

# Definition of the spin current: The angular spin current and its physical consequences

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We find that in order to completely describe the spin transport, apart from spin current (or linear spin current), one has to introduce the angular spin current. The two spin currents, respectively, describe the translational and rotational motion (precession) of a spin. The definitions of these spin current densities are given and their physical properties are discussed. Both spin current densities appear naturally in the spin continuity equation. In particular, we predict that the angular spin current (or the spin torque as called in previous works), similar to the linear spin current, can also induce an electric field  $\vec{E}$ . The formula for the induced electric field  $\vec{E}$  by the angular spin current element is derived, playing the role of “Biot-Savart law” or “Ampere law.” When at large distance  $r$ , this induced electric field  $\vec{E}$  scales as  $1/r^2$ , whereas the  $\vec{E}$  field generated from the linear spin current goes as  $1/r^3$ .

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## I. INTRODUCTION

Recently, a new subdiscipline of condensed matter physics, spintronics, is emerging rapidly and generating great interests.<sup>1,2</sup> The spin current, the most important physical quantity in spintronics, has been extensively studied. Many interesting and fundamental phenomena, such as the spin Hall effect<sup>3-6</sup> and the spin precession<sup>7,8</sup> in systems with spin-orbit coupling, have been discovered and are under further study.

As for the charge current, the definition of the local charge current density  $\vec{j}_e(\mathbf{r}, t) = \text{Re}[\Psi^\dagger(\mathbf{r}, t)e\vec{v}\Psi(\mathbf{r}, t)]$  and its continuity equation  $(d/dt)\rho^e(\mathbf{r}, t) + \nabla \cdot \vec{j}_e(\mathbf{r}, t) = 0$  is well known in physics. Here  $\Psi(\mathbf{r}, t)$  is the electronic wave function,  $\vec{v} = \hat{\mathbf{r}}$  is the velocity operator, and  $\rho^e(\mathbf{r}, t) = e\Psi^\dagger\Psi$  is the charge density. This continuity equation is the consequence of charge invariance, i.e., when an electron moves from one place to another, its charge remains the same. However, in the spin transport, there are still a lot of debates over what is the correct definition for spin current.<sup>9,10</sup> The problem stems from that the spin  $\vec{s}$  is no invariant quantity in the spin transport, so that the conventional defining of the spin current  $\langle \vec{v}\vec{s} \rangle$  is no conservative. Recently, some studies have begun investigation in this direction,<sup>11-13</sup> e.g., a semiclassical description of the spin continuity equation has been proposed,<sup>11,12</sup> as well as studies introducing a conserved spin current under special circumstances.<sup>9</sup>

In this paper, we study the definition of local spin current density. We find that due to the spin is vector and it has the translational and rotational motion, one has to use two quantities, the linear spin current and the angular spin current, to completely describe the spin transport. Here the linear spin current describe the translational motion of a spin, and the angular spin current is for the rotational motion. The definition of two spin current densities are given and they appear naturally in the quantum spin continuity equation. Moreover, we predict that the angular spin current can generate an elec-

tric field similar as with the linear spin current, and thus contains physical consequences.

The paper is organized as follows. In Sec. II, we first discuss the flow of a classical vector. The flow of a quantum spin is investigated in Sec. III. In Sec. IV and Sec. V, we study the problem of an induced electric field and the heat produced by spin currents, respectively. Finally, a brief summary is given in Sec. VI.

## II. THE FLOW OF A CLASSICAL VECTOR

Before studying the spin current in a quantum system, we first consider the classical case. Consider a classical particle having a vector  $\vec{m}$  (e.g., the classical magnetic moment, etc.) with its magnitude  $|\vec{m}|$  fixed under the particle motion. To completely describe this vector flow [see Fig. 1(c)], in addition to the local vector density  $\vec{M}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\vec{m}(\mathbf{r}, t)$ , one needs two quantities: the linear velocity  $\vec{v}(\mathbf{r}, t)$  and the angular velocity  $\vec{\omega}(\mathbf{r}, t)$ . Here  $\rho(\mathbf{r}, t)$  is the particle density, and  $\vec{v}$  and  $\vec{\omega}$  describe the translational and rotational motions, re-

