)$ is our desired function.

## References

1. N. U. Arakeljan, *Entire functions of finite order with an infinite set of deficient values*, Soviet Math. Dokl. 7 (1966), 1303–1306.

2. W. H. J. Fuchs, *Théorie de l'approximation des fonctions d'une variable complexe*, Université de Montréal, 1968.

3. A. A. Goldberg and I. V. Ostrovskiĭ, *The distribution of values of meromorphic functions*, "Nauka", Moscow, 1970. (Russian)

4. W. K. Hayman, *Meromorphic functions*, Clarendon, Oxford, 1964.

5. S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, Uspehi Mat. Nauk **7** (1952), no. 2 (48), 31–122; English transl., Amer. Math. Soc. Transl. **101** (1954), 1–99.

6. ______, *On the completeness of systems of analytic functions*, Uspehi Mat. Nauk **8** (1953), no. 4 (56), 3–63; English transl., Amer. Math. Soc. Transl. (2) **19** (1962), 109–166.

7. Sanford S. Miller, *Complex analysis* (Proc. Conf. S.U.N.Y. Brockport, 1978), Dekker, New York, 1978.

8. Yang Lo and Zhang Guanghou, *Deficient values and asymptotic values of entire functions*, Sci. Sinica, Special Issue (II) (1979), 190–203.

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