

DEFINABILITY AND STABILITY OF MULTISCALE DECOMPOSITIONS FOR MANIFOLD-VALUED DATA.

PHILIPP GROHS AND JOHANNES WALLNER

ABSTRACT. We discuss multiscale representations of discrete manifold-valued data. As it turns out that we cannot expect general manifold-analogues of biorthogonal wavelets to possess perfect reconstruction, we focus our attention on those constructions which are based on upscaling operators which are either interpolating or midpoint-interpolating. For definable multiscale decompositions we obtain a stability result.

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1. INTRODUCTION

1.1. The problem area. The correct multiscale representation of manifold-valued data is a basic question whenever one wishes to eliminate the arbitrariness in choosing coordinates for such data, and to avoid artifacts caused by applying linear methods to the ensuing coordinate representations of data. This question appears to have been proposed first by D. Donoho [2]. The detailed paper [11] describes different constructions, including most of ours, and states results inferred from numerical experiments, but without giving proofs. A series of papers, starting with [12], has since dealt with the systematic analysis of upscaling operations on discrete data – also known under the name *subdivision rules* – in the case that data live in Lie groups, Riemannian manifolds, and other nonlinear geometries. Regarding smoothness of limits, a satisfactory solution has been achieved by means of the method of *proximity inequalities* which also play a role in the present paper. Multiscale decompositions in particular have been investigated by [6] (characterizing smoothness by decay of detail coefficients) and [8] (stability).

The present paper studies multiscale decompositions which are analogous to linear biorthogonal wavelets and reviews the known examples based on interpolatory and midpoint-interpolating subdivision rules including the simple Haar wavelets. It turns out, however, that it seems unlikely that a rather general way of defining manifold analogues of linear constructions can have perfect reconstruction, which is the first main result of this paper, even if it turns out to be rather vague. For those multiscale decompositions which exist, we show a stability theorem which represents the second main result of the paper. We further discuss averaging procedures which work in manifolds equipped with an exponential mapping and which generalize the well known Riemannian center of mass. This discussion does not contain substantial new results, but it is included because we need this construction for the definition of nonlinear up- and downscaling rules, as well as for converting continuous data to discrete data in the first place.

1.2. Biorthogonal wavelets revisited. We begin by briefly reviewing the notion of biorthogonal Riesz wavelets, but we are content with the properties relevant for the following sections. We start with real-valued sequences $\alpha = (\alpha_i)_{i \in \mathbb{Z}}$ with finite support which are called *filters* and define the *upsampling rule*, or *subdivision rule* associated with the filter α by

$$(S_\alpha c)_k := \sum_{l \in \mathbb{Z}} \alpha_{k-2l} c_l.$$

Here $c : \mathbb{Z} \rightarrow V$ is any sequence with values in a vector space. The transpose of the upscaling rule (we skip the definition of *transpose*) shall be the *downscaling rule* D associated with the filter β , via

$$(D_\beta c)_k := \sum_{l \in \mathbb{Z}} \beta_{l-2k} c_l.$$

Upsampling and downscaling commutes with the left shift operator $(Lc)_k = c_{k+1}$ in the following way:

$$S_\alpha L = L^2 S_\alpha, \quad D_\beta L^2 = L D_\beta.$$

The most basic rules are defined by the delta sequence: S_δ inserts zeros between the elements of the original sequence, and D_δ deletes every other element. All rules can be expressed in terms of S_δ , D_δ , and convolution:

$$\begin{aligned} S_\delta c &= (\dots, c_0, 0, c_1, 0, c_2, \dots), & D_\delta c &= (\dots, c_0, c_2, c_4, \dots) \\ \implies S_\alpha c &= (S_\delta c) * \alpha, & D_\beta c &= D_\delta(c * \beta). \end{aligned}$$

We suppress the indices α, β from now on. We assume a further upscaling rule R and a downscaling rule Q which shall be high pass filters in contrast to low-pass filters S and D .¹

Any sequence $c^{(j)}$, which is interpreted as *data at level j* may be recursively decomposed into a low-frequency-part $c^{(j-1)}$ (data at level $j - 1$) and a high-frequency-part $d^{(j)}$ (details at level j) by letting

$$(1) \quad c^{(j-1)} = Dc^{(j)}, \quad d^{(j)} = Qc^{(j)}.$$

This process can be iterated in order to obtain a pyramid consisting of *coarse data* $c^{(0)}$ and *wavelet coefficients* $d^{(1)}, \dots, d^{(j)}$. Data at level j shall be reconstructed by

$$(2) \quad c^{(j)} = Sc^{(j-1)} + Rd^{(j)},$$

which works precisely if the so-called *quadrature mirror filter equation*,

$$(3) \quad SD + RQ = \text{id},$$

holds. It makes sense to require certain further ('biorthogonality') properties like $QR = \text{id}$. In particular, high pass downscaling should annihilate everything generated by low pass upscaling:

$$(4) \quad QS = 0.$$

An important consequence of the previous properties is that we can rewrite (1) in the form

$$(5) \quad c^{(j-1)} = Dc^{(j)}, \quad d^{(j)} = Q(c^{(j)} - Sc^{(j-1)}).$$

There are many examples of biorthogonal wavelet decompositions. In the following we give some examples.

1.3. Examples: interpolating and midpoint-interpolating schemes.

Example 1.1. An upscaling scheme is called *interpolating*, if it keeps the original data, which is expressed by

$$(Sc)_{2k} = c_k \iff D_\delta S = \text{id}.$$

For interpolating schemes, downscaling is simply $D = D_\delta$. Then detail coefficients are the difference between data c and the prediction gained via upscaling of Dc . With the left shift operator, we can write

$$Qc = DL(c - SDc).$$

If we define detail coefficients via (5), then we can also employ the modified downscaling operator

$$Q^{\text{modif}} = DL.$$

Reconstruction works via a basic upscaling rule:

$$R = L^{-1}S_\delta$$

It is easy to check that we have indeed perfect reconstruction. An example is furnished by the four-point scheme [4] defined by $\alpha_{\{-3, \dots, 3\}} = (-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16})$. The action $c^{(j-1)} = Dc^{(j)}$ of the decimation operator is consistent with the interpretation of discrete data $c_k^{(j)}$ as *samples* of a continuous function $f(t)$ at the parameter value $t = \frac{1}{2^j}k$.

¹usually formulated in terms of Fourier transforms.

