DEFINABILITY UNDER DUALITY

PANDELIS DODOS

Abstract. It is shown that if A is an analytic class of separable Banach spaces with separable dual, then the set $A^* = \{Y : \exists X \in A \text{ with } Y \cong X^*\}$ is analytic. The corresponding result for pre-duals is false.

1. Introduction

(A) All separable Banach spaces can be realized, up to isometry, as subspaces of $C(2^{\mathbb{N}})$. Denoting by SB the set of all closed linear subspaces of $C(2^{\mathbb{N}})$ and endowing SB with the relative Effros–Borel structure, the set SB becomes the standard Borel space of all separable Banach spaces (see [AD, AGR, Bos, Ke]). By identifying any class of separable Banach spaces with a subset of SB, the space SB provides the appropriate frame for studying structural properties of classes of Banach spaces. This identification is ultimately related to universality problems in Banach space theory. This is justified by a number of results (see, e.g., [AD, DF, D, DLo]) of which the following one, taken from [DF], is a sample.

If A is an analytic subset of SB *such that every $X \in A$ is reflexive, then there exists a reflexive Banach space Y, with a Schauder basis, that contains isomorphic copies of every $X \in A$.*

To see how such a result is used, let us consider the set UC consisting of all $X \in$ SB which are uniformly convex. It is a classical fact (see, e.g., [LT]) that UC contains only reflexive spaces. Moreover, it is easily checked that UC is a Borel subset of SB. Applying the above result, we recover a recent result of Odell and Schlumprecht [OS] asserting the existence of a separable reflexive space R containing an isomorphic copy of every separable uniformly convex Banach space. The problem of the existence of such a space was posed by Jean Bourgain [Bou2].

(B) As we have already indicated, in applications one has to decide whether a given class of separable Banach spaces is analytic or not. Sometimes this is straightforward to check invoking, simply, the definition of the class. There are classes, however, which are defined implicitly using a certain Banach space operation. In these cases, usually, deeper arguments are involved.

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This note is concerned with the question whether analyticity is preserved under duality, a very basic operation encountered in Banach space theory. Precisely, the following two questions are naturally asked in such a context.

- **(Q1)** If A is an analytic class of separable dual Banach spaces, then is the set $A_* := \{X \in \mathrm{SB} : \exists Y \in A \text{ with } X^* \cong Y\}$ analytic?
- **(Q2)** If A is an analytic class of separable Banach spaces with separable dual, then is the set $A^* := \{Y \in \mathrm{SB} : \exists X \in A \text{ with } Y \cong X^*\}$ analytic?

Question (Q1) has a negative answer and a counterexample is the set $A = \{Y \in \mathrm{SB} : Y \cong \ell_1\}$, that is, the isomorphic class of ℓ_1 (a more detailed explanation will be given later on). However, for question (Q2) we do have a positive result.

Theorem 1. *Let A be an analytic class of separable Banach spaces with separable dual. Then the set $A^* = \{Y \in \mathrm{SB} : \exists X \in A \text{ with } Y \cong X^*\}$ is analytic.*

The proof of Theorem 1 is based on a selection result which is perhaps of independent interest. To state it, let $H = [-1,1]^{\mathbb{N}}$ be equipped with the product topology. That is, H is the closed unit ball of ℓ_∞ with the weak* topology. A subset S of H will be called *norm separable* if it is separable with respect to the metric induced by the supremum norm $\|\cdot\|_\infty$. The selection result we need is the following.

Proposition 2. *Let Z be a standard Borel space and let $A \subseteq Z \times H$ be Borel such that the following hold.*

1. *For every $z \in Z$ the section A_z is nonempty and compact.*
2. *For every $z \in Z$ the section A_z is norm separable.*

Then there exists a sequence (f_n) of Borel selectors of A such that for every $z \in Z$ the sequence $\big(f_n(z)\big)$ is norm dense in A_z.

As usual, a map $f\colon Z \to H$ is said to be a *Borel selector* of A if f is a Borel map such that $\big(z, f(z)\big) \in A$ for every $z \in Z$. The proof of Proposition 2 is based on a Szlenk type index defined on all norm-separable compact subsets of H. Actually, what we use is the fact that this index has nice definability properties (it is a co-analytic rank) and it satisfies boundedness.