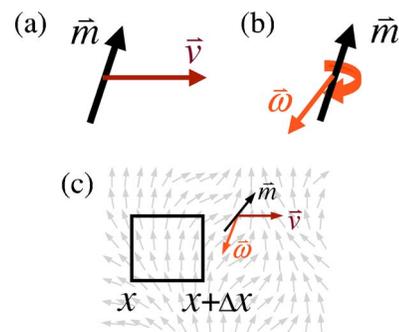


FIG. 1. (Color online) (a) and (b) are the schematic diagram for the translational motion and the rotational motion of the classic vector  $\vec{m}$ , respectively. (c) Schematic diagram for a classic vector flow.

spectively [see Figs. 1(a) and 1(b)]. In contrast with the flow of a scalar quantity in which one only needs one quantity, namely, the local velocity  $\vec{v}(\mathbf{r}, t)$ , to describe the translational motion, it is essential to use two quantities  $\vec{v}(\mathbf{r}, t)$  and  $\vec{\omega}(\mathbf{r}, t)$  for the vector flow. We emphasize that it is impossible to use one vector to describe both translational and rotational motions altogether.

Since  $|\vec{m}|$  is a constant, the change of the local vector  $\vec{M}(\mathbf{r}, t)$  in the volume element  $\Delta V = \Delta x \Delta y \Delta z$  with the time from  $t$  to  $t + dt$  can be obtained [see Fig. 1(c)]:

$$d[\vec{M}(\mathbf{r}, t) \Delta V] = \sum_{i=x,y,z} \left[ \frac{\Delta V}{\Delta i} v_i(\mathbf{r}, t) dt \vec{M}(\mathbf{r}, t) - \frac{\Delta V}{\Delta i} v_i(\mathbf{r} + \Delta \hat{i}, t) dt \vec{M}(\mathbf{r} + \Delta \hat{i}, t) \right] + \vec{\omega}(\mathbf{r}, t) \times \vec{M}(\mathbf{r}, t) \Delta V dt. \quad (1)$$

The first and the second terms on the right describe the classical vector flowing in or out the volume element  $\Delta V$  and its rotational motion respectively, and both can cause a change in the local vector density. When  $\Delta V$  goes to zero, we have the vector continuity equation

$$\frac{d}{dt} \vec{M}(\mathbf{r}, t) = -\nabla \cdot \vec{v}(\mathbf{r}, t) \vec{M}(\mathbf{r}, t) + \vec{\omega}(\mathbf{r}, t) \times \vec{M}(\mathbf{r}, t), \quad (2)$$

where  $\vec{v} \vec{M}$  is a tensor, and its element  $(\vec{v} \vec{M})_{ij} = v_i M_j$ . Note that this vector continuity equation is from the kinematics and the invariance of  $|\vec{m}|$ , and it is independent of the dynamic laws. It is well known that the scalar (e.g., charge  $e$ ) continuity equation  $(d/dt)\rho^e + \nabla \cdot \vec{j}_e = 0$  is from the kinematics and the invariance of the charge  $e$ . It is independent of the external force  $F$  as well dynamic laws. In other words, even if the acceleration  $a \neq F/m$ , the continuity equation still survives. It is complete same with the vector continuity Eqs. (2) or (3). It is also from the kinematics and the invariance of  $|\vec{m}|$ . In particular, it is independent of the external force and the torque acting on the vector, as well as the dynamic laws.

Introducing  $\vec{j}_s(\mathbf{r}, t) = \vec{v}(\mathbf{r}, t) \vec{M}(\mathbf{r}, t)$  and  $\vec{j}_\omega(\mathbf{r}, t) = \vec{\omega}(\mathbf{r}, t) \times \vec{M}(\mathbf{r}, t)$ , then Eq. (2) reduces to

$$\frac{d}{dt} \vec{M}(\mathbf{r}, t) = -\nabla \cdot \vec{j}_s(\mathbf{r}, t) + \vec{j}_\omega(\mathbf{r}, t). \quad (3)$$

Here  $\vec{j}_s = \vec{v} \vec{M}$  is from the translational motion of the classical vector  $\vec{m}$ , and  $\vec{j}_\omega = \vec{\omega} \times \vec{M}$  describes its rotational motion. Since  $\vec{v}$  and  $\vec{\omega}$  are called as the linear velocity and the angular velocity respectively, it is natural to name  $\vec{j}_s$  and  $\vec{j}_\omega$  as the linear and the angular current densities. Notice although the unit of  $\vec{j}_\omega$  is different from that of the linear spin current. However,  $\vec{j}_\omega$  is indeed the current in the angular space.