Example 1.2. The *Haar scheme* is defined by the rules

$$S = (L + \text{id})S_\delta, \quad D = \frac{1}{2}D_\delta(L + \text{id}), \quad R = (\text{id} - L)S_\delta, \quad Q = \frac{1}{2}D_\delta(\text{id} - L),$$

which operate as follows:

$$\begin{aligned} Sc &= (\dots, c_0, c_0, c_1, c_1, \dots), \\ Rd &= (\dots, d_0, -d_0, d_1, -d_1, \dots), \\ Dc &= (\dots, \frac{c_0 + c_1}{2}, \frac{c_2 + c_3}{2}, \dots), \\ Qc &= (\dots, \frac{c_0 - c_1}{2}, \frac{c_2 - c_3}{2}, \dots). \end{aligned}$$

Example 1.3. A subdivision scheme S is called *midpoint-interpolating*, if it is a right inverse of the decimation operator D which computes midpoints and which is also used for the Haar wavelets of Example 1.2:

$$DS = \text{id}, \quad \text{where} \quad Dc = (\dots, \frac{c_0 + c_1}{2}, \frac{c_2 + c_3}{2}, \dots).$$

The detail coefficients are the difference between that actual data c and the imputation SDc found by upscaling the decimated data. Since $c - SDc$ is by construction in the kernel of D (i.e., is an alternating sequence), it contains redundant information. We thus complete our definitions by letting

$$\begin{aligned} Qc &= D_\delta(c - SDc) = (\dots, (c - SDc)_0, (c - SDc)_2, \dots), \\ Rd &= (\text{id} - L)S_\delta d = (\dots, d_0, -d_0, d_1, -d_1, \dots). \end{aligned}$$

If we define detail coefficients via (5), then a much simpler downscaling operator for details can be employed:

$$Q^{\text{modif}} = D_\delta.$$

The action $c^{(j-1)} = Dc^{(j)}$ of the decimation rule is consistent with the interpretation of discrete data $c_k^{(j)}$ as an *average* of continuous data over the interval $\frac{1}{2^j} \cdot [k, k + 1]$.

The defining relation implies that any such S can be turned into an interpolating subdivision rule \tilde{S} by adding one round of midpoint computation:

$$\tilde{S} = \frac{1}{2}(L + \text{id})S.$$

\tilde{S} is interpolatory, since $D_\delta\tilde{S} = \frac{1}{2}(D_\delta L + D_\delta) = DS = \text{id}$. The relation $S = 2(L + \text{id})^{-1}\tilde{S}$ leads to a way of finding midpoint-interpolating schemes from interpolatory ones, since it can be turned into an effective computation by the use of symbols [5]. For more information on that kind of schemes, see e.g. [3].

2. BIORTHOGONAL DECOMPOSITIONS FOR MANIFOLD-VALUED DATA

2.1. Manifold analogues of linear elementary constructions. The main idea to apply the previous constructions to manifold-valued data is to find replacements for the elementary operations they are composed of. These are the operations $-$ (“vector is difference of points”), $+$ (“point plus vector is a point”), and computing the weighted average of points, which again yields a point. As to which kind of data are points and which are vectors, data $c^{(j)}$ at level j shall be manifold-valued sequences of points, while detail coefficients $d^{(j)}$ shall be sequences with values in vector spaces associated with the manifold.

For data with values in a Lie group G , with associated Lie algebra \mathfrak{g} , we let

$$p \oplus v := p \exp(v), \quad q \ominus p := \log(p^{-1}q) \in \mathfrak{g},$$

where \exp is the group exponential function and \log is its inverse. For matrix groups, we have $\exp(x) = \sum_{k \geq 0} x^k/k!$ as usual (see e.g. [1] for Lie theory). In a surface or Riemannian manifold M , we use the exponential mapping \exp_p which maps a vector v in the tangent space T_pM to the endpoint of a geodesic of length $\|v\|$ which emanates from p with initial tangent vector v :

$$p \oplus v := \exp_p(v), \quad q \ominus p := \exp_p^{-1}(q) \in T_pM.$$

We have thus found analogues \oplus and \ominus of the $+$ and $-$ operations, respectively. An average with weights of total sum 1 is in Euclidean space equivalently definable by

$$(6) \quad m = \sum \alpha_j x_j \iff \sum \alpha_j (x_j - m) = 0 \iff \sum \alpha_j \text{dist}(x_j, m)^2 = \min.$$

The middle definition carries over to both Lie groups and Riemannian manifolds (provided m is unique, which it locally is):

$$(7) \quad \sum \alpha_j (x_j \ominus m) = 0.$$

In Riemannian manifolds, this average is the same as the one defined by the right hand condition. These constructions have been employed to define operations on manifold-valued data before, in particular subdivision processes. For more details the reader is referred to [8].

Another way of redefining averages is by means of an auxiliary base point: In a vector space, we have

$$\sum \alpha_j = 1 \implies \sum \alpha_j x_j = x + \sum \alpha_j (x_j - x),$$

for any choice of x . This leads to the definition

$$(8) \quad x \oplus \left(\sum \alpha_j (x_j \ominus x) \right)$$

of manifold average which involves the choice of an additional base point.

Example 2.1. It is not difficult to see that the weights $\alpha_0 = \alpha_1 = \frac{1}{2}$ lead to a symmetric average $m = \mu(x_0, x_1) = x_0 \oplus \frac{1}{2}(x_1 \ominus x_0) = x_1 \oplus \frac{1}{2}(x_0 \ominus x_1)$, which can be taken as the manifold-midpoint of x_0 and x_1 . It fulfills the balance condition $(x_1 \ominus m) + (x_0 \ominus m) = 0$.

An obvious generalization, where the averaging process possibly works with a continuum of values is defined as follows: Consider a set X which is equipped with some probability measure. For instance we could take the unit interval $X = [0, 1]$ with Lebesgue measure. The weighted average m of data $(f(t))_{t \in X}$ with values in a vector space is defined by the following equivalent definitions

$$(9) \quad m = \int_X f(x) \iff \int_X (f(x) - m) = 0 \iff \int_X \text{dist}(f(x), m)^2 = \min.$$

In the case that X is the integers, and the measure means giving each $i \in \mathbb{Z}$ the weight α_i , then this definition reduces to (6). Also the integral version of the average can be made to work for manifold-valued data, by defining m via

$$(10) \quad \int_X (f(x) \ominus m) = 0.$$

In the Riemannian case, which has been thoroughly discussed by Karcher [9], this is equivalent to $\int_X \text{dist}(f(x), m)^2 = \min$. It is then called the Riemannian center of mass (see Section IX.2 of [10]).

