Defining Ambiguity and Ambiguity Attitude

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According to the well-known distinction attributed to Knight (1921), there are two kinds of uncertainty. The first, called "risk," corresponds to situations in which all events relevant to decision making are associated with obvious probability assignments (which every decision maker agrees to). The second, called "(Knightian) uncertainty" or (following Ellsberg (1961)) "ambiguity," corresponds to situations in which some events do not have an obvious, unanimously agreeable, probability assignment. As Chapter 1 makes clear, this collection focuses on the issues related to decision making under ambiguity. In this Chapter, I briefly discuss the issue of the formal definition of ambiguity and ambiguity attitude.

In his seminal paper on the CEU model (1989), Schmeidler proposed a behavioral definition of ambiguity aversion, showing that it is represented mathematically by the convexity of the decision maker's capacity v. The property he proposed can be understood by means of the example of the two coins used in Chapter 1. Assume that the decision maker places bets that depend on the result of two coin flips, the first of a coin that she is very familiar with, the second of a coin provided by somebody else. Given that she is not familiar with the second coin, it is possible that she would consider"ambiguous" all the bets whose payoff depends on the result of the second flip. (For instance, a bet that pays \$1 if the second coin lands with heads up, or equivalently if the event {HH, TH} obtains.) If she is averse to ambiguity, she may therefore see such bets as somewhat less desirable that bets that are "unambiguous," i.e., only depend on the result of the first flip. (For instance, a bet that pays \$1 if the first coin lands with heads up, or equivalently if the event {HH, HT} obtains.)

However, suppose that we give the decision maker the possibility of buying *shares* of each bet. Then, if she is offered a bet that pays \$0.50 on {HH} and \$0.50 on {HT}, she may prefer it to either of the two ambiguous bets. In fact, such a bet has the same contingent payoffs as a bet which pays \$0.50 if the first coin lands with heads up, which is unambiguous. That is, a decision maker who is averse to ambiguity may prefer the equal-probability "mixture" of two ambiguous acts to either of the acts. In contrast, a decision maker who is attracted to ambiguity may prefer to choose one of the ambiguous acts.

Formally, Schmeidler called *ambiguity averse* a decision maker who prefers the even mixture[1] $(1/2)f + (1/2)g$ of two acts that she finds indifferent to either of the two acts. That is, $(1/2)f +$

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[1]Recall that Schmeidler used the Anscombe-Aumann setting, in which mixtures of acts can be defined state-by-state. Also, he used the term "uncertainty" averse rather than ambiguity averse.

$(1/2)g \succcurlyeq f$ for all f and g such that $f \sim g$. As recalled earlier, if the decision maker has CEU preferences, this property implies that her capacity v is convex. If, instead, she has MMEU preferences, then she satisifies this property automatically (indeed, it is one of the axioms that characterize the model).

While this is certainly a compelling definition, it does not seem to be fully satisfactory as a definition of ambiguity aversion. First of all, it explicitly relies on the availability of mixtures of acts, and thus apparently on the existence of objective randomizing devices. This is not a serious problem, for it has been shown by Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2001) that mixtures can be defined without invoking randomizing devices, provided the set of prizes is rich and preferences satisfy some mild restrictions. (Moreover, Casadesus-Masanell, Klibanoff, and Ozdenoren (2000) show that Schmeidler's definition can be formulated in a Savage setting which does not explicitly involve mixtures.) Second —and more important— Schmeidler's definition is not satisfied by preferences that *do* seem to embody ambiguity aversion, as illustrated by the following example.

Example 1 Consider again the decision maker facing the set S ={HH,HT,TH,TT} of results of flips of a familiar and an unfamiliar coin. Suppose that she has CEU preferences represented by a capacity v on S which:

- assigns 1/8 to each singleton state, i.e.,

$$v(\{\text{HH}\}) = v(\{\text{HT}\}) = v(\{\text{TH}\}) = v(\{\text{TT}\}) = 1/8;$$

- assigns 1/2 to the results of the familiar coin flip, i.e.,

$$v(\{\text{HH,HT}\}) = v(\{\text{TH,TT}) = 1/2;$$

- assigns 9/16 to any 3-state event (like {HH,HT,TH}) and 1 to the whole state space;

- assigns the sum of the weights of its (singleton) elements to each other event.