It is worth to mention the following two points. (1) If to consider that there are many particles in the volume element  $\Delta V$  (i.e., the volume element  $\Delta V$  is very small macroscopically but very large microscopically), the vector direction  $\vec{m}_i$ , the velocity  $\vec{v}_i$ , and the angular velocity  $\vec{\omega}_i$  for each particle may be different, however, the vector continuity

equation (3) is still valid:  $(d/dt)\vec{M} = -\nabla \cdot \vec{j}_s + \vec{j}_\omega$ , with  $\vec{j}_s(\mathbf{r}, t) \equiv \langle \vec{v} \vec{M} \rangle = \lim_{\Delta V \rightarrow 0} (\sum_i \vec{v}_i \vec{m}_i / \Delta V)$  and  $\vec{j}_\omega(\mathbf{r}, t) \equiv \langle \vec{\omega} \times \vec{M} \rangle = \lim_{\Delta V \rightarrow 0} (\sum_i \vec{\omega}_i \times \vec{m}_i / \Delta V)$ . Here  $\vec{j}_s(\mathbf{r}, t)$  and  $\vec{j}_\omega(\mathbf{r}, t)$  still describe the translational and the rotational motions of the classical vector. (2) If one adds an arbitrary curl  $\nabla \times \vec{A}_{\text{arb}}$  to the current  $\vec{j}_s = \vec{v} \vec{M}$ , the continuity equation does not change. Does this imply that the linear spin current density can be defined as  $\vec{j}_s = \vec{v} \vec{M} + (\nabla \times \vec{A}_{\text{arb}}) \vec{C}$  with a constant vector  $\vec{C}$ ? In our opinion, this does not because the local spin current density has physical meanings. This reason is completely same with the charge current density that cannot be redefined as  $\vec{j}_e = e\vec{v} + \nabla \times \vec{A}_{\text{arb}}$ .

In order to describe a scalar (e.g., charge) flow, one local current  $\vec{j}_e(\mathbf{r}, t)$  is sufficient. Why is it required to introduce two quantities instead of one to describe a vector flow? The reason is that the scalar quantity only has the translational motion, but the vector quantity has two kinds of motion, the translational and the rotational. So one has to use two quantities, the velocity  $\vec{v}$  and the angular velocity  $\vec{\omega}$ , to describe the motion of a single vector. Correspondingly, two quantities  $\vec{j}_s = \vec{v} \vec{M}$  and  $\vec{j}_\omega = \vec{\omega} \times \vec{M}$  are necessary to describe the vector flow.

In the steady state case, the scalar (e.g., charge) continuity equation reduces into  $\nabla \cdot \vec{j}_e = 0$ , so the scalar current  $\vec{j}_e$  is a conserved quantity. But for a vector flow, the linear vector current  $\vec{j}_s$  is not conserved since  $\nabla \cdot \vec{j}_s = \vec{j}_\omega$ . Whether it is possible to have a conserved vector current through redefinition? Of course, this redefined vector current should have a clear physical meaning and is measurable. In our opinion, this is almost impossible in the three-dimensional space. The reasons are as follows. (i) One cannot use a three-dimensional vector to combine both  $\vec{v}$  and  $\vec{\omega}$ . Therefore one can also not use a three-dimensional tensor to combine both  $\vec{j}_s = \vec{v} \vec{M}$  and  $\vec{j}_\omega = \vec{\omega} \times \vec{M}$ . (ii) Consider an example, as shown in Fig. 3(a), a one-dimensional classical vector flowing along the  $x$  axis. When  $x < 0$ , the vector's direction is in the  $+x$  axis. After  $0 < x < L$ , the vector rotates in accompany with its translational motion. When  $x > L$ , its direction is along the  $+y$  axis. Since for  $x < 0$  and  $x > L$  the vector has no rotational motion, the definition of the vector current is unambiguous, and the nonzero element is  $j_{xx}$  for  $x < 0$ , and  $j_{xy}$  for  $x > L$ . Therefore, the vector current is obviously different for  $x < 0$  and  $x > L$ , and the vector current is nonconservative.