2.2. Manifold versions of filters. We now define nonlinear analogues of the up- and down-scaling rules S, D, Q, R . In order to distinguish them from the corresponding nonlinear rules, we write the latter as $S_{lin}, D_{lin}, Q_{lin}, R_{lin}$. The symbols $\mathcal{S}, \mathcal{D}, \mathcal{Q}, \mathcal{R}$ denote nonlinear up- and downscaling operators which like the linear ones commute with the left shift operator in the following way:

$$SL = L^2S, \quad \mathcal{D}L^2 = L\mathcal{D}, \quad \mathcal{R}L = L^2\mathcal{R}, \quad \mathcal{Q}L^2 = L\mathcal{Q}.$$

We now decompose manifold-valued data ‘at level j ’, which are denoted by the symbol $c^{(j)}$ in a manner similar to (5):

$$(11) \quad c^{(j-1)} = \mathcal{D}c^{(j)} \quad d^{(j)} = \mathcal{Q}(c^{(j)} \ominus \mathcal{S}\mathcal{D}c^{(j)}).$$

By iteration we arrive at data $c^{(0)}$ at the coarsest scale together with a pyramid of detail coefficients $d^{(1)}, \dots, d^{(j)}$. In order to obtain perfect reconstruction via

$$(12) \quad c^{(j)} = \mathcal{S}c^{(j-1)} \oplus \mathcal{R}d^{(j)}$$

we impose the following condition on the nonlinear operators which could be interpreted as a nonlinear quadrature mirror filter equation:

$$(13) \quad \mathcal{S}\mathcal{D}c \oplus (\mathcal{R}\mathcal{Q}(c \ominus \mathcal{S}\mathcal{D}c)) = c \quad \text{for all } c.$$

2.3. Examples: interpolating and midpoint-interpolating schemes.

Example 2.2. (manifold version of Example 1.2) We show how the Haar scheme can be made to work in groups and in Riemannian manifolds. With the midpoint $\mu(p, q)$ of Example 2.1 we let

$$\begin{aligned} \mathcal{S}c &= S_{lin}c = (\dots, c_0, c_0, c_1, c_1, \dots), \\ \mathcal{D}c &= (\dots, \mu(c_0, c_1), \mu(c_1, c_2), \dots) \end{aligned}$$

while $\mathcal{Q} = Q_{lin}$ and $\mathcal{R} = R_{lin}$. Indeed, $c \ominus \mathcal{S}\mathcal{D}c$ is an alternating sequence of vectors, and the detail coefficients associated with data c are given by

$$\begin{aligned} d &= Q_{lin}(c \ominus \mathcal{S}\mathcal{D}c) \\ &= Q_{lin}(\dots, c_0 \ominus \mu(c_0, c_1), c_1 \ominus \mu(c_0, c_1), c_2 \ominus \mu(c_2, c_3), \dots), \\ &= (\dots, c_0 \ominus \mu(c_0, c_1), c_2 \ominus \mu(c_2, c_3), \dots). \end{aligned}$$

It is obvious that with this definition, $\mathcal{S}\mathcal{D}c \oplus \mathcal{R}d = c$, so we have perfect reconstruction.

Example 2.3. (manifold version of Example 1.1) To find a nonlinear analogue \mathcal{S} of a linear upscaling rule defined by affine averages, we can employ geometric averages instead. In this way the interpolating scheme $S_{lin} = S_\alpha$ can be transferred to the geometric setting, by letting

$$(\mathcal{S}c)_{2k} = c_k, \quad \sum_{r \in \mathbb{Z}} \alpha_{2r+1} (c_{k-r} \ominus (\mathcal{S}c)_{2k+1}) = 0.$$

The remaining rules can be taken from the linear case (using the fact that the simplest rules can be applied to *any* sequence, as its elements do not undergo computations).

$$\mathcal{D} = D_{lin} = D_\delta, \quad \mathcal{Q} = Q_{lin} = Q^{\text{modif}} = LD_\delta, \quad \mathcal{R} = R_{lin} = L^{-1}S_\delta.$$

From the interpolating property of \mathcal{S} we see that we have perfect reconstruction.

Example 2.4. (manifold version of Example 1.3) In order to make a midpoint-interpolating rule S_{lin} work on manifolds, we define an upscaling operator \mathcal{S} which retains the crucial property that c_k is the midpoint of $(\mathcal{S}c)_{2k}$ and $(\mathcal{S}c)_{2k+1}$. For this purpose we use (8). We introduce the following notation for sequences c, v and a point $x \in M$:

$$(c \ominus x)_k := c_k \ominus x, \quad (x \oplus v)_k := x \oplus v_k,$$

and define

$$(\mathcal{S}c)_{2k} = c_k \oplus (S_{lin}(c \ominus c_k))_{2k}, \quad (\mathcal{S}c)_{2k+1} = c_k \oplus (S_{lin}(c \ominus c_k))_{2k+1}.$$

It is clear from $(c \ominus c_k)_k = 0$ and the midpoint-interpolating property of S_{lin} , that \mathcal{S} is also midpoint-interpolating:

$$\mu((\mathcal{S}c)_{2k}, (\mathcal{S}c)_{2k+1}) = c_k.$$

We use the same downscaling operators \mathcal{Q}, \mathcal{D} as in the Haar case of Example 2.2, which yields

$$d_k^{(j)} = (c^{(j)} \ominus \mathcal{S}c^{(j-1)})_{2k}.$$

By midpoint interpolation, $c^{(j-1)}$ and $d^{(j)}$ together determine the original data $c^{(j)}$: With the geodesic reflection $\sigma_x(y)$ of y in the point x defined by

$$\sigma_x(y) = x \oplus (- (y \ominus x)) \quad \text{or, locally equivalently,} \quad \mu(y, \sigma_x(y)) = x,$$

we have

$$c_{2k}^{(j)} = (\mathcal{S}c^{(j-1)})_{2k} \oplus d_k^{(j)}, \quad c_{2k+1}^{(j)} = \sigma_{c_k^{(j-1)}}(c_{2k}^{(j)}).$$

This construction is already contained in [11]. A nonlinear upscaling operator \mathcal{R} which effects exactly this construction via $c^{(j)} = c^{(j-1)} \oplus \mathcal{R}d^{(j)}$ necessarily depends on the data and may be defined by

$$(\mathcal{R}d)_{2k} = d_k, \quad (\mathcal{R}d)_{2k+1} = \sigma_{c_k^{(j-1)}}\left((\mathcal{S}c^{(j-1)})_{2k} \oplus d_k^{(j)}\right) \ominus \mathcal{S}c_{2k+1}^{(j)}.$$

In Riemannian geometry we cannot further simplify this expression. In the case of matrix groups, we employ the fact that $\sigma_x(y) = xy^{-1}x$ and that successive points with indices $2k, 2k+1$ of $\mathcal{S}c$ are converted into each other by geodesic reflection in the point c_k :

$$\begin{aligned} (\mathcal{R}d)_{2k+1} &= \log \left[\left(\mathcal{S}c_{2k+1}^{(j-1)} \right)^{-1} \left(c_k^{(j-1)} \right) \left(\mathcal{S}c_{2k}^{(j-1)} \exp d_k^{(j)} \right)^{-1} \left(c_k^{(j-1)} \right) \right] \\ &= \log \left[\left(c_k^{(j-1)} \right)^{-1} \left(\mathcal{S}c_{2k}^{(j-1)} \right) \exp \left(-d_k^{(j)} \right) \left(\mathcal{S}c_{2k}^{(j-1)} \right)^{-1} \left(c_k^{(j-1)} \right) \right] \\ &= -\text{Ad}_{(c_k^{(j-1)})^{-1}(\mathcal{S}c_{2k}^{(j-1)})} (d_k^{(j)}) = -\text{Ad}_{\exp((\mathcal{S}c_{2k}^{(j-1)}) \ominus (c_k^{(j-1)}))} (d_k^{(j)}) \\ &= -\text{Ad}_{\exp(S_{lin}(c^{(j-1)} \ominus c_k^{(j-1)})_{2k})} (d_k^{(j)}). \end{aligned}$$

Here we have used the notation $\text{Ad}_g(v) = gvg^{-1}$. Note that in abelian groups and especially in Euclidean space, where $g \oplus v = g + v$, this formula reduces to $\mathcal{R}d_{2k+1} = -\mathcal{R}d_{2k}$.

