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DEFINING NORMAL SUBGROUPS OF UNIPOTENT ALGEBRAIC GROUPS

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ABSTRACT. Let G be a connected unipotent algebraic group defined over the perfect field k. We show that polynomial generators $x_1, \cdots, x_n$ for the ring $k[G]$ can be chosen so that if N is any connected normal k-closed subgroup of G, then $I(N)$ can be generated by codim N p-polynomials in $x_1, \cdots, x_n$ where $p = \text{char } k$. Moreover $k[G/N]$ can also be generated as a polynomial algebra over k by p-polynomials.

Introduction. These results are essentially an extension of a theorem of Rosenlicht [4, Theorem 1].

We use the notation and conventions of [1] throughout this paper.

Recall that a p-polynomial in $k[T]$ is a linear form if $p = 0$ and a polynomial all of whose exponents are powers of p if $p > 0$. A p-polynomial in $k[x_1, \cdots, x_n]$ is a sum of p-polynomials in each of the single variables $x_1, \cdots, x_n$. A function $f \in k[G]$ will be called *additive* if $f(ab) = f(a) + f(b)$ for all closed points a, b in G.

1. **Frattini coordinates.** Let G be a unipotent algebraic group. The *Frattini subgroup* of G is the intersection of all closed subgroups of codimension one. We shall denote this group by $Fr(G)$.

Proposition 1. *If G is a unipotent algebraic group then $Fr(G)$ is a closed characteristic subgroup of G. If G is connected and defined over the perfect field k, then $Fr(G)$ is connected and defined over k. Moreover in the connected case $G/Fr(G)$ has the structure of a vector group (over k if G is defined over k) and is the maximal such quotient.*

Proof. The first assertion is immediate. Let $H \subset G$ be a closed subgroup of G of codimension one. Since $G/H \simeq G_a$, H contains the commutator subgroup of G and the group generated by the pth powers of the elements of G. It follows that $Fr(G)$ also contains these subgroups.

Thus $G/Fr(G)$ is connected, commutative and of exponent p hence by

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[3, Proposition 2] has the structure of a vector group. If $N \subset G$ is any normal subgroup such that G/N is isomorphic to G_a^r for some integer r, then consider the natural map $G \to G/N \simeq G_a^r$ followed by projection Π_i onto each factor, $i = 1, 2, \cdots, r$. Each Π_i is a homomorphism with kernel say H_i and $\bigcap H_i = N$. Since H_i has codimension one, $N \supset Fr(G)$ and G/N is an image of $G/Fr(G)$ which establishes the maximality assertion.

As for rationality and connectedness, let N be the closed normal subgroup generated by the commutator subgroup and pth powers of the elements of G. Then $N \subset Fr(G)$, N is connected and G/N has the structure of a vector group [3, Proposition 2] so $N = Fr(G)$. Since N is defined over k so is G/N [1, 6.8]. This completes the proof.

Now let G be a connected unipotent algebraic group and $Fr(G)$ the Frattini subgroup of G. If G is defined over the perfect field k then by [4, Corollary 2 of Theorem 1], $k[G]$ is k-isomorphic to $k[G/Fr(G)] \otimes k[Fr(G)]$. Let $x_1, \cdots, x_r$ be additive coordinates for the vector k-group $G/Fr(G)$ (cf. [3, §1]). Then $k[G] = k[x_1, \cdots, x_r] \otimes k[Fr(G)]$. By the proposition $F_1 = Fr(G)$ is again connected and defined over k and we may continue this process until we arrive at a complete set of polynomial generators for $k[G]$. A set of polynomial generators $x_1, \cdots, x_n$ obtained in this way will be called a set of *Frattini* coordinates for G.

In case G itself has the structure of a vector group, these coordinates have essentially been studied by Rosenlicht [3], [4] and Tits [7, III, 3.3]. In particular the following proposition is easily deduced from their results.

Proposition 2. *Let V be a connected unipotent algebraic group defined over the perfect field k. Suppose V has the structure of a vector group over k and $x_1, \cdots, x_n$ are Frattini coordinates for V. Then*

(i) *if W is any k-closed subgroup of V then $I(W)$ is generated by* codim W *p-polynomials in $x_1, \cdots, x_n$;*

(ii) *the Frattini coordinates of $k[V/W] \subset k[V]$ are p-polynomials in the Frattini coordinates of V.*

Now let G be any connected unipotent group defined over the perfect field k. Let $N \subset Fr(G) = F$ be a k-closed normal subgroup of G. Then since $G/N \simeq G/F \times F/N$ we have $k[G/N] \simeq k[G/F] \otimes k[F/N]$. It follows from (ii) above that if $Fr(F) = e$ then a set of Frattini coordinates for G/N may be taken to be p-polynomials in any fixed set of Frattini coordinates of G.