In our opinion,  $\vec{j}_s = \vec{v} \vec{M}$  and  $\vec{j}_\omega = \vec{\omega} \times \vec{M}$  already have clear physical meanings. They also completely and sufficiently describe a vector flow, and they can determine any physical effects caused by the vector flow (see Secs. IV and V). One may not need to enforce a conserved current. In particular, as shown in the example of Fig. 3(a), sometimes it is impossible to introduce a conserved current.<sup>14</sup>

### III. THE FLOW OF A QUANTUM SPIN

Now we study the electronic spin  $\vec{s}$  in the quantum case. Consider an arbitrary wave function  $\Psi(\mathbf{r}, t)$ . The local spin density  $\vec{s}$  at the position  $\mathbf{r}$  and time  $t$  is  $\vec{s}(\mathbf{r}, t)$

$=\Psi^\dagger(\mathbf{r},t)\hat{s}\Psi(\mathbf{r},t)$ , where  $\hat{s}=(\hbar/2)\hat{\sigma}$  with  $\hat{\sigma}$  being the Pauli matrices. The time-derivative of  $\vec{s}(\mathbf{r},t)$  is

$$\frac{d}{dt}\vec{s}(\mathbf{r},t)=\frac{\hbar}{2}\left\{\left[\frac{d}{dt}\Psi^\dagger\right]\hat{\sigma}\Psi+\Psi^\dagger\hat{\sigma}\frac{d}{dt}\Psi\right\}. \quad (4)$$

From the Schrödinger equation, we have  $(d/dt)\Psi(\mathbf{r},t)=(1/i\hbar)H\Psi(\mathbf{r},t)$  and  $(d/dt)\Psi^\dagger(\mathbf{r},t)=(1/-i\hbar)[H\Psi(\mathbf{r},t)]^\dagger$ . Notice here the transposition in the symbol  $\dagger$  only acts on the spin indexes. By using the above two equations, Eq. (4) changes into

$$(d/dt)\vec{s}(\mathbf{r},t)=[\Psi^\dagger\hat{\sigma}H\Psi-(H\Psi)^\dagger\hat{\sigma}\Psi]/2i. \quad (5)$$

In the derivation below, we use the following Hamiltonian:

$$H=\frac{\vec{p}^2}{2m}+V(\mathbf{r})+\hat{\sigma}\cdot\vec{B}+\frac{\alpha}{\hbar}\hat{z}\cdot(\hat{\sigma}\times\vec{p}). \quad (6)$$

Note that our results are independent of this specific choice of the Hamiltonian. In Eq. (6) the first and second terms are the kinetic energy and potential energy. The third term is the Zeeman energy due to a magnetic field, and the last term is the Rashba spin-orbit coupling,<sup>15,16</sup> which has been extensively studied recently.<sup>4,7,8</sup> Next we substitute the Hamiltonian (6) into Eq. (5), and Eq. (5) reduces to

$$\begin{aligned} \frac{d}{dt}\vec{s} &= -\frac{\hbar}{2}\nabla\cdot\text{Re}\left\{\Psi^\dagger\left[\frac{\vec{p}}{m}+\frac{\alpha}{\hbar}(\hat{z}\times\hat{\sigma})\right]\hat{\sigma}\Psi\right\} \\ &+ \text{Re}\left\{\Psi^\dagger\left[\vec{B}+\frac{\alpha}{\hbar}\vec{p}\times\hat{z}\right]\times\hat{\sigma}\Psi\right\}. \end{aligned} \quad (7)$$

Introducing a tensor  $\mathbf{j}_s(\mathbf{r},t)$  and a vector  $\vec{j}_\omega(\mathbf{r},t)$ :

$$\mathbf{j}_s(\mathbf{r},t)=\text{Re}\left\{\Psi^\dagger\left[\frac{\vec{p}}{m}+\frac{\alpha}{\hbar}(\hat{z}\times\hat{\sigma})\right]\hat{s}\Psi\right\}, \quad (8)$$

$$\vec{j}_\omega(\mathbf{r},t)=\text{Re}\left\{\Psi^\dagger\frac{2}{\hbar}\left[\vec{B}+\frac{\alpha}{\hbar}(\vec{p}\times\hat{z})\right]\times\hat{s}\Psi\right\}, \quad (9)$$

then Eq. (7) reduces to

$$\frac{d}{dt}\vec{s}(\mathbf{r},t)=-\nabla\cdot\mathbf{j}_s(\mathbf{r},t)+\vec{j}_\omega(\mathbf{r},t), \quad (10)$$

or it can also be rewritten in the integral form

$$\frac{d}{dt}\int\int\int_V\vec{s}dV=-\oint_S d\vec{S}\cdot\mathbf{j}_s+\int\int\int_V\vec{j}_\omega dV. \quad (11)$$