Such a preference embodies a dislike of ambiguity: The decision maker prefers to bet on the familiar coin rather than on the unfamiliar one (notice that $v(\{\text{HH,TH}\}) = 1/4 < 1/2 = v(\{\text{HH,HT}\})$). However, the capacity v is not convex,[2] so that she is not ambiguity averse according to Schmeidler's definition. ◇

Comparative Foundations to Ambiguity Aversion

Motivated by these problems with Schmeidler's definition, Epstein (1999) tried a different approach to defining aversion to ambiguity, inspired by Yaari's (1969) general definition of risk aversion for non-expected utility preferences. He suggested using a two-stage approach, first defining a notion of comparative ambiguity aversion, and then calling averse to ambiguity any

[2]For instance, we have that $v(\{\text{HH,HT,TH}\}) = 9/16 < 10/16 = v(\{\text{HH,HT}\}) + v(\{\text{TH}\})$.

preference which is more averse than (what we establish to be) an ambiguity neutral preference. Ghirardato and Marinacci (2002, GM) followed his example, employing a different comparative notion and a different definition of ambiguity neutrality. For reasons that will become clear presently, I shall discuss these contributions in inverse chronological order.

GM depart from the observation that preferences that obey the classical EUT are intuitively ambiguity neutral, and propose using such preferences as the benchmark to measure ambiguity aversion. As to the comparative ambiguity aversion notion, they suggest calling a preference $\succcurlyeq_2$ *more ambiguity averse than* a preference $\succcurlyeq_1$ if both preferences are represented by the same utility function[3] and given any constant act x and any act f, we have that whenever the first preference favors the (certainly unambiguous) constant x to the (possibly ambiguous) f, the second does the same; that is,

$$x \succcurlyeq_1 (\succ_1) f \Longrightarrow x \succcurlyeq_2 (\succ_2) f. \tag{1}$$

Thus, a preference is *ambiguity averse* if it is more averse to ambiguity than some EUT preference. GM show that every MMEU preference is averse to ambiguity in this sense (while "maximax EU" preferences are ambiguity seeking). In contrast, a CEU preference is ambiguity averse if and only if its capacity v has a non-empty core, a strictly weaker property than convexity. Therefore, GM conclude that Schmeidler's definition captures *strictly more* than aversion to ambiguity. (Notice that the capacity v in Example 1 does have a non-empty core; the uniform probability on S is in $Core(v)$.)

This definition is simple and it has intuitive characterizations,[4] but it can be criticized in an important respect. It does not distinguish between those departures from EUT which are unrelated to ambiguity (like the celebrated "Allais paradox") —in the terminology of Chapter 1, the violations of the third tenet of Bayesianism— and those which are. Every departure from the EUT benchmark is attributed to the presence of ambiguity. To see why this may be an issue, consider the following example.

Example 2 Using again the two-coin example, consider a decision maker with CEU (indeed, RDEU) preferences and the capacity v' defined by : $v'(S) = 1$ and $v'(A) = P(A)/2$ for $A \neq S$, where P is the uniform probability on the state space S. At first blush, we may invoke aversion to ambiguity (recall that the second coin is the unfamiliar one) to explain the fact that $v'(\{\text{TH,HH}\}) = v'(\{\text{TT,HT}\}) = 1/4$. However, we also see that $v'(\{\text{HH,HT}\}) = v'(\{\text{TH,TT}\}) = 1/4$; that is, the decision maker is similarly unwilling to bet on the familiar, unambiguous, coin. What we are observing is a dislike of uncertainty which is more general than just aversion to ambiguity: The decision maker treats even events with "known" probability 1/2 as if they really had probability 1/4. This is a trait usually called *probabilistic risk*

[3] The preferences considered in GM, called "biseparable preferences," induce state-independent and cardinally unique utilities. They include CEU and MMEU preferences (and other models as well) as special cases.