We remark that the use of boundedness in selection theorems is common in descriptive set theory (it is used, for instance, in the proof of the strategic uniformization theorem—see [Ke, Theorem 35.32]). Also we notice that the transfinite manipulations used in the proof of Proposition 2 are similar to the ones in the selection theorems of Jayne and Rogers [JR], and Ghoussoub, Maurey and Schachermayer [GMS]. We point out, however, that the crucial definability considerations in the proof of Proposition 2 do not appear in [JR, GMS].

1.1. **Notation.** By $\mathbb{N} = \{0,1,2,\dots\}$ we denote the natural numbers. For every Polish space X by $K(X)$ we denote the set of all compact subsets of X (the empty set is included). We equip $K(X)$ with the Vietoris topology τ_V, that is, the topology generated by the sets

$$\{K \in K(X) : K \cap U \neq \emptyset\} \ \text{ and } \ \{K \in K(X) : K \subseteq U\}$$

where U ranges over all nonempty open subsets of X. It is well-known (see, e.g., [Ke]) that the space $(K(X), \tau_V)$ is Polish. A map $D\colon K(X) \to K(X)$ is said to be a *derivative* on $K(X)$ provided that $D(K) \subseteq K$ and $D(K_1) \subseteq D(K_2)$ if $K_1 \subseteq K_2$. For every $K \in K(X)$ by transfinite recursion one defines the *iterated derivatives* $D^{(\xi)}(K)$ of K by the rule

$$D^{(0)}(K) = K,\ D^{(\xi+1)}(K) = D\big(D^{(\xi)}(K)\big) \text{ and } D^{(\lambda)} = \bigcap_{\xi<\lambda} D^{(\xi)}(K) \text{ if } \lambda \text{ is limit.}$$

The *D-rank* of K is the least ordinal ξ for which $D^{(\xi)}(K) = D^{(\xi+1)}(K)$. It is denoted by $|K|_D$. Moreover, set $D^{(\infty)}(K) := D^{|K|_D}(K)$. If X, Y are sets and $A \subseteq X \times Y$, then for every $x \in X$ by A_x we denote the section of A at x, that is, the set $\{y : (x, y) \in A\}$. All the other pieces of notation we use are standard (see, for instance, [Ke, LT]).

1.2. **The counterexample to question (Q1).** We have already mentioned that the counterexample is the isomorphic class of ℓ_1, that is, the set $A = \{Y : Y \cong \ell_1\}$. Since the equivalence relation of isomorphism $\cong$ is analytic in $\mathrm{SB} \times \mathrm{SB}$ (see [Bos]), the set A is analytic. We will show that the set

$$A_* := \{X : \exists Y \in A \text{ with } X^* \cong Y\} = \{X : X^* \cong \ell_1\}$$

is not analytic. The argument below goes back to the fundamental work of Bourgain on $C(K)$ spaces, with K countable compact (see [Bou1]). Specifically, there exists a Borel map $\Phi\colon K(2^{\mathbb{N}}) \to \mathrm{SB}$ such that for every $K \in K(2^{\mathbb{N}})$ the space $\Phi(K)$ is isomorphic to $C(K)$ (see [Ke, page 263]). Denote by $K_\omega(2^{\mathbb{N}})$ the set of all countable compact subsets of $2^{\mathbb{N}}$. It follows that

$$K \in K_\omega(2^{\mathbb{N}}) \Leftrightarrow C(K)^* \cong \ell_1 \Leftrightarrow \Phi(K) \in A_*.$$

Therefore, $K_\omega(2^{\mathbb{N}}) = \Phi^{-1}(A_*)$. By a classical result of Hurewicz (see [Ke, Theorem 27.5]), the set $K_\omega(2^{\mathbb{N}})$ is co-analytic non-Borel and so the set A_* is not analytic (for if not, we would have that $K_\omega(2^{\mathbb{N}})$ is analytic). In descriptive set-theoretic terms, the above argument shows that the set A_* is Borel $\mathbf{\Pi}^1_1$-hard.

2. Proof of Proposition 2

In what follows, by H we shall denote the set $[-1, 1]^{\mathbb{N}}$ equipped with the product topology. We recall the following well-known topological lemma (see, e.g., [GM, Ro] and the references therein). For the sake of completeness we include a proof.