Due to the fact that  $\hat{v}=(d/dt)\mathbf{r}=\vec{p}/m+(\alpha/\hbar)(\hat{z}\times\hat{\sigma})$  and  $(d/dt)\hat{\sigma}=(1/i\hbar)[\hat{\sigma},H]=(2/\hbar)[\vec{B}+(\alpha/\hbar)\vec{p}\times\hat{z}]\times\hat{\sigma}$ , Eqs. (8) and (9) become

$$\mathbf{j}_s(\mathbf{r},t)=\text{Re}\{\Psi^\dagger(\mathbf{r},t)\hat{v}\hat{s}\Psi(\mathbf{r},t)\}, \quad (12)$$

$$\vec{j}_\omega(\mathbf{r},t)=\text{Re}\{\Psi^\dagger(d\hat{s}/dt)\Psi\}=\text{Re}\{\Psi^\dagger\hat{\omega}\times\hat{s}\Psi\}, \quad (13)$$

where  $\hat{\omega}\equiv(2/\hbar)[\vec{B}+(\alpha/\hbar)(\vec{p}\times\hat{z})]$  is the angular velocity operator. We emphasize that those results, (10), (12), and (13),

are valid in general. They are independent of the special choice of Hamiltonian (6). For example, in the case with a vector potential  $\vec{A}$ , the general spin-orbit coupling  $\alpha\hat{\sigma}\cdot[\vec{p}\times\nabla V(\mathbf{r})]$ , and so on,<sup>17</sup> the results still hold. Notice that for the Hamiltonian (6), one has  $\text{Re}\{\Psi^\dagger\hat{v}_i\hat{s}_j\Psi\}=\text{Re}\{\Psi^\dagger\hat{s}_j\hat{v}_i\Psi\}$  and  $\text{Re}\{\Psi^\dagger\hat{\omega}\times\hat{s}\Psi\}=-\text{Re}\{\Psi^\dagger\hat{s}\times\hat{\omega}\Psi\}$ . If  $\text{Re}\{\Psi^\dagger\hat{v}_i\hat{s}_j\Psi\}\neq\text{Re}\{\Psi^\dagger\hat{s}_j\hat{v}_i\Psi\}$  and  $\text{Re}\{\Psi^\dagger\hat{\omega}\times\hat{s}\Psi\}\neq-\text{Re}\{\Psi^\dagger\hat{s}\times\hat{\omega}\Psi\}$  for certain Hamiltonians, Eqs. (12) and (13) will change to

$$\mathbf{j}_s(\mathbf{r},t)=\text{Re}\left\{\Psi^\dagger\frac{1}{2}[\hat{v}\hat{s}+(\hat{s}\hat{v})^T]\Psi\right\}, \quad (14)$$

$$\vec{j}_\omega(\mathbf{r},t)=\text{Re}\{\Psi^\dagger(d\hat{s}/dt)\Psi\}=\text{Re}\left\{\Psi^\dagger\frac{1}{2}[\hat{\omega}\times\hat{s}-\hat{s}\times\hat{\omega}]\Psi\right\}. \quad (15)$$

Equation (10) is the quantum spin continuity equation, which is the same with the classic vector continuity equation (3) although the derivation process is very different. In some previous works, this equation has also been obtained in the semiclassical case.<sup>11,12</sup> Here we emphasize that this spin continuity equation (10) is the consequence of invariance of the spin magnitude  $|\vec{s}|$ , i.e., when an electron makes a motion, either translation or rotation, its spin magnitude  $|\vec{s}|=\hbar/2$  remains a constant. Equation (10) should also be independent with the force (i.e., the potential) and the torque, as well the dynamic law. The two quantities  $\mathbf{j}_s(\mathbf{r},t)$  and  $\vec{j}_\omega(\mathbf{r},t)$  in Eq. (10), which are defined in Eqs. (12) and (13) respectively, describe the translational and rotational motion (precession) of a spin at the location  $\mathbf{r}$  and the time  $t$ . They will be named the linear and the angular spin current densities accordingly, similar as  $\vec{v}$  and  $\vec{\omega}$  are called the linear and the angular velocities. In fact,  $\vec{j}_\omega$  is called the spin torque in a recent work.<sup>11</sup> We consider both  $\vec{j}_\omega$  and  $\mathbf{j}_s$  describing the motion of a spin, capable of inducing an electric field (see Sec. IV), and so on. Namely, both of them play a parallel role in contributing to a physical quantity. Therefore, it is better to name  $\mathbf{j}_s$  and  $\vec{j}_\omega$  both as spin currents. Otherwise, when one calculates the contribution of spin current to a given physical quantity, one may forget to include the contribution by the angular spin current.