[4] The idea that non-emptiness of the core could be a more appropriate formalization of ambiguity aversion for CEU preferences had already been suggested by Montesano and Giovannoni (1996).

aversion; the decision maker appears in fact to be *neutral* to the ambiguity in this problem. Moreover, observe that the capacity v' is convex, casting some doubt on the relation between convexity and ambiguity aversion. ◇

Epstein (1999) offers a definition that avoids this problem, carefully distinguishing between "risk-based" behavioral traits and "ambiguity-based" ones. The key idea is to use a set $\mathcal{A}$ of events which are exogenously known to be considered unambiguous by every decision maker, like the results of the flips of the familiar coin in the example above. Acts which only depend on the events in $\mathcal{A}$ are called *unambiguous*. The comparative definition is then modified as follows: Say that preference $\succcurlyeq_2$ is *more ambiguity averse than* preference $\succcurlyeq_1$ if for any act f and any unambiguous act h, we have

$$h \succcurlyeq_1 (\succ_1) f \Longrightarrow h \succcurlyeq_2 (\succ_2) f. \tag{2}$$

Notice that this definition is strictly stronger than GM's, as constant acts are unambiguous, while in general (i.e., for nontrivial $\mathcal{A}$) there will be unambiguous acts which are not constant. As long as the set $\mathcal{A}$ (and hence the set of unambiguous acts) is sufficiently rich, this implies that the two preferences have identical utility functions *as well as* identical probabilistic risk aversion. For instance, the CEU decision maker with capacity v in Example 1 cannot be compared to the one with capacity v' in Example 2; their willingness to bet on the unambiguous results of the flips of the second coin are different. A CEU preference comparable to that capacity v' must also "transform" an objective probability of 1/2 into a 1/4.

The choice of the benchmark with respect to which ambiguity aversion has to be measured is made consistently with this modified comparative notion. EUT preferences are probabilistic risk neutral, and do not "transform" the probabilities of unambiguous events, so they cannot be compared to preferences like the CEU preference with capacity v'. Epstein uses preferences which satisfy Machina and Schmeidler's (1992) *probabilistic sophistication* model, which allows non-expected utility preferences as long as their ranking of bets on events can be represented by a probability.[5] He calls a decision maker *ambiguity averse* if his preference is more averse to ambiguity than a probabilistically sophisticated preference. His characterization results are not as clear-cut as those in GM: While basically every MMEU preference is ambiguity averse, the characterization of CEU preferences is less straightforward. Epstein does provide a full characterization for those CEU preferences that satisfy a certain smoothness condition, which he calls "eventwise differentiability." I refer the reader to his paper for details.

Epstein's definition of ambiguity aversion is limited by the requirement of a rich set $\mathcal{A}$ of exogenously unambiguous events. Suppose that we observe a decision maker who has CEU preferences with capacity v' as in Example 2, but we *do not know* what the decision maker knows about these two coins. Can we conclude that he is ambiguity neutral and probabilistic risk averse? If both coins were unfamiliar, his capacity would instead reflect ambiguity aversion

[5] For instance, a CEU preference is probabilistically sophisticated if its capacity v is ordinally equivalent to a probability; i.e., if it is RDEU. Such is the case of the preference with capacity v' in Example 2.

—for all we know, he may even have EUT preferences (i.e., be probabilistic risk *neutral*) when betting on familiar coins. The problem is that in this case the set $\mathcal{A}$ is just the trivial $\{\emptyset, S\}$, too poor to enable us to distinguish between "pure" ambiguity aversion and probabilistic risk aversion. (As a consequence, the observation that the capacity v' is convex yet induces behavior that is not *intuitively* ambiguity averse, may be in need of reconsideration.)