2.4. On the general feasibility of the construction. The examples of geometric and nonlinear multiscale decompositions given above are special cases, which are based on interpolatory subdivision rules, or at midpoint-interpolating rules. It is not clear how perfect reconstruction can be achieved in general. We shall presently see that there are some basic obstructions which disappear in the linear case. For simplicity we consider only periodic sequences, because then the upscaling and downscaling rules have a finite-dimensional domain of definition.

Prop. 2.5. *Smooth rules $\mathcal{S}, \mathcal{D}, \mathcal{Q}, \mathcal{R}$ can lead to detail coefficients with perfect reconstruction for periodic data $c \in M^{2n}$ only if the rank of the mapping $c \mapsto c \ominus \mathcal{S}\mathcal{D}c$ equals $n \cdot \dim M$, which is half the generic rank of such a mapping.*

Proof. Equation (13), which expresses perfect reconstruction, is equivalent to

$$\mathcal{R}\mathcal{Q}x = x, \quad \text{where } x = c \ominus \mathcal{S}\mathcal{D}c.$$

It follows that the mapping $c \mapsto c \ominus \mathcal{S}\mathcal{D}c = \mathcal{R}\mathcal{Q}(c \ominus \mathcal{S}\mathcal{D}c)$ has rank $\leq n \cdot \dim M$, because \mathcal{Q} , mapping $2n$ data items to n detail coefficients, has this property. As to the mapping $c \mapsto \mathcal{S}\mathcal{D}c$, its rank does not exceed $n \cdot \dim M$, because \mathcal{D} has this property. In case the rank is less than $n \cdot \dim M$, the mapping $\text{id}_{M^{2n}} : c \mapsto \mathcal{S}\mathcal{D}c \oplus (c \ominus \mathcal{S}\mathcal{D}c)$ would have rank $< 2n \cdot \dim M$, a contradiction. \square

The condition of rank $n \cdot \dim M$ which is necessary for perfect reconstruction as mentioned in Prop. 2.5 is unlikely to be satisfied if both upscaling by \mathcal{S} and downscaling by \mathcal{D} are defined via geometric averaging rules derived from linear rules S_α and D_β . The following discussion of derivatives should make this clear: We have

$$(14) \quad \sum_l \alpha_{k-2l}(c_l \ominus \mathcal{S}c_k) = 0, \quad \sum_l \beta_{l-2k}(c_l \ominus \mathcal{D}c_k) = 0,$$

and we are interested in the change in $(\mathcal{S}\mathcal{D}c)_k$ if each c_l undergoes a 1-parameter variation. We use the abbreviations ϕ and ψ for the derivatives of \ominus with respect to the first and second argument, respectively. In the Lie group case, where all tangent vectors are represented by elements of the Lie algebra \mathfrak{g} , both ϕ and ψ are linear endomorphisms of \mathfrak{g} . In case of Riemannian manifolds, where $\ominus : M \times M \rightarrow TM$, both ϕ, ψ map to $T_{p \ominus q}(TM)$. As the next formula shows it is not necessary to look closer at this abstract tangent space, because we always combine ψ^{-1} with ϕ and the image of ϕ occurs only implicitly. Differentiation of (14) implies that

$$\frac{d}{dt}(\mathcal{D}c)_k = -\left(\sum_l \beta_{l-2k} \psi_{c_l, \mathcal{D}c_k}\right)^{-1} \left(\sum_l \beta_{l-2k} \phi_{c_l, \mathcal{D}c_k} \frac{d}{dt}c_l\right).$$

and further

$$\begin{aligned} \frac{d}{dt}(\mathcal{S}\mathcal{D}c)_k &= \left(\sum_l \alpha_{k-2l} \psi_{\mathcal{D}c_l, \mathcal{S}\mathcal{D}c_k}\right)^{-1} \\ &\cdot \left(\sum_l \alpha_{k-2l} \phi_{\mathcal{D}c_l, \mathcal{S}\mathcal{D}c_k} \left(\sum_r \beta_{r-2l} \psi_{c_r, \mathcal{D}c_l}\right)^{-1} \left(\sum_r \beta_{r-2l} \phi_{c_r, \mathcal{D}c_l} \frac{d}{dt}c_r\right)\right). \end{aligned}$$

The precise form of this equation is not relevant, but by observing that the differentials of \ominus have to be evaluated at *many more* independent locations than the desired rank $n \cdot \dim M$ would suggest, it is clear that only very special filters can lead to rank $n \cdot \dim M$. The situation in the linear case is different: The differentials of \ominus are constant, and the condition that the previous formula defines a mapping of rank n is an algebraic condition involving the coefficients of filters α, β .

Similar considerations show that also the so-called log-exponential construction, where a nonlinear rule is constructed via (8) (see Ex. 2.4) do not in general yield the rank condition expressed by Prop. 2.5.

3. STABILITY ANALYSIS

The point of going through the trouble of decomposing a signal is that one expects many detail coefficients $d_l^{(k)}$ to be small and therefore to be negligible. This is the basis of thresholding in order to compress data, which makes sense only if one can control the change in reconstructed data if we change the detail coefficients by resetting some of them to zero. Similarly, quantizing data will result in deviation from the original. Again, it is important to control that change. It is the purpose of this section to establish a *stability* result for nonlinear rules which applies to such situations.

3.1. Coordinate representations of nonlinear rules. For the stability analysis we transfer all manifold operations to a local coordinate chart. This is justified only if the constructions we are going to analyze are local. The linear upscaling and downscaling rules defined previously have this property, and so have the nonlinear ones mentioned in the examples above.