Lemma 3. *Let $K \subseteq H$ be nonempty and compact. If K is norm separable, then for every $\varepsilon > 0$ there exists an open subset U of H such that $K \cap U \neq \emptyset$ and $\|\cdot\|_\infty - \mathrm{diam}(K \cap U) \leqslant \varepsilon$.*

Proof. We fix a compatible metric ρ for H with $\rho - \mathrm{diam}(H) \leqslant 1$ (notice that such a metric ρ is necessarily complete). Assume, towards a contradiction, that the lemma is false. Hence, we can construct a family (V_t) $(t \in 2^{<\mathbb{N}})$ of nonempty relatively open subsets of K such that the following are satisfied.

(a) For every $t \in 2^{<\mathbb{N}}$ we have $\overline{V}_{t^\frown 0} \cap \overline{V}_{t^\frown 1} = \emptyset$, $\big(\overline{V}_{t^\frown 0} \cup \overline{V}_{t^\frown 1}\big) \subseteq V_t$ and $\rho - \mathrm{diam}(V_t) \leqslant 2^{-|t|}$.

(b) For every $n \in \mathbb{N}$ with $n \geqslant 1$, every $t, s \in 2^n$ with $t \neq s$ and every pair $(f,g) \in V_t \times V_s$ we have $\|f - g\|_\infty > \varepsilon$.

We set $P \coloneqq \bigcup_{\sigma \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} V_{\sigma|n}$. By (a) above, we see that P is a perfect subset of K. On the other hand, by (b), we have that $\|f - g\|_\infty > \varepsilon$ for every $f, g \in P$ with $f \neq g$. That is, K is not norm separable, a contradiction. The proof is completed. □

Lemma 3 suggests a canonical derivative operation on compact subsets of H, similar to the derivative operation appearing in Szlenk's analysis of separable dual spaces [Sz]. Actually, our interest on it stems from the fact that it has the right definability properties.

To define this derivative, let (U_n) be an enumeration of a countable basis of H (we will assume that every U_n is nonempty). This basis will be fixed. Let $\varepsilon > 0$ be arbitrary. Define $D_{n,\varepsilon} \colon K(H) \to K(H)$ by

$$D_{n,\varepsilon}(K) = \begin{cases} K \setminus U_n & \text{if } K \cap U_n \neq \emptyset \text{ and } \|\cdot\|_\infty - \mathrm{diam}(K \cap U_n) \leqslant \varepsilon, \\ K & \text{otherwise.} \end{cases}$$

Notice that $D_{n,\varepsilon}$ is a derivative on $K(H)$. Now define $\mathbb{D}_\varepsilon \colon K(H) \to K(H)$ by $\mathbb{D}_\varepsilon(K) = \bigcap_n D_{n,\varepsilon}(K)$. That is,

$$\mathbb{D}_\varepsilon(K) = K \setminus \bigcup \{U_n : K \cap U_n \neq \emptyset \text{ and } \|\cdot\|_\infty - \mathrm{diam}(K \cap U_n) \leqslant \varepsilon\}.$$

Clearly $\mathbb{D}_\varepsilon$ is derivative on $K(H)$ too.

Lemma 4. *Let $\varepsilon > 0$. Then the following hold.*

(i) *For every $n \in \mathbb{N}$ the map $D_{n,\varepsilon}$ is Borel.*

(ii) *The map $\mathbb{D}_\varepsilon$ is a Borel derivative.*

Proof. (i) Fix $n \in \mathbb{N}$ and set

$$A_n \coloneqq \{K \in K(H) : K \cap U_n \neq \emptyset \text{ and } \|\cdot\|_\infty - \mathrm{diam}(K \cap U_n) \leqslant \varepsilon\}.$$

Then A_n is Borel (actually it is the complement of a K_σ set) in $K(H)$, since

$$\begin{aligned} K \notin A_n \quad \Leftrightarrow \quad & (K \cap U_n = \emptyset) \text{ or } \big(\exists f, g \in K \cap U_n \\ & \exists l \in \mathbb{N}\ \exists m \in \mathbb{N} \text{ with } |f(l) - g(l)| \geqslant \varepsilon + \frac{1}{m+1}\big). \end{aligned}$$

Now observe that $D_{n,\varepsilon}(K) = K$ if $K \notin A_n$ and $D_{n,\varepsilon}(K) = K \setminus U_n$ if $K \in A_n$. This easily implies that $D_{n,\varepsilon}$ is Borel.