We reach the conclusion that a theory of "pure" ambiguity aversion (as opposed to what is measured by GM) must be founded on an endogenous theory of ambiguity, if it is to be generally valid. This is what Epstein next turned his attention to; it is discussed in the next subsection.

Before closing this discussion on the comparative foundation to ambiguity aversion, I remark that, while Epstein (1999) is the earliest paper to use a comparative approach to provide an *absolute* notion of ambiguity aversion, there are earlier papers that discuss comparative ambiguity aversion. Tversky and Wakker (1995) present and characterize some different comparative notions related to ambiguity and probabilistic risk aversion. Kelsey and Nandeibam (1996) propose a comparative notion similar to GM's, implicitly assuming the equality of utility, and show its characterization for CEU and MMEU preferences.

What Is Ambiguity?

As observed earlier, the quest for the distinction of ambiguity aversion and behavioral traits unrelated to the presence of ambiguity was a driving force behind the more recent attempts (like Epstein and Zhang (2001)) at understanding the behavioral consequences of the presence of ambiguity. However, there have been earlier papers that addressed the definition of ambiguity.

Fishburn (1993) considers a primitive ambiguity relation over events, and discusses its properties and representation by an ambiguity measure. Nehring (1999) defines an event A unambiguous for a MMEU preference with set of priors C if $P(A) = P'(A)$ for every $P, P' \in C$. As to CEU preferences, Nehring recalls that any capacity v on a finite state space $S = \{s_1, s_2, \ldots, s_n\}$ can be canonically associated with the set $\mathcal{C}_v$ of the probabilities P_σ defined as follows. Let σ denote a permutation of the indices $\{1, \ldots, n\}$, and define[6]

$$P_\sigma(s_{\sigma(i)}) = v(\{s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(i)}\}) - v(\{s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(i-1)}\}).$$

Using this fact allows him to define ambiguity of events analogously to the MMEU case, with $\mathcal{C}_v$ in place of C. In both cases, an event is *unambiguous* if it is given identical weight in the evaluation of any act. Nehring shows that while for MMEU preferences the set of unambiguous events is a λ-*system* (a class closed with respect to complements and disjoint unions), for CEU preferences it is an algebra (i.e., it is also closed with respect to intersections). As there are situations in which the set of unambiguous events is not an algebra, this suggests that CEU

[6] Given utility u and an act f, it can be seen from the definition of Choquet integral that if σ is such that $u(f(s_{\sigma(1)})) \geq u(f(s_{\sigma(2)})) \geq \cdots \geq u(f(s_{\sigma(n)}))$, then $\int u(f)\, dv = \int u(f)\, dP_\sigma$.

preferences cannot be used to model all decision problems under ambiguity.[7]

A notion of ambiguity for events that holds for a wider class of preferences was introduced in Zhang (2002). Loosely put, Zhang calls *unambiguous* an event A such that Savage's sure-thing principle holds for acts separated on the partition $\{A, A^c\}$. He then shows that the set of such events is a λ-system, and that for a subset of CEU preferences (those which induce an exact v; details are found in GM) it has a simple representation in terms of the capacity v: It is the set of the A's such that $v(A) + v(A^c) = 1$.