Theorem. *Let the connected unipotent algebraic group G be defined over the perfect field k. Let $x_1, \cdots, x_n$ be a fixed set of Frattini coordi-*

nates for G. Suppose Z is a closed connected central one dimensional subgroup of G defined over k. Then there exists a set of Frattini coordinates of G/Z *in* $R = k[G/Z] \subset k[G]$ *which consists of p-polynomials in* $x_1, \cdots, x_n$.

Proof. Let $F_0 = G \supset F_1 = Fr(G) \supset \cdots \supset F_s \supset e$ be the Frattini series of G. We argue by induction on the length, s, of the series. Thus, suppose $s = 1$. If $Z \subset F_1$ then $G/Z \simeq G/F \times F/Z$, hence $k[G/Z] \simeq k[G/F] \otimes k[F/Z]$. But by the remarks above, $k[G/Z]$ has p-polynomials in $x_1, \cdots, x_n$ as Frattini coordinates.

If $Z \cap F_1$ is finite we distinguish two cases.

Case 1. $Z \cap F_1 = e$. Then ZF_1/F_1 is a direct factor of G/F_1 and is not equal to G/F_1 since Z is contained in a subgroup of codimension one.

Let $L \supset F_1$ be a connected k-closed subgroup of G such that L/F_1 is a complement of ZF_1/F_1 in G/F_1 [3, Proposition 1]. Then codim $L = 1$, hence L is normal in G. If $N = L \cap Z$ then $NF_1/F_1 = e$, hence $N \subset Z \cap F_1$. Thus $L \cap Z = e$ and clearly $LZ = G$.

Consider the commutative diagram

$$\begin{array}{ccccccc} e \longrightarrow F_1 & \xrightarrow{\ i\ } & G & \xrightarrow{\ \pi\ } & G/F_1 & \longrightarrow & e \\ & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m'} & & \\ & & Z \times L & \xrightarrow{\ \nu\ } & ZF_1/F_1 \times L/F_1 & \longrightarrow & e \end{array}$$

where i is inclusion, π the quotient morphism, m and m' multiplication, and $\nu = \pi|_Z \times \pi|_L$.

We obtain a commutative diagram of Lie algebras:

$$\begin{array}{ccccc} \mathfrak{L}(F_1) & \xrightarrow{\ di\ } & \mathfrak{L}(G) & \xrightarrow{\ d\pi\ } & \mathfrak{L}(G/F_1) \longrightarrow 0 \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle dm} & & \uparrow{\scriptstyle dm'} \\ (\mathrm{Ker}\, d(\pi/A)) \oplus \mathfrak{L}(F_1) & \longrightarrow & \mathfrak{L}(Z) \oplus \mathfrak{L}(L) & \xrightarrow{\ d\nu\ } & \mathfrak{L}(ZF_1/F_1) \oplus \mathfrak{L}(L/F_1) \end{array}$$

Since α and dm' are surjective so is dm. Thus dm is an isomorphism and m is separable. It follows from [1, Chapter II, 6.1] that $m: Z \times L \longrightarrow G$ is an isomorphism.

Now choose new Frattini coordinates $y_1, \cdots, y_r, x_{r+1}, \cdots, x_n$ such that $V(y_1, \cdots, y_r) = F_1$ and $V(y_1) = L$. Then $y_1, \cdots, y_r$ are p-polynomials in $x_1, \cdots, x_r$ and $k[G/Z] = k[L] = k[y_2, \cdots, y_r, x_{r+1}, \cdots, x_n]$.

Case 2. $\Lambda = Z \cap F \neq e$. Then in G/Λ we have the conditions of Case 1.

Hence $k[G/Z] = k[G/\Lambda/Z/\Lambda] \subset k[G/\Lambda]$ is generated by p-polynomials in any set of Frattini coordinates of G/Λ. But we may assume these last are p-polynomials in $x_1, \cdots, x_n$. Hence $k[G/Z]$ has a set of Frattini coordinates consisting of p-polynomials in $x_1, \cdots, x_n$ and the case $s = 1$ is done.

If $s > 1$ we form the chain

$$G \supset F_1' \supset F_1 \supset F_2' \supset F_2 \supset \cdots \supset F_l' \supset Z \supset e$$

where F_i'/Z is the ith term in the Frattini series of G/Z, and F_l'/Z has the structure of a vector group over k. Each F_i' may be taken to be connected, closed and defined over k.

Suppose $l \geq 2$. Then $F_1 \supset Z$. By induction $k[F_1/Z] \subset k[F_1]$ has a set of Frattini coordinates which consists of p-polynomials in $x_{r+1}, \cdots, x_n$. It then follows as before from the isomorphism $k[G/Z] \simeq k[G/F_1] \otimes k[F_1/Z]$ that G/Z has the desired property.