The operators \oplus, \ominus are replaced by their respective coordinate representations, which are denoted by the same symbols and which are defined in open subsets of suitable coordinate vector space: We assume that \oplus maps from $V \times W$ into V , and \ominus maps from $V \times V$ into W . Besides smoothness they are assumed to fulfill the compatibility condition

$$(15) \quad p \oplus (q \ominus p) = q.$$

We further assume that \oplus, \ominus are Lipschitz functions, i.e., there exist constants A, B with

$$(16) \quad A\|p - q\| \leq \|p \ominus q\| \leq B\|p - q\|.$$

Locally this is always the case. Our analysis of stability requires that the operators $\mathcal{S}, \mathcal{D}, \mathcal{Q}, \mathcal{R}$ (we do not introduce new symbols for their coordinate representations) fulfill some reasonable assumptions which are listed below. Notation makes use of the symbol “ \lesssim ” which means that there is a uniform constant such that the left hand side is less than or equal to that constant times the right hand side. For a sequence $w = (w_i)_{i \in \mathbb{Z}}$ we use the notation $\|w\| := \sup_{i \in \mathbb{Z}} \|w_i\|$.

- *Boundedness of \mathcal{Q}, \mathcal{R} :* The mappings \mathcal{Q}, \mathcal{R} operate on W -valued sequences w , which are generated as the difference of point sequences. They are supposed to satisfy $\|\mathcal{Q}w\|, \|\mathcal{R}w\| \lesssim \|w\|$, with respect to some norm W is equipped with.
- *Reproduction of constants:* For constant data we require that $\mathcal{S}c = c$ and $\mathcal{D}c = c$.
- Each of $\mathcal{S}, \mathcal{R}, \mathcal{D}, \mathcal{Q}$ shall be as smooth as is needed (in general a little more than C^1 will suffice).
- *First-order linearity of \mathcal{S}, \mathcal{D} on constant data:* For constant sequences we require that

$$(17) \quad d\mathcal{S}|_c = S_{lin}, \quad d\mathcal{D}|_c = D_{lin}$$

for some low-pass upscaling and downscaling operators S_{lin}, D_{lin} operating on V -valued sequences, and where S_{lin} is a convergent subdivision rule. The only exception shall be Haar case, where $S_{lin} = S_\delta$ shall be the splitting rule (see Ex. 1.1). This condition is natural when one considers \mathcal{S}, \mathcal{D} as geometric analogues of linear constructions which are defined by replacing affine averages by geometric averages, or by replacing the $+$ and $-$ operations by \oplus and \ominus .

3.2. Stability Results. The aim of this section is to prove the following stability theorem:

Theorem 3.1. *Suppose that $\mathcal{S}, \mathcal{D}, \mathcal{Q}, \mathcal{R}$ are upscaling and downscaling operators which fulfill the nonlinear version (13) of the quadrature mirror filter equation, and which also fulfill the technical conditions listed above. Consider a data pyramid $(c^{(j)})_{j \geq 0}$ with $c^{(j-1)} = \mathcal{D}c^{(j)}$ which enjoys the weak contractivity property*

$$(18) \quad \|\Delta c_j\| \lesssim \mu^j \quad (\mu < 1).$$

Then the reconstruction procedure of data $c^{(j)}$ at level j from coarse data $c^{(0)}$ and details $d^{(1)}, \dots, d^{(j)}$ is stable in the sense that there are constants D, E_1, E_2 such that for all j and any further data pyramid $\tilde{c}^{(i)}$ with details $\tilde{d}^{(i)}$ we have

$$(19) \quad \|c^{(0)} - \tilde{c}^{(0)}\| \leq E_1, \quad \|d^{(k)} - \tilde{d}^{(k)}\| \leq E_2 \mu^k \text{ for all } k$$

$$(20) \quad \implies \|c^{(j)} - \tilde{c}^{(j)}\| \leq D(\|c^{(0)} - \tilde{c}^{(0)}\| + \sum_{k=1}^j \|d^{(k)} - \tilde{d}^{(k)}\|).$$

The assumption of decay given by (18) is fulfilled for any finite data pyramid (simply adjust the constant which is implied by using the symbol “ \lesssim ”).

3.3. Proofs. The remaining part of this section is devoted to the proof of this statement. Our arguments closely follow the ones in [8] which will enable us to occasionally skip over some purely technical details and focus on the main ideas.

The crux is to show that the differentials of the reconstruction mappings are uniformly bounded. We shall go about this task by using perturbation arguments. The justification of this approach lies in the fact that by our assumptions the nonlinear reconstruction procedure agrees with a linear one up to first order on constant data. Indeed, our assumptions already imply that \mathcal{S} satisfies a *proximity condition* with S_{lin} in the sense of [12]:

Lemma 3.2. *With the above assumptions we have the inequalities*

$$(21) \quad \|\mathcal{S}c - S_{lin}c\| \lesssim \|\Delta c\|^2, \quad \|\mathcal{D}c - D_{lin}c\| \lesssim \|\Delta c\|^2.$$

Proof. We use a first order Taylor expansion of \mathcal{S} . For any constant sequence e we have $S_{lin}e = Se = e$, so

$$\begin{aligned} \mathcal{S}c &= \mathcal{S}e + d\mathcal{S}|_e(c - e) + O(\|c - e\|^2) \\ &= e + S_{lin}(c - e) + O(\|c - e\|^2) = S_{lin}c + O(\|c - e\|^2). \end{aligned}$$

Since \mathcal{S} and S_{lin} are local operators we may choose e such that

$$\|c - e\| \lesssim \|\Delta c\|.$$

This proves the first equation. The proof of the second one is the same. \square

We now show that for all initial data $c^{(j)}$ with exponential decay of $\|\Delta c^{(j)}\|$, the associated detail coefficients experience the same type of decay.

Lemma 3.3. *Assume that (18) holds for $(c^{(j)})_{j \geq 0}$. Then*

$$(22) \quad \|d^{(j)}\| \lesssim \mu^j.$$

Proof. We use the boundedness of \mathcal{Q} and Lemma 3.2 to estimate the norm of detail coefficients:

$$\begin{aligned} \|d^{(j)}\| &= \|\mathcal{Q}c^{(j)} \ominus \mathcal{S}c^{(j-1)}\| \lesssim \|c^{(j)} \ominus \mathcal{S}c^{(j-1)}\| \lesssim \|c^{(j)} - \mathcal{S}c^{(j-1)}\| \\ &\leq \|c^{(j)} - S_{lin}\mathcal{D}c^{(j)}\| + \|S_{lin}c^{(j-1)} - \mathcal{S}c^{(j-1)}\| \\ &\lesssim \|c^{(j)} - S_{lin}D_{lin}c^{(j)}\| + \|S_{lin}(\mathcal{D}c^{(j)} - D_{lin}c^{(j)})\| + \|S_{lin}c^{(j-1)} - \mathcal{S}c^{(j-1)}\| \\ &\lesssim \|c^{(j)} - S_{lin}D_{lin}c^{(j)}\| + \mu^{2j}. \end{aligned}$$

It remains to estimate $\|c^{(j)} - S_{lin}D_{lin}c^{(j)}\|$. Reproduction of constants implies that for any constant sequence e ,

$$\|c^{(j)} - S_{lin}D_{lin}c^{(j)}\| = \|c^{(j)} - e + S_{lin}D_{lin}(c^{(j)} - e)\| \lesssim \|c^{(j)} - e\|.$$

By the locality of S_{lin} and D_{lin} we can pick e such that $\|c^{(j)} - e\| \lesssim \|\Delta c^{(j)}\|$. This concludes the proof. \square

For later use we record the following two facts. The first one is a perturbation theorem which has been shown in [12].