Zhang's definition of unambiguous event was later modified in Epstein and Zhang (2001, EZ), the announced attempt to endogenize the class of unambiguous events used in Epstein's definition of ambiguity aversion. The idea of EZ's definition is similar to Zhang's (2002), though it yields a larger collection of unambiguous events. Axioms on the decision maker's preferences are introduced which guarantee that the resulting collection of events is a λ-system and that the preferences over the sets of unambiguous acts (those which are measurable with respect to unambiguous events) are probabilistically sophisticated in the sense of Machina and Schmeidler (1992). This yields an interesting extension of Machina and Schmeidler's and Savage's models, wherein the set of events on which the decision maker satisfies the first and second tenet of Bayesianism is determined endogenously.[8] However, it does not fully solve the problem of screening "risk-based" behavioral traits. In fact, if a preference is probabilistically sophisticated then *every* event is unambiguous in the EZ sense. It follows that the decision maker with CEU preferences and capacity v' in Example 2 (who, recall, is probabilistically sophisticated) considers every event unambiguous and is probabilistic risk averse. This is *regardless of the information that is available to him*; it does not matter whether he is betting on familiar or unfamiliar coins. The problem is that EZ's definition does not distinguish between the really unambiguous events and those which appear to be. It seems likely that such distinction could only be assessed by enriching the decision framework; i.e., allowing the theorist to observe more than just the decision maker's preferences over acts.

Going back to the two-coins flip example, regardless of what a decision maker thinks about the unfamiliar coin she may believe that the event that it lands heads up on a single flip is more likely than the event that it lands heads up twice in a row. That is, she may hold that a bet on one head in two flips is "unambiguously better than" a bet on two heads in two flips. All the notions of ambiguity introduced thus far cannot formally capture this possibility. In an unpublished 1996 conference talk, Nehring suggested doing so using the largest subrelation of $\succcurlyeq$ that satisfies independence, that I shall label $\succcurlyeq^I$. He argued that if S is finite, for a class of preferences[9] the results in Bewley (2002) can be used to show that $\succcurlyeq^I$ has a multiple priors

[7] The fact that unambiguous events should form λ-systems and not algebras was observed earlier in Zhang (2002), whose first version predates Nehring's.

[8] Further extensions in this spirit are found in Kopylov (2002). In that paper it is also shown that in general the sets of unambiguous events of Zhang and EZ are not λ-systems, but less structured families called "mosaics."

[9] Those that have linear utility among the preferences that satisfy all the axioms in Gilboa and Schmeidler (1989) but their "uncertainty aversion" axiom. The latter are called *invariant biseparable preferences* by Ghirardato, Maccheroni, and Marinacci (2002).

with unanimity representation, with a set of priors D. In particular, when the decision maker satisfies MMEU with set of priors C we have $C = D$, while $D = \mathcal{C}_v$ when she satisfies CEU with capacity v.

Although the relation $\succcurlyeq^I$ thus obtained can in principle be constructed using only behavioral data, its derivation is not simple. Independently, Nehring (2001) and Ghirardato, Maccheroni, and Marinacci (2002) proposed to derive from the decision maker's preference an *unambiguous preference* relation as follows: Say that act f is *unambiguously preferred to* act g, which is denoted $f \succcurlyeq^* g$, if $\alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h$ for every α and every h. That is, $f \succcurlyeq^* g$ if the preference of f over g cannot be overturned by mixing them with another act h, regardless of whether the latter allows to hedge (or speculate on) ambiguity. It turns out that $\succcurlyeq^* = \succcurlyeq^I$, providing a more immediate behavioral foundation to the approach proposed by Nehring in his 1996 talk. The set of priors D representing $\succcurlyeq^*$ by unanimity is naturally interpreted as the ambiguity that the decision maker perceives —better, appears to perceive— in her problem. The events on which all probabilities in D agree (which can simply characterized in terms of the primitive $\succcurlyeq$; see Ghirardato *et al.* (2002, Prop. 24)) are natural candidates for being called *unambiguous*, and the collection of unambiguous events forms a λ-system.

Unlike his 1996 talk, Nehring (2001) considers a countably infinite S and preferences whose induced $\succcurlyeq^*$ is represented by a D satisfying a "range convexity" condition. Among various consequences of such range convexity, he shows the characterization of two intuitive notions of absolute ambiguity aversion. In particular, say that a preference relation is *weakly ambiguity averse* if for every pair of partitions of S, $\{A_1, A_2, \ldots, A_n\}$ and $\{T_1, T_2, \ldots, T_n\}$, such that each T_i is unambiguous, we cannot have that the decision maker prefers betting on A_i over betting on T_i for every i. Under Nehring's assumptions, a decision maker is weakly ambiguity averse iff her ranking of bets can be represented by a capacity v with a non-empty core. A stronger property, which Nehring calls "ambiguity aversion," is shown instead to be equivalent to the fact that the decision maker's ranking of bets is represented by the lower envelope of D.