(ii) Consider the map $F\colon K(H)^{\mathbb{N}} \to K(H)^{\mathbb{N}}$ defined by

$$F\big((K_n)\big) = \big(D_{n,\varepsilon}(K_n)\big).$$

By part (i), we have that F is Borel. Moreover, by [Ke, Lemma 34.11], the map $\bigcap : K(H)^{\mathbb{N}} \to K(H)$ defined by $\bigcap\big((K_n)\big) = \bigcap_n K_n$ is Borel too. Finally, let $I\colon K(H) \to K(H)^{\mathbb{N}}$ be defined by $I(K) = (K_n)$ with $K_n = K$ for every n. Clearly I is continuous. Noticing that $\mathbb{D}_\varepsilon(K) = \bigcap\big(F(I(K))\big)$, the result follows. □

We will need the following well-known result concerning sets in product spaces with compact sections (see [Ke, Theorem 28.8]).

Theorem 5. *Let Z be a standard Borel space, let H be a Polish space and let $A \subseteq Z \times H$ with compact sections. Let $\Phi_A\colon Z \to K(H)$ be defined by $\Phi_A(z) = A_z$ for every $z \in Z$. Then A is Borel in $Z \times H$ if and only if Φ_A is a Borel map.*

Now let $B \subseteq H$ and $\varepsilon > 0$. We say that a subset S of B is *norm ε-dense* in B if for every $g \in B$ there exists $f \in S$ with $\|f - g\|_\infty \leqslant \varepsilon$.

Lemma 6. *Let Z and A be as in Proposition 2. Also let $\varepsilon > 0$ and let $\tilde{A} \subseteq Z \times H$ be Borel with $\tilde{A} \subseteq A$ and such that for every $z \in Z$ the section $\tilde{A}_z$ is a (possibly empty) compact set. Then there exists a sequence (f_n) of Borel selectors of A such that for every $z \in Z$ if the section $\tilde{A}_z$ is nonempty, then the set $\{f_n(z) : f_n(z) \in \tilde{A}_z \setminus \mathbb{D}_\varepsilon(\tilde{A}_z)\}$ is nonempty and norm ε-dense in $\tilde{A}_z \setminus \mathbb{D}_\varepsilon(\tilde{A}_z)$.*

Proof. Let $n \in \mathbb{N}$. By Theorem 5, the map $\Phi_{\tilde{A}}$ is Borel. Set

$$Z_n \coloneqq \{z \in Z : \tilde{A}_z \cap U_n \neq \emptyset \text{ and } \|\cdot\|_\infty - \mathrm{diam}(\tilde{A}_z \cap U_n) \leqslant \varepsilon\}.$$

Then Z_n is Borel in Z. To see this, notice that $Z_n = \Phi_{\tilde{A}}^{-1}(A_n)$ where A_n is defined in the proof of part (i) of Lemma 4. Now let $\tilde{A}_n \subseteq Z \times H$ be defined by the rule

$$(z,f) \in \tilde{A}_n \quad \Leftrightarrow \quad \big(z \in Z_n \text{ and } f \in U_n \text{ and } (z,f) \in \tilde{A}\big) \text{ or } \big(z \notin Z_n \text{ and } (z,f) \in A\big).$$

It is easy to see that for every $n \in \mathbb{N}$ the set $\tilde{A}_n$ is a Borel set with nonempty σ-compact sections. By the Arsenin–Kunugui theorem (see [Ke, Theorem 35.46]), there exists a Borel map $f_n\colon Z \to H$ such that $\big(z, f_n(z)\big) \in \tilde{A}_n$ for every $z \in Z$. We claim that the sequence (f_n) is the desired one. Clearly it is a sequence of Borel selectors of A. What remains is to check that it has the desired property. So, let $z \in Z$ such that $\tilde{A}_z$ is nonempty and let $f \in \tilde{A}_z \setminus \mathbb{D}_\varepsilon(\tilde{A}_z)$. It follows readily by the definition of $\mathbb{D}_\varepsilon$ that there exists $n_0 \in \mathbb{N}$ such that $z \in Z_{n_0}$ and $(z,f) \in \tilde{A}_{n_0}$. The definition of $\tilde{A}_{n_0}$ yields that the set $\{h : (z,h) \in \tilde{A}_{n_0}\}$ has norm diameter less or equal to ε. Since $(z, f_{n_0}(z)) \in \tilde{A}_{n_0}$, we conclude that $\|f - f_{n_0}(z)\|_\infty \leqslant \varepsilon$ and the proof is completed. □