The linear spin current  $\mathbf{j}_s(\mathbf{r},t)$  is identical with the conventional spin current investigated in recent studies.<sup>4</sup> From  $\mathbf{j}_s(\mathbf{r},t)$ , the total linear spin current along  $i$  direction ( $i=x,y,z$ ) is  $\vec{I}_{si}(i,t)=\int\int\int dS\hat{i}\cdot\mathbf{j}_s(\mathbf{r},t)$ . To assume  $\vec{I}_{si}(i,t)$  independent on  $t$  and  $i$  (e.g., in the case of the steady state and without spin flip), one has  $\vec{I}_{si}=(1/L)\int\int\int dV\hat{i}\cdot\mathbf{j}_s(\mathbf{r},t)=(1/L)\int\int\int dV\text{Re}\Psi^\dagger\hat{v}_i\hat{s}\Psi=(1/L)\int\int\int dV\Psi^\dagger\frac{1}{2}(\hat{v}_i\hat{s}+\hat{s}\hat{v}_i)\Psi=\langle\frac{1}{2}(\hat{v}_i\hat{s}+\hat{s}\hat{v}_i)\rangle$ , where  $L$  is sample length in the  $i$  direction. This definition is the same as in recent publications.<sup>4</sup>

Next, we discuss certain properties of  $\mathbf{j}_s(\mathbf{r},t)$  and  $\vec{j}_\omega(\mathbf{r},t)$ . Notice that  $\vec{j}_\omega(\mathbf{r},t)$  which describe the rotational motion (precession) of the spin plays a parallel role in comparison with

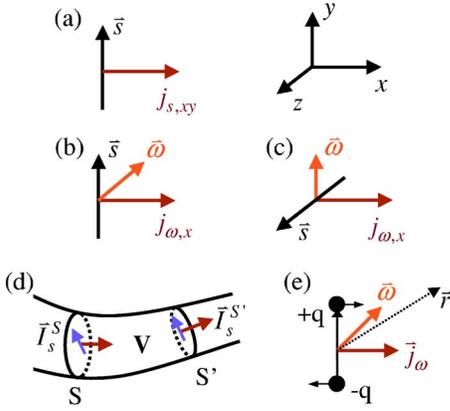


FIG. 2. (Color online) (a) The linear spin current element  $j_{s,xy}$ . (b) and (c) The angular spin current element  $j_{\omega,x}$ . (d) The spin current in a quasi 1D quantum wire. (e) The currents of two magnetic charges that are equivalent to an angular MM current.

the conventional linear spin current  $\mathbf{j}_s(\mathbf{r}, t)$  for the spin transport. (1) Similar to the classical case, it is necessary to introduce the two quantities  $\mathbf{j}_s(\mathbf{r}, t)$  and  $\vec{j}_\omega(\mathbf{r}, t)$  to completely describe the motion of a quantum spin. (2) The linear spin current is a tensor. Its element, e.g.,  $j_{s,xy}$ , represents an electron moving along the  $x$  direction with its spin in the  $y$  direction [see Fig. 2(a)]. The angular spin current  $\vec{j}_\omega$  is a vector. In Fig. 2(b), its element  $j_{\omega,x}$  describes the rotational motion of the spin in the  $y$  direction and the angular velocity  $\vec{\omega}$  in the  $-z$  direction. (3) From the linear spin current density  $\mathbf{j}_s(\mathbf{r}, t)$ , one can calculate (or say how much) the linear spin current  $\vec{I}_s$  flowing through a surface  $S$  [see Fig. 2(d)]:  $\vec{I}_s^S = \int \int_S d\vec{S} \cdot \mathbf{j}_s$ . However, the behavior for the angular spin current is different. From the density  $\vec{j}_\omega(\mathbf{r}, t)$ , it is meaningless to determine how much the angular spin current flowing through a surface  $S$  because  $\vec{j}_\omega$  is the current in the angular space. On the other hand, one can calculate the total angular spin current  $\vec{I}_\omega^V$  in a volume  $V$  from  $\vec{j}_\omega$ :  $\vec{I}_\omega^V = \int \int \int_V \vec{j}_\omega(\mathbf{r}, t) dV$ . (4) It is easy to prove that the spin currents in the present definitions of Eqs. (12) and (13) are invariant under a space coordinate transformation as well the gauge transformation. (5) If the system is in a steady state,  $\mathbf{j}_s$  and  $\vec{j}_\omega$  are independent of the time  $t$ , and  $(d/dt)\vec{s}(\mathbf{r}, t) = 0$ . Then the spin continuity equation (10) reduces to  $\nabla \cdot \mathbf{j}_s = \vec{j}_\omega$  or  $\oint_S d\vec{S} \cdot \mathbf{j}_s = \int \int \int_V \vec{j}_\omega dV$ . This means that the total linear spin current flowing out of a closed surface is equal to the total angular spin current enclosed. If to further consider a quasi-one-dimensional (1D) system [see Fig. 2(d)], then one has  $\vec{I}_s^S - \vec{I}_s^{S'} = \vec{I}_\omega^V$ . (6) The linear spin current density  $\mathbf{j}_s = \text{Re}\{\Psi^\dagger \hat{v} \hat{s} \Psi\}$  gives both the spin direction and the direction of spin movement, so it completely describes the translational motion. However, the angular spin current density,  $\vec{j}_\omega = \text{Re}\{\Psi^\dagger (\hat{d}\hat{s}/dt) \Psi\} = \text{Re}\{\Psi^\dagger \hat{\omega} \times \hat{s} \Psi\}$  involves the vector product of  $\hat{\omega} \times \hat{s}$ , not the tensor  $\hat{\omega} \hat{s}$ . Is it correct or sufficient to describe the rotational motion? For example, the rotational motion of Fig. 2(b) with the spin  $\vec{s}$  in the  $y$  direction and the angular velocity  $\vec{\omega}$  in the  $-z$  direction is different from the one in Fig. 2(c) in which  $\vec{s}$  is in the  $z$  direction and