If $l = 1$ then F_1'/Z has the structure of a vector group. But then $F_1 \subset F_1'$ and if $Z \subset F_1$ we are done arguing as above. If not, $Z \cap F_1$ is finite and $F_1 \to F_1/Z \cap F_1$ is an isogeny whose image is a vector group. Hence F_1 itself has the structure of a vector group so $s = 1$ a contradiction. This completes the proof.

Corollary 1. *Let G be a connected unipotent group defined over the perfect field k. Let N be a connected closed normal subgroup of G also defined over k. Then the Frattini coordinates of G/N in $k[G/N]$ can be taken to be p-polynomials in any fixed set of Frattini coordinates of G.*

Proof. Any connected closed subgroup normal in G and defined over k contains a central connected subgroup of dimension one defined over k by [5]. The corollary now follows by induction on the dimension of N.

Corollary 2. *Suppose G and k are as above. Then every normal closed connected subgroup N of G which is defined over k can be defined by $d =$ codim N p-polynomials in $x_1, \cdots, x_n$. Moreover these may be chosen so as to generate the ideal $I(N)$.*

Proof. We have $G/N \times N \simeq G$ and by Corollary 1, $k[G/N]$ is generated by p-polynomials in a fixed set of Frattini coordinates for G. Say $k[G/N] = k[f_1, \cdots, f_d] \subset k[G]$ where $d = \operatorname{codim} N$ and the f_i, $i = 1, \cdots, d$, are p-polynomials in $x_1, \cdots, x_n$. Then each f_i is constant on the fibres of $\pi: G \to G/N$ and vanishes on N.

Since $k[G/N] \to k[G/N] \otimes k[N] = k[G]$ is a polynomial extension by [4, Corollary 1 of Theorem 1] the ideal $(f_1, \cdots, f_d)k[G]$ is prime in $k[G]$. Hence $I(N) = (f_1, \cdots, f_d)k[G]$.

Remarks. 1. Corollary 2 is false without the assumption of normality on $N \subset G$. Consider the following example suggested by Rosenlicht.

Let G be the group of 3×3 upper triangular unipotent matrices

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : \; x, y, z \in K, \text{ char } K \neq 2 \right\}.$$

Let x, y and z be the obvious Frattini coordinates. Then

$$N = \left\{ \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} : \; t \in K \right\}$$

is a connected subgroup of G. The ideal $I(N) = (x - y, z - x^2/2)$ is clearly not generated by two p-polynomials.

Moreover if $H \subset G$ is the subgroup defined by $x - y = 0$ and $N \subset H$ is defined by $x^p - x = z - x^2/2 = 0$, then N is a finite normal subgroup of H which cannot be defined by two p-polynomials in the Frattini coordinates x, z of H. Thus the assumption of connectivity is also necessary in Corollary 2.

2. If G and k are as in Theorem 1 and H is any k-closed subgroup of codimension one (connected or not), then H can be defined by a single p-polynomial in any set of Frattini coordinates. More generally, any k-closed subgroup of G containing the Frattini subgroup of G can be defined by p-polynomials. Simply note that $\operatorname{codim}_G N = \operatorname{codim}_{G/Fr(G)} N/Fr(G)$ and apply Proposition 2(i) and (ii).

3. An interesting application of Frattini coordinates is the following theorem of Sullivan.

Theorem [6, Theorem 4]. *A connected unipotent algebraic group defined over a field of characteristic $p > 0$ is conservative if and only if it has dimension one.*

Proof. Recall that an algebraic group is conservative if the following condition holds.

Let W be the group of all algebraic group automorphisms of G. If $f \in K[G]$ then $V_f = \{w_*(f) : w \in W\}$ is finite dimensional.

By [2, §1] this is equivalent to saying that W may be given the structure of an algebraic group in such a way that the natural map $W \times G \rightarrow G$ is a morphism of varieties.

Now let $\dim G > 1$ and $K[G] = K[x_1, \cdots, x_n]$ where x_i, $i = 1, \cdots, n$, are Frattini coordinates for G. Then it is easily checked (cf. [4, Corollary 2, p. 101]) that the assignments

$$x_i \longrightarrow x_i, \qquad i = 1, \cdots, n-1,$$
$$x_n \longrightarrow x_n + P(x_1), \qquad P \text{ a } p\text{-polynomial in } x_1,$$

give an automorphism of G. In particular V_{x_n} is not finite dimensional. It is well known that $\mathrm{Aut}_{\mathrm{Alg\ group}}(G_a) = G_m$ the multiplicative group.

4. If char $K = 0$, then with respect to the isomorphism of G with $\underline{A}^n$ given by a fixed set of Frattini coordinates, every normal subgroup is a linear subvariety.

5. The converse of Corollary 2 is easily seen to be false. If G is the group of Remark 1 above and H is the subgroup $y = z = 0$, it is easily seen that H is not normal in G.

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