Before we proceed to the proof of Proposition 2, we need the following facts about the derivative operation $\mathbb{D}_\varepsilon$ described above. By Lemma 4, the map $\mathbb{D}_\varepsilon$ is a Borel derivative on $K(H)$. By [Ke, Theorem 34.10], it follows that the set

$$\Omega_{\mathbb{D}_\varepsilon} := \{K \in K(H) : \mathbb{D}_\varepsilon^{(\infty)}(K) = \emptyset\}$$

is co-analytic and that the map $K \to |K|_{\mathbb{D}_\varepsilon}$ is a co-analytic rank on $\Omega_{\mathbb{D}_\varepsilon}$ (a $\mathbf{\Pi}^1_1$-rank in the technical logical jargon—see [Ke] for the definition and the properties of co-analytic ranks). We are particulary interested in the following important property which is shared by all co-analytic ranks (see [Ke, Theorem 35.22]): *if S is an analytic subset of $\Omega_{\mathbb{D}_\varepsilon}$, then*

$$\sup\{|K|_{\mathbb{D}_\varepsilon} : K \in S\} < \omega_1.$$

(This property is known as boundedness.) We are now ready to give the proof of Proposition 2.

Proof of Proposition 2. Let $A \subseteq Z \times H$ be as in the statement of the proposition. By Theorem 5, the map $\Phi_A \colon Z \to K(H)$ defined by $\Phi_A(z) = A_z$ is Borel, and so, the set $\{A_z : z \in Z\}$ is an analytic subset of $K(H)$.

Now, let $\varepsilon > 0$ be arbitrary and consider the derivative operation $\mathbb{D}_\varepsilon$. By our assumptions on A and Lemma 3, we see that for every $z \in Z$ and every $\xi < \omega_1$ if $\mathbb{D}_\varepsilon^{(\xi)}(A_z) \neq \emptyset$, then $\mathbb{D}_\varepsilon^{(\xi+1)}(A_z) \subsetneq \mathbb{D}_\varepsilon^{(\xi)}(A_z)$. It follows that the transfinite sequence $\big(\mathbb{D}_\varepsilon^{(\xi)}(A_z)\big)$ $(\xi < \omega_1)$ must be stabilized at $\emptyset$, and so, $\{A_z : z \in Z\} \subseteq \Omega_{\mathbb{D}_\varepsilon}$. Hence, by boundedness, we obtain that

$$\sup\big\{|A_z|_{\mathbb{D}_\varepsilon} : z \in Z\big\} = \xi_\varepsilon < \omega_1.$$

For every $\xi < \xi_\varepsilon$ we define recursively $A^\xi \subseteq Z \times H$ as follows. First we set $A^0 := A$. If $\xi = \zeta + 1$ is a successor ordinal, then define A^ξ by the rule

$$(z, f) \in A^\xi \Leftrightarrow f \in \mathbb{D}_\varepsilon\big((A^\zeta)_z\big)$$

where $(A^\zeta)_z$ is the section $\{f : (z, f) \in A^\zeta\}$ of A^ζ. If ξ is limit, then set

$$(z, f) \in A^\xi \Leftrightarrow (z, f) \in \bigcap_{\zeta<\xi} A^\zeta.$$

Claim. *The following hold.*

(1) *For every $\xi < \xi_\varepsilon$ the set A^ξ is a Borel subset of A with compact sections.*
(2) *For every $(z, f) \in Z \times H$ with $(z, f) \in A$ there exists a unique ordinal $\xi < \xi_\varepsilon$ such that $(z, f) \in A^\xi \setminus A^{\xi+1}$, equivalently, $f \in (A^\xi)_z \setminus \mathbb{D}_\varepsilon\big((A^\xi)_z\big)$.*

Proof of the claim. (1) By induction on all ordinals less than ξ_ε. For $\xi = 0$ it is straightforward. If $\xi = \zeta+1$ is a successor ordinal, then, by our inductive hypothesis and Theorem 5, the map $z \mapsto (A^\zeta)_z$ is Borel. By part (ii) of Lemma 4, the map $z \mapsto \mathbb{D}_\varepsilon\big((A^\zeta)_z\big)$ is Borel too. By the definition of $A^\xi = A^{\zeta+1}$ and invoking Theorem 5 once more, we conclude that A^ξ is a Borel subset of A with compact sections. If

ξ is limit, then this is an immediate consequence of our inductive hypothesis and the definition of A^ξ.