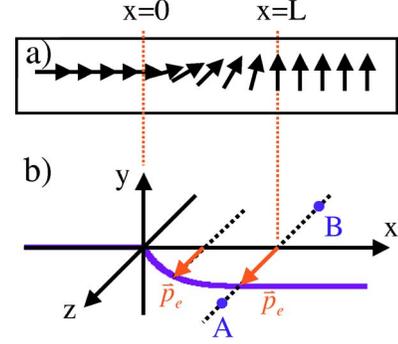


FIG. 3. (Color online) (a) Schematic diagram for the spin moving along the  $x$  axis, with the spin precession (rotational motion) in the  $x$ - $y$  plane while  $0 < x < L$ . (b) A 1D wire of electric dipole moment  $\vec{p}_e$ . This configuration will generate an electric field equivalent to the field from the spin currents in (a).

$\vec{\omega}$  is in the  $y$  direction, but their angular spin currents are completely the same. Shall we distinguish them? It turns out that the physical results produced by the above two rotational motions (Figs. 2(b) and 2(c)) are indeed the same. For instance, the induced electric field by them is identical since a spin  $\vec{s}$  has only the direction but no size (see detail discussion below). Thus, the vector  $\vec{j}_\omega$  is sufficient to describe the rotational motion, and no tensor is necessary.

Now we give an example of applying the above formulas, (12) and (13), to calculate the spin currents. Let us consider a quasi-1D quantum wire having the Rashba spin orbit coupling, and its Hamiltonian is

$$H = \frac{\vec{p}^2}{2m} + V(y, z) + \frac{\sigma_z}{2\hbar} [\alpha(x)p_x + p_x\alpha(x)] + \frac{\hbar^2 k_R^2}{2m}, \quad (16)$$

where  $k_R(x) \equiv \alpha(x)m/\hbar^2$ .  $\alpha(x) = 0$  for  $x < 0$  and  $x > L$ , and  $\alpha(x) \neq 0$  while  $0 < x < L$ . The other Rashba term  $-(\alpha/\hbar)\sigma_x p_z$  is neglected because the  $z$  direction is quantized.<sup>7</sup> Let  $\Psi$  be a stationary wave function

$$\Psi(\mathbf{r}) = \frac{\sqrt{2}}{2} e^{ikx} \begin{pmatrix} e^{-i\int_0^x k_R(x) dx} \\ e^{i\int_0^x k_R(x) dx} \end{pmatrix} \varphi(y, z), \quad (17)$$