Theorem 3.4. *Assume that S_{lin} is a convergent linear subdivision scheme and that \mathcal{S} satisfies $d\mathcal{S}|_c = S_{lin}$ for all constant data c . Then there exists $\mu < 1$ such that*

$$(23) \quad \|\Delta \mathcal{S}^j c\| \lesssim \mu^j$$

for all initial data c with $\|\Delta c\|$ small enough.

We do not want to go into details concerning the precise meaning of ‘small enough’. The reader who is interested in the considerable technical subtleties arising from this restriction and also the fact that \mathcal{S} is usually not globally defined is referred to our previous work [7, 8, 6] where these issues are rigorously taken into account and the appropriate bounds for $\|\Delta c\|$ are derived.

The second result is also a perturbation result which has been shown in [8].

Lemma 3.5. *Let A_i, U_i be operators on a normed vector space. Assume exponential decay $\|U_i\| \lesssim \mu^i$, for some $\mu < 1$. Then uniform boundedness of $\|A_1 \cdots A_k\|$ implies uniform boundedness of $\|(A_1 + U_1) \cdots (A_k + U_k)\|$.*

We continue with the proof of Theorem 3.1 by showing that the decay property (18) we assumed for the data pyramid $c^{(j)}$ also holds for the perturbed data pyramid $\tilde{c}^{(j)}$.

Lemma 3.6. *Under the assumptions of Theorem 3.1, further assume that S_{lin} is a convergent subdivision scheme. Then there exist constants s_1, s_2 such that for all j , and any choice of data $\tilde{c}^{(j)}$ we have*

$$(24) \quad \|\Delta \tilde{c}^{(0)}\| \leq s_1, \quad \|\tilde{d}^{(k)}\| \leq s_2 \mu^k \text{ for all } k \quad \implies \quad \|\Delta \tilde{c}^{(j)}\| \lesssim (\mu + \varepsilon)^j.$$

Here for each $\varepsilon > 0$ the implied constant is uniform.

Proof. (Sketch) We make the simplifying assumption that for all initial data c which occur in the course of the proof we have

$$(25) \quad \|\Delta \mathcal{S}c\| \leq \mu \|\Delta c\|.$$

This is no big restriction as it can be shown that such an equation always holds for some iterate \mathcal{S}^N of \mathcal{S} and initial data with $\|\Delta c\|$ small enough, provided S_{lin} is convergent [12]. In case that only

$$\|\Delta \mathcal{S}c\| \leq \bar{\mu} \|\Delta c\|$$

for some $\bar{\mu} \in (\mu, 1)$ we make the initial μ larger. This does not change the substance of Theorem 3.1. With the Lipschitz constants r, r' defined by $\|\mathcal{R}c\| \leq r\|c\|$, $\|a \oplus b - a\| \leq r'\|a \oplus b\|$ we now estimate:

$$\begin{aligned} \|\Delta \tilde{c}^{(1)}\| &\leq \|\Delta \mathcal{S} \tilde{c}^{(0)}\| + 2\|(\mathcal{S} \tilde{c}^{(0)} \oplus \mathcal{R} \tilde{d}^{(1)}) - \mathcal{S} \tilde{c}^{(0)}\| \\ &\leq \mu \|\Delta \tilde{c}^{(0)}\| + 2r' \|\mathcal{R} \tilde{d}^{(1)}\| \leq \mu s_1 + 2rr' s_2 \mu \end{aligned}$$

Iteration of this argument gives the inequality

$$\|\Delta \tilde{c}^{(n)}\| \leq s_1 \mu^n + 2nrr' s_2 \mu^n \lesssim (\mu + \varepsilon)^n$$

for all $\varepsilon > 0$, which we wanted to show. In case (25) does not hold for \mathcal{S} , but only for an iterate \mathcal{S}^N , a similar argument is required which we would like to skip. The reason for requiring s_1, s_2 to be ‘small enough’ is that (25) usually only holds for data c in some set

$$P_{M,\delta} := \{c \mid c_k \in M \ \forall k, \text{ and } \|\Delta c\| < \delta\}.$$

In general we need to ensure that all $c^{(i)}$ ’s lie in the set $P_{M,\delta}$ if the only information on the data is the size of detail coefficients. This rather technical step is where we the restrictions on the constants s_1, s_2 come in. We chose to skip the technical details regarding this issue, since we do not find them particularly enlightening and they have already been treated in full detail in previous work [8, 6, 7]. \square

We are finally in a position to prove Theorem 3.1.

Proof (of Theorem 3.1). The mapping which computes data $c^{(k)}$ at level k by way of reconstruction is denoted by P_k . We use the following notation and definition:

$$(26) \quad X_j := (c^{(0)}, d^{(1)}, \dots, d^{(k)}) \in \ell(V) \times \ell(W)^k$$

$$(27) \quad P_k(X_k) := \mathcal{S}P_{k-1}(X_{k-1}) \oplus \mathcal{R}d^{(k)}, \quad P_0 = \text{id}.$$

We first treat the case that S_{lin} is a convergent subdivision scheme and later deal with the Haar case.

Observe that we can without loss of generality assume that both $\|\Delta c^{(0)}\|$ and the implied constant in (22) are arbitrarily small. This is because we can simply do a re-indexing $(c')^{(i)} = c^{(i+j_0)}$ and we assumed exponential decay of $\Delta c^{(j)}$. In particular,

$$\|\Delta c^{(0)}\| \leq f_1 < s_1, \quad \|d^{(k)}\| \leq f_2 \mu^k, \quad f_2 < s_2,$$

with the constants s_1, s_2 from Lemma 3.6. By Lemma 3.3, $\|d^{(j)}\|$ is likewise of exponential decay. By the same argument we can make the implied constant arbitrarily small.

Pick the constants E_1, E_2 such that $f_1 + E_1 \leq s_1$ and $f_2 + E_2 \leq s_2$, and consider coarse data $\tilde{c}^{(0)}$ and detail coefficients $\tilde{d}^{(1)}, \dots, \tilde{d}^{(j)}$ which obey the assumption (19) made in the statement of the theorem. Lemma 3.6 implies that we have exponential decay of $\|\Delta \tilde{c}^{(j)}\|$.