(2) For every $z \in Z$ let $\xi_z = |A_z|_{\mathbb{D}_\varepsilon} \leqslant \xi_\varepsilon$. Notice that A_z is partitioned into the disjoint sets $\mathbb{D}_\varepsilon^{(\xi)}(A_z) \setminus \mathbb{D}_\varepsilon^{(\xi+1)}(A_z)$ with $\xi < \xi_z$. By transfinite induction, one easily shows that $(A^\xi)_z = \mathbb{D}_\varepsilon^{(\xi)}(A_z)$ for every $\xi < \xi_z$. It follows that

$$\mathbb{D}_\varepsilon^{(\xi)}(A_z) \setminus \mathbb{D}_\varepsilon^{(\xi+1)}(A_z) = (A^\xi)_z \setminus (A^{\xi+1})_z = (A^\xi)_z \setminus \mathbb{D}_\varepsilon\big((A^\xi)_z\big).$$

The claim is proved. □

By part (1) of the claim, for every $\xi < \xi_\varepsilon$ we may apply Lemma 6 for the set A^ξ, and we obtain for every $\xi < \xi_\varepsilon$ a sequence (f_n^ξ) of Borel selectors of A as described in Lemma 6. Enumerate the sequence (f_n^ξ) $(\xi < \xi_\varepsilon, n \in \mathbb{N})$ in a single sequence, say as (f_n). Clearly the sequence (f_n) is a sequence of Borel selectors of A. Moreover, by part (2) of the above claim and the properties of the sequence obtained by Lemma 6, we see that for every $z \in Z$ the set $\{f_n(z) : n \in \mathbb{N}\}$ is norm ε-dense in A_z. Applying the above for $\varepsilon = (m+1)^{-1}$ with $m \in \mathbb{N}$, the result follows. □

3. Proof of Theorem 1

Before we embark into the proof, we need to discuss some standard facts (see, e.g., [Ke, page 264]). First we notice that, by the Kuratowski–Ryll-Nardzewski selection theorem (see [Ke, Theorem 12.13]), there exists a sequence $d_n \colon \mathrm{SB} \to C(2^{\mathbb{N}})$ $(n \in \mathbb{N})$ of Borel functions such that for every $X \in \mathrm{SB}$ the sequence $\big(d_n(X)\big)$ is dense in X and closed under rational linear combinations.

Using this, for every $X \in \mathrm{SB}$ we can identify the closed unit ball $B_1(X^*)$ of X^* with a compact subset K_{X^*} of $H = [-1,1]^{\mathbb{N}}$. In particular, we view every element $x^* \in B_1(X^*)$ as an element $f \in H$ by identifying it with the sequence $n \mapsto \frac{x^*(d_n(X))}{\|d_n(X)\|}$ (if $d_n(X) = 0$, then we define this ratio to be 0). There are two crucial properties established with this identification.

(P1) The set $D \subseteq \mathrm{SB} \times H$ defined by

$$(X, f) \in D \Leftrightarrow f \in K_{X^*}$$

is Borel. Indeed, notice that

$$\begin{aligned}(X,f) \in \mathrm{D} \quad \Leftrightarrow \quad & \forall n,m,k \in \mathbb{N}\ \forall p,q \in \mathbb{Q} \text{ we have} \\ & \big[p \cdot d_n(X) + q \cdot d_m(X) = d_k(X) \Rightarrow \\ & p \cdot \|d_n(X)\| \cdot f(n) + q \cdot \|d_m(X)\| \cdot f(m) = \|d_k(X)\| \cdot f(k)\big].\end{aligned}$$

(P2) If $f_0,\dots,f_k \in K_{X^*}$ and $x_0^*,\dots,x_k^*$ denote the corresponding elements in $B_1(X^*)$, then for every $a_0,\dots,a_k\in\mathbb{R}$ we have

$$\begin{aligned}\|\sum_{i=0}^k a_i x_i^*\| &= \sup\Big\{\Big|\sum_{i=0}^k a_i \frac{x_i^*(d_n(X))}{\|d_n(X)\|}\Big| : d_n(X)\neq 0\Big\}\\ &= \sup\Big\{\Big|\sum_{i=0}^k a_i f_i(n)\Big| : n\in\mathbb{N}\Big\} = \|\sum_{i=0}^k a_i f_i\|_\infty.\end{aligned}$$

In other words, this identification of $B_1(X^*)$ with K_{X^*} is isometric.