Ghirardato *et al.* (2002) consider an arbitrary S and a different class of preferences.[10] They show that the set D representing $\succcurlyeq^*$ can be also obtained as an (appropriately defined) "derivative" of the functional that represents the preferences. In particular, when the state S is finite this characterization implies that D is the (closed convex hull of the) set of all the Gateaux derivatives of the preference functional, where they exist. This result generalizes the (EUT) intuition that a decision maker's subjective probability of state s is the shadow price for changes in the utility received in state s, by allowing a multiplicity of shadow prices. A consequence is the extension to preferences with nonlinear utility of Nehring's 1996 result that D corresponds to C (resp. $\mathcal{C}_v$) in the MMEU (resp. CEU) case —which in turn implies that the set of unambiguous events coincides with that defined for such preferences in Nehring (1999).

[10]Invariant biseparable preferences (see footnote 9). Such preferences do not yield specific restrictions on D (beyond convexity, nonemptiness and closedness), but they embody a mild restriction that Nehring (2001) calls "utility sophistication." Nehring shows that under range convexity, it is possible to define an unambiguous *likelihood* relation on events even without utility sophistication. See that paper for details.

Ghirardato *et al.* also prove that the preferences they study can in general be given a representation which is a generalization of the MMEU representation. More precisely, an act f is evaluated via

$$a(f)\min_{P\in D}\int u(f(s))\,dP(s)+(1-a(f))\max_{P\in D}\int u(f(s))\,dP(s),$$

where $a(\cdot)$ is a function taking values in [0,1] which represents the decision maker's aversion to perceived ambiguity in the sense of GM. They also axiomatize the so-called α-*maxmin EU* model, in which $a(\cdot)\equiv\alpha$.[11] The interesting aspect of this representation is its clear separation of ambiguity (represented by D) and ambiguity attitude (represented by $a(\cdot)$), and it is encouraging that the model does not impose cross-restrictions between these two aspects of the representation.

As can be seen from the foregoing discussion, the "relation-based" approach to modelling ambiguity is, at least in terms of its consequences, a significant improvement over the previous "event-based" approaches. It has also yielded some interesting new perspectives on the characterization of ambiguity aversion and love. On the other hand, it is important to stress that this approach suffers of the same shortcoming as GM's theory of ambiguity aversion: It does not really describe "pure" ambiguity aversion, rather the conjunction of all those behavioral features that induce departures from the independence axiom of EUT. In the terminology of Chapter 1 it does not distinguish between the violations of the first and the third tenets of Bayesianism. As observed earlier, it is not obvious that a solution to this identification problem can be reached without departing from a purely behavioral approach. Besides, a difficulty with such a departure is that it would require some prejudgment as to what really constitutes ambiguity, which is the very question that we set to answer.

Another limitation of the "relation-based" approach due to its purely behavioral nature is the identification of ambiguity neutrality with lack of ambiguity. If a decision maker's preference satisfy EUT, she is deemed to perceive no ambiguity, while it may be the case that she perceives ambiguity and is neutral with respect to it. Clearly, distinctions could be drawn if we considered ancillary information about the ambiguity present in the problem, at the mentioned cost of prejudging the nature of ambiguity. On the other hand, this is not as serious a concern as the one mentioned above, for ultimately our interest is modelling ambiguity as it affects decision makers' behavior, and not otherwise.

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[11] Variants of this representation are well-known at least since the seminal work of Hurwicz (published in Arrow and Hurwicz (1972)). See, in particular, Jaffray (1989).

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