The estimates gathered so far enable us to show that there exists a constant C such that for all j, k and all perturbed arguments

$$\tilde{X}_j = (\tilde{c}^{(0)}, \tilde{d}^{(1)}, \dots, \tilde{d}^{(j)}),$$

we have the bound

$$(28) \quad \left\| \frac{\partial}{\partial d^{(k)}} \Big|_{\tilde{X}_j} P_j \right\|, \quad \left\| \frac{\partial}{\partial c^{(0)}} \Big|_{\tilde{X}_j} P_j \right\| \leq C.$$

Indeed, using the chain rule on the recursive definition (27), we see that

$$(29) \quad \frac{\partial}{\partial c^{(0)}} \Big|_{\tilde{X}_j} P_j = \left(d_1 \oplus \Big|_{(\mathcal{S}P_{j-1}, \mathcal{R}d^{(j)})} \right) \left(d\mathcal{S} \Big|_{\tilde{c}^{(j-1)}} \right) \left(\frac{\partial}{\partial c^{(0)}} \Big|_{\tilde{X}_{j-1}} P_{j-1} \right).$$

Our assumptions on smoothness (here: \oplus is C^2) and the compatibility relation (15) together imply that

$$d_1 \oplus \Big|_{(\mathcal{S}\tilde{c}^{(j-1)}, \mathcal{R}d^{(j)})} = d_1 \oplus \Big|_{(\mathcal{S}\tilde{c}^{(j-1)}, 0)} + \left(d_1 \oplus \Big|_{(\mathcal{S}\tilde{c}^{(j-1)}, \mathcal{R}d^{(j)})} - d_1 \oplus \Big|_{(\mathcal{S}\tilde{c}^{(j-1)}, 0)} \right) = I + V_j$$

with $\|V_j\| \lesssim \|\mathcal{R}d^{(j)}\| \lesssim \mu^j$. In order to estimate the term $d\mathcal{S} \Big|_{\tilde{c}^{(j-1)}}$, we note that $d\mathcal{S} = S_{lin}$ implies that $\|d\mathcal{S} \Big|_c - S_{lin}\| \lesssim \|\Delta c\|$ for all initial data c , see [8]. Hence we can write

$$d\mathcal{S} \Big|_{\tilde{c}^{(j-1)}} = S_{lin} + W_j, \quad \text{where } \|W_j\| \lesssim \|\tilde{c}^{(j-1)}\| \lesssim (\mu + \varepsilon)^j$$

for any $\varepsilon > 0$. It is a well known fact that for a convergent subdivision scheme S_{lin} , there is a constant M with $\sup_j \|S_{lin}^j\| \leq M$. The previous discussion and iterative application of (29) implies

$$\frac{\partial}{\partial c^{(0)}} \Big|_{\tilde{X}_j} P_j = (S_{lin} + U_1) \cdots (S_{lin} + U_j), \quad \text{where } \|U_k\| \lesssim (\mu + \varepsilon)^k.$$

Now we invoke Lemma 3.5 and see that indeed the partial derivatives of P_k with respect to $\tilde{c}^{(0)}$ at \tilde{X}_j are uniformly bounded, independent of j . The derivatives with respect to \tilde{d}^k can be handled in an analogous manner. This shows (28), from which it is easy to see (20).

Having concluded the proof in the case that S_{lin} is a convergent subdivision scheme, we turn to the Haar case. It is analogous, but because we have $\mathcal{S} = S_{lin}$ we do not need the perturbation inequalities at all to estimate differentials (in particular we do not need Lemma 3.6). \square

Remark 3.7. The only place where the constants E_1, E_2 come into play is the assumption (25) which is usually only satisfied for data in some set $P_{M,\delta}$ – see the discussion in the proof of Lemma 3.6. It is easy to see that if \mathcal{S} is defined and contractive for *all* initial data, then the constants E_1, E_2 can be arbitrarily large.

4. OBTAINING DISCRETE DATA

4.1. Convolution and smoothing of manifold-valued data. Here we are going to investigate further properties of the geometric average which was defined by Equations (7) and (10). They will become important in Section 4.2. This material is already contained in Karcher's paper [9] as far as surfaces and Riemannian geometry are concerned. Here we also show the extension to Lie groups, which is not difficult once the Riemannian case is known.

Convolution with a function ψ with $\int \psi = 1$ can be interpreted as an average. This applies to multivariate functions as well as to univariate ones, which are our main concern. In order to fit the previous definitions, we give an equivalent construction of the convolution $g * \psi$ for vector-valued functions g , and at the same time a definition of $(f \circledast \psi)(u)$ for manifold-valued functions $f : \mathbb{R}^d \rightarrow M$.

$$(30) \quad m = (g * \psi)(u) \iff m = \int_{\mathbb{R}^d} g(x) \psi(u - x) dx \iff \int_{\mathbb{R}^d} (g(x) - m) \psi(u - x) dx = 0,$$

$$(31) \quad m = (f \circledast \psi)(u) \iff \int_{\mathbb{R}^d} (f \ominus m) \psi(u - x) dx = 0.$$

The even more general case where the domain of functions are manifolds has been discussed in [9]. It turns out that basically any nonnegative kernel function ψ supported in the cube $[-1, 1]^d$ can be used for smoothing in the following way: For each $\rho > 0$, we let

$$(32) \quad f^\rho = f \circledast \psi^\rho, \quad \text{where } \psi^\rho(x) = \frac{1}{\rho^d} \psi\left(\frac{x}{\rho}\right).$$

We want to show that f and its differential df are approximated by f^ρ and df^ρ as ρ approaches zero. The proofs consist of revisiting the proofs given in [9] which apply to the Riemannian case.

Theorem 4.1. *Consider the smoothed functions f^ρ defined by a function $f : \mathbb{R}^d \rightarrow M$ and a kernel ψ as above. Then*

$$\lim_{\rho \rightarrow 0} f^\rho = f, \quad \lim_{\rho \rightarrow 0} df^\rho = df.$$

In case f is Lipschitz differentiable, then this convergence is linear.