We proceed to the proof of Theorem 1.

Proof of Theorem 1. Let A be an analytic subset of SB such that every $X\in A$ has separable dual. Denote by SD the set of all $X\in$ SB with separable dual. It is co-analytic (see, e.g., [Ke, Theorem 33.24]). Hence, by Lusin's separation theorem (see [Ke, Theorem 14.7]), there exists $Z\subseteq$ SD Borel with $A\subseteq Z$. Define $G\subseteq Z\times H$ by the rule

$$(X,f)\in G \Leftrightarrow f\in K_{X^*}.$$

By property (P1) above, it follows that G is a Borel set such that for every $X\in Z$ the section G_X of G at X is nonempty, compact and norm-separable. We apply Proposition 2 and we obtain a sequence $f_n\colon Z\to H$ $(n\in\mathbb{N})$ of Borel selectors of G such that for every $X\in Z$ the sequence $\big(f_n(X)\big)$ is norm dense in $G_X=K_{X^*}$. Notice that, by property (P2) above, for every $Y\in$ SB and every $X\in Z$ we have

$$\begin{aligned} Y\cong X^* \quad\Leftrightarrow\quad & \exists (y_n)\in Y^{\mathbb{N}}\ \exists k\geqslant 1 \text{ with } \overline{\mathrm{span}}\{y_n : n\in\mathbb{N}\} = Y\\ & \text{and } (y_n)\overset{k}{\sim}\big(f_n(X)\big)\end{aligned}$$

where $(y_n)\overset{k}{\sim}\big(f_n(X)\big)$ if for every $m\in\mathbb{N}$ and every $a_0,\dots,a_m\in\mathbb{R}$ we have

$$\frac{1}{k}\,\|\sum_{n=0}^m a_n y_n\| \leqslant \|\sum_{n=0}^m a_n f_n(X)\|_\infty \leqslant k\,\|\sum_{n=0}^m a_n y_n\|.$$

For every $k\in\mathbb{N}$ with $k\geqslant 1$ consider the relation E_k in $C(2^{\mathbb{N}})^{\mathbb{N}}\times H^{\mathbb{N}}$ defined by

$$\big((y_n),(h_n)\big)\in E_k \Leftrightarrow (y_n)\overset{k}{\sim}(h_n).$$

Then E_k is Borel since

$$\begin{aligned}(y_n)\overset{k}{\sim}(h_n) \quad\Leftrightarrow\quad & \forall m\ \forall a_0,\dots,a_m\in\mathbb{Q}\ \Big(\forall l\ \big|\sum_{n=0}^m a_n h_n(l)\big| \leqslant k\|\sum_{n=0}^m a_n y_n\|\Big)\\ & \text{and } \Big(\forall p\ \exists i\ \frac{1}{k}\,\|\sum_{n=0}^m a_n y_n\| - \frac{1}{p+1} \leqslant \big|\sum_{n=0}^m a_n h_n(i)\big|\Big).\end{aligned}$$

The sequence (f_n) consists of Borel functions, and so, the relation I_k in $C(2^{\mathbb{N}})^{\mathbb{N}}\times Z$ defined by the rule

$$\big((y_n),X\big)\in I_k \Leftrightarrow \big((y_n),(f_n(X))\big)\in E_k$$

is Borel. Finally, the relation S in $\mathrm{SB}\times C(2^{\mathbb{N}})^{\mathbb{N}}$ defined by

$$\big(Y,(y_n)\big)\in S \Leftrightarrow (\forall n\ y_n\in Y) \text{ and } \overline{\mathrm{span}}\{y_n : n\in\mathbb{N}\}=Y$$

is Borel (see [Bos, Lemma 2.6]). Now set $A^*=\{Y\in \mathrm{SB} : \exists X\in A \text{ with } X^*\cong Y\}$. By the above discussion, it follows that

$$\begin{aligned} Y\in A^* \quad \Leftrightarrow \quad & \exists X\in A\ \exists (y_n)\in C(2^{\mathbb{N}})^{\mathbb{N}}\ \exists k\geqslant 1 \text{ with } \big(Y,(y_n)\big)\in S \\ & \text{and } \big((y_n),X\big)\in I_k. \end{aligned}$$

Clearly the above formula gives an analytic definition of A^*, as desired. □

4. Further Consequences

The following proposition is a second application of Proposition 2. It implies that, although question (Q1) stated in the introduction is false, its relativized version to any analytic subset of SD is true. Specifically, we have the following proposition.

Proposition 7. *Let A be an analytic class of separable dual spaces. Also let B be an analytic subset of* SD. *Then the set $A_*(B):=\{X\in B : \exists Y\in A \text{ with } X^*\cong Y\}$ is analytic.*

Proof. Arguing as in the proof of Theorem 1, we select a Borel subset Z of SD such that $B\subseteq Z$. Define $G\subseteq Z\times H$ by setting $(X,f)\in G$ if and only if $f\in K_{X^*}$. Then G is Borel. Let $f_n\colon Z\to H$ ($n\in\mathbb{N}$) be the sequence of Borel selectors of G obtained by Proposition 2. Also let I_k ($k\in\mathbb{N}$) and S be the relations defined in the proof of Theorem 1. Now observe that

$$\begin{aligned} X\in A_*(B) \quad \Leftrightarrow \quad & (X\in B) \text{ and } \big[\exists Y\in A\ \exists (y_n)\in C(2^{\mathbb{N}})^{\mathbb{N}}\ \exists k\geqslant 1 \text{ with} \\ & \big(Y,(y_n)\big)\in S \text{ and } \big((y_n),X\big)\in I_k\big]. \end{aligned}$$

Therefore, the set $A_*(B)$ is analytic, as desired. □

Remark 1. Related to Proposition 7 the following question is open to us. Let ϕ be a co-analytic rank on SD. Also let A be an analytic class of separable dual spaces such that for every $Y\in A$ there exists $\xi_Y<\omega_1$ with $\sup\{\phi(X) : X^*\cong Y\}<\xi_Y$. Is, in this case, the set $A_*=\{X\in\mathrm{SB} : \exists Y\in A \text{ with } X^*\cong Y\}$ analytic? If this is true, then the counterexample to question (Q1), presented in the introduction, is (in a sense) unique. We notice that if we further assume that $\sup\{\xi_Y : Y\in A\}<\omega_1$, then Proposition 7 implies that the answer is positive.

For every Banach space X denote by $\mathrm{Sz}(X)$ the Szlenk index of X (see [Sz]). Let ξ be a countable ordinal and consider the class

$$\mathcal{S}_\xi:=\big\{X\in\mathrm{SB} : \max\{\mathrm{Sz}(X),\mathrm{Sz}(X^*)\}\leqslant\xi\big\}.$$

By Theorem 1 and Proposition 7, we have the following corollary.

Corollary 8. *For every countable ordinal ξ the class $\mathcal{S}_\xi$ is analytic.*

Proof. We fix a countable ordinal ξ. As in the proof of Theorem 1, consider the subset SD of SB consisting of all Banach spaces with separable dual. We set $B \coloneqq \{X \in \mathrm{SD} : \mathrm{Sz}(X) \leqslant \xi\}$ and $A \coloneqq B \cap B^*$. Notice that

$$A = \big\{Y \in \mathrm{SB} : \mathrm{Sz}(Y) \leqslant \xi \text{ and } (\exists X \in \mathrm{SB} \text{ with } \mathrm{Sz}(X) \leqslant \xi \text{ and } Y \cong X^*)\big\}.$$

By [Bos, Theorem 4.11], the map $X \mapsto \mathrm{Sz}(X)$ is a co-analytic rank on SD. It follows that the set B is analytic (in fact Borel—see [Ke]). By Theorem 1, so is the set A. By Proposition 7, we see that the set $A_*(B)$ is analytic. Since $A_*(B) = \mathcal{S}_\xi$, the result follows. □

Let REFL be the subset of SD consisting of all separable reflexive spaces. Recently, Odell, Schlumprecht and Zsák have shown [OSZ, Theorem D] that for every countable ordinal ξ the class

$$\mathcal{C}_\xi \coloneqq \big\{X \in \mathrm{REFL} : \max\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\} \leqslant \xi\big\}$$

is also analytic. Their proof is based on Corollary 8 above, as well as, on a deep refinement of Zippin's embedding theorem [Z] and on a sharp universality result concerning the classes $\{\mathcal{C}_{\omega^{\xi\cdot\omega}} : \xi < \omega_1\}$ (see [OSZ, Theorems B and C]).

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National Technical University of Athens, Faculty of Applied Sciences, Department of Mathematics, Zografou Campus, 157 80, Athens, Greece.

E-mail address: pdodos@math.ntua.gr