Proof. We skip convergence of f^ρ and show only convergence of df^ρ . The proof is in the spirit of Lemma 4.2 and Theorem 4.4 of [9], the difference being that the domain of f is a vector space. We define $V : \mathbb{R}^d \times M \rightarrow \mathbb{R}^{\dim M}$ by letting

$$V(u, p) := \int (f(x) \ominus p) \psi^\rho(u - x) dx.$$

By definition, $V(u, f^\rho(u)) = 0$. This implies the following equation of derivatives:

$$(33) \quad d_1 V_{u, f^\rho(u)} + D_2 V_{u, f^\rho(u)} \circ df_u^\rho = 0.$$

The capital D indicates the fact that in the Riemannian case we employ a covariant derivative. The partial derivatives of V have the form

$$\begin{aligned} d_1|_{u,p} V(\dot{u}) &= \frac{d}{dt} \Big|_{t=0} \int (f(y) \ominus p) \psi^\rho(u(t) - y) dy \\ &= \frac{d}{dt} \Big|_{t=0} \int (f(x - u + u(t)) \ominus p) \psi^\rho(u - x) dx, \\ D_2|_{u,p} V(\dot{p}) &= \frac{D}{dt} \Big|_{t=0} \int (f(x) \ominus p(t)) \psi^\rho(u - x) dx \end{aligned}$$

Using the functions $E_{p,q}(\dot{q}) = -\frac{D}{dt}(p \ominus q(t))$ and $F_{p,q}(\dot{p}) = \frac{d}{dt}(p(t) \ominus q)$, we get

$$(34) \quad d_1|_{u,p} V(\dot{u}) + D_2|_{u,p} V(\dot{p}) = \int (F_{f(x),p}(df_x(\dot{u})) - E_{f(x),p}(\dot{p})) \psi^\rho(u - x) dx.$$

It is shown in [9] that in the Riemannian case the functions $E_{p,q}$ and $F_{p,q}$ can be bounded in terms of sectional curvature K , and the parallel transport operator $\text{Pt}_{\text{from}}^{to}$:

$$E_{p,q}(v) = v + R, \quad F_{p,q}(v) = \text{Pt}_p^q(v) + R',$$

where $\|R\| \leq \|v\| \text{const}(\min K, \max K) \cdot \text{dist}(p, q)^2$ and $\|R'\| \leq \|F_{p,q}(v)\| \text{const}(\max |K|) \cdot \text{dist}(p, q)^2$. Letting $p = f^\rho(u)$ and $\dot{p} = df^\rho(\dot{u})$, we convert (33) and (34) into the integral

$$0 = \int (\text{Pt}_{f(x)}^{f^\rho(u)} df_x(\dot{u}) + R'(x) - df^\rho(\dot{u}) - R(x)) \psi^\rho(u - x) dx,$$

without indicating the dependence of the remainder terms R, R' on x . The assumption that f is C^1 implies that for all x with contribute to the integral (i.e., $\psi^\rho(u - x) \neq 0$), we have $x \rightarrow u, df_x(\dot{u}) \rightarrow df_u(\dot{u}), f^\rho(x) \rightarrow f(u), \text{Pt} \rightarrow \text{id}, R \rightarrow 0, R' \rightarrow 0$. Observe that all these limits have at least linear convergence rate, provided df is Lipschitz. With $\int \psi = 1$, we obtain

$$\lim_{\rho \rightarrow 0} (df_x + df^\rho)(\dot{u}) \rightarrow 0,$$

where the limit is linear if df is Lipschitz. This concludes the proof in the Riemannian case.

In the Lie group case, it is not difficult to compute the derivatives $E_{p,q}(\dot{p})$ and $F_{p,q}(\dot{q})$ by means of the Baker-Campbell-Hausdorff formula which says $\log(e^x e^y) = x + y + \frac{1}{2}[x, y] + \dots$, where the dots indicate terms of third and higher order expressible by Lie brackets. When p and $q = pe^z$ undergo 1-parameter variations of the form $p(t) = pe^{tw}$ and $q(t) = qe^{tw}$ with $w \in \mathfrak{g}$, then

$$\begin{aligned} p(t) \ominus q &= \log(e^z e^{tw}) = z + tw + \frac{1}{2}[z, tw] + \dots \\ p \ominus q(t) &= \log(e^{-tw} e^z) = -tw + z + \frac{1}{2}[-tw, z] + \dots \end{aligned}$$

This implies

$$\begin{aligned} F_{p,q}(w) &= w + \frac{1}{2}[z, w] + \dots, \\ E_{p,q}(w) &= w + \frac{1}{2}[w, z] + \dots \end{aligned}$$

Similar to the Riemannian case above, we convert (33) and (34) into the integral

$$\int \left(df_x(\dot{u}) + \frac{1}{2} \left[f^\rho(u) \ominus f(x), df_x(\dot{u}) + df^\rho(u) \right] - df^\rho(u) + \dots \right) \psi^\rho(u - x) dx = 0$$

in the Lie algebra. The same arguments imply $x \rightarrow u$, $f^\rho(x) \rightarrow f(u)$, $df_x(\dot{u}) \rightarrow df_u(\dot{u})$, and as a consequence $df^\rho \rightarrow df$ as $\rho \rightarrow 0$. This concludes the proof of Theorem 4.1 in the Lie group case. \square

4.2. The passage from continuous to discrete data. In the analysis of multiscale decompositions one frequently assumes an infinite detail pyramid. In practice a vector-valued or manifold-valued function $f(t)$ which depends on a parameter $t \in \mathbb{R}$ is given by finitely many measurements. Such measurements might be samples at parameters $t_i = ih$, for some small h ; or measurements might be modeled as averages of the form $f \circledast \phi(\cdot - ih)_{i \in \mathbb{Z}}$ where ϕ is some kernel with $\int \phi = 1$ and $\text{supp}(\phi)$ small (in fact physics excludes the kind of measurement we called *samples* and permits only ϕ to approach the Dirac delta).

In the linear case any multiscale decomposition based on midpoint-interpolation and especially the Haar scheme are well adapted to deal with averages: The decimation operator D in this case is consistent with the definition of discrete data as follows:

$$\begin{aligned} \psi^{(j)} &= 2^j 1_{[0,1]}(2^j \cdot) = 2^j 1_{[0,2^{-j}]}, \quad f^{(j)} = f * \psi^{(j)}, \quad c^{(j)} = f^{(j)}|_{2^{-j}\mathbb{Z}} \\ \implies c^{(j-1)} &= Dc^{(j)}. \end{aligned}$$

We have no analogous relation for manifold-valued multiscale decompositions. Nevertheless we may let

$$f^{(j)} = f \circledast \psi^{(j)}, \quad c^{(j)} = f^{(j)}|_{2^{-j}\mathbb{Z}}.$$

In view of Theorem 4.1, this yields discrete data whose discrete derivatives $\Delta c^{(j)}$ approximate the derivatives of f . Assuming f to be C^2 , we have

$$\Delta c_k^{(j)} := 2^j (c_{k+1}^{(j)} - c_k^{(j)}) \implies \Delta c_k^{(j)} = \frac{d}{dt} f^{(j)}|_{k2^{-j}} + O(2^{-j}) = \frac{d}{dt} f|_{k2^{-j}} + O(2^{-j}).$$

The previous equation is to be interpreted in any smooth coordinate chart of the manifold under consideration.

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P. GROHS: KING ABDULLAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, SAUDI ARABIA, AND TU GRAZ, AUSTRIA. PGROHS@TUGRAZ.AT

J. WALLNER: TU GRAZ, AUSTRIA. EMAIL J.WALLNER@TUGRAZ.AT