where  $\varphi(y, z)$  is the bound state wave function in the confined  $y$  and  $z$  directions.  $\Psi(\mathbf{r})$  represents the spin motion as shown in Fig. 3(a), in which the spin moves along the  $x$  axis, as well the spin precession in the  $x$ - $y$  plane in the region  $0 < x < L$ .<sup>7,8</sup> Using Eqs. (12) and (13), the spin current densities of the wave function  $\Psi(\mathbf{r})$  are easily obtained. There are only two nonzero elements of  $\mathbf{j}_s(\mathbf{r})$ :

$$j_{sxx}(\mathbf{r}) = \frac{\hbar^2 k}{2m} |\varphi(y, z)|^2 \cos 2\phi(x), \quad (18)$$

$$j_{sxy}(\mathbf{r}) = \frac{\hbar^2 k}{2m} |\varphi(y, z)|^2 \sin 2\phi(x). \quad (19)$$

The nonzero elements of  $\vec{j}_\omega(\mathbf{r})$  are

$$j_{\omega x}(\mathbf{r}) = -\frac{\hbar^2 k k_R(x)}{m} |\varphi(y,z)|^2 \sin 2\phi(x), \quad (20)$$

$$j_{\omega y}(\mathbf{r}) = \frac{\hbar^2 k k_R}{m} |\varphi(y,z)|^2 \cos 2\phi(x), \quad (21)$$

where  $\phi(x) = \int_0^x k_R(x) dx$ . Those spin current densities confirm with the intuitive picture of an electron motion, precession in the  $x$ - $y$  plane in  $0 < x < L$  and movement in the  $x$  direction [see Fig. 3(a)].

In particular, in the region of  $x < 0$  and  $x > L$ ,  $\alpha(x) = k_R(x) = 0$ , and  $\hat{\sigma}$  is a good quantum number, hence,  $\vec{j}_\omega = 0$ . In this case, the definition of the spin current  $\mathbf{j}_s$  is unambiguous. However, the spin currents are different in  $x < 0$  and  $x > L$  except for  $\phi(L) = n\pi$  ( $n=0, \pm 1, \pm 2, \dots$ ). This is clearly seen from Fig. 3(a). Therefore, through this example, one can conclude that it is sometimes impossible to define a conserved spin current.<sup>14</sup> The example of Fig. 3(a) indeed exists and has been studied before.<sup>7,8</sup>

Above discussion shows that the linear spin current  $\mathbf{j}_s = \text{Re}\{\Psi^\dagger \hat{v} \hat{s} \Psi\}$  and the angular spin current  $\vec{j}_\omega = \text{Re}\{\Psi^\dagger \hat{\omega} \times \hat{s} \Psi\}$  have clear physical meanings, representing the translational motion and the rotational motion (precession) respectively. They completely describe the flow of a quantum spin. Any physical effects of the spin currents, such as the induced electric field, can be expressed by  $\mathbf{j}_s$  and  $\vec{j}_\omega$ .

#### IV. SPIN CURRENTS INDUCED ELECTRIC FIELDS

Recently, theoretic studies have suggested that the (linear) spin current can induce an electric field  $\vec{E}$ .<sup>18-20</sup> Can the angular spin current also induce an electric field? If so, this gives a way of detecting the angular spin current. Following, we study this question by using the method of equivalent magnetic charge.<sup>21</sup> Let us consider a steady-state angular spin current element  $\vec{j}_\omega dV$  at the origin. Associated with the spin  $\vec{s}$ , there is a magnetic moment (MM)  $\vec{m} = g\mu_B \vec{\sigma} = (2g\mu_B/\hbar)\vec{s}$  where  $\mu_B$  is the Bohr magneton. Thus, corresponding to  $\vec{j}_\omega$ , there is also a angular MM current  $\vec{j}_{m\omega} dV = (2g\mu_B/\hbar)\vec{j}_\omega dV$ . From above discussions, we already know that  $\vec{j}_{m\omega}$  (or  $\vec{j}_\omega$ ) comes from the rotational motion of a MM  $\vec{m}$  (or  $\vec{s}$ ) [see Figs. 2(b) and 2(c)], and  $\vec{j}_{m\omega} = \vec{\omega} \times \vec{m}$  (or  $\vec{j}_\omega = \vec{\omega} \times \vec{s}$ ). Under the method of equivalent magnetic charge, the MM  $\vec{m}$  is equivalent to two magnetic charges: one with magnetic charge  $+q$  located at  $\delta\hat{n}_m$  and the other with  $-q$  at  $-\delta\hat{n}_m$  [see Fig. 2(e)].  $\hat{n}_m$  is the unit vector of  $\vec{m}$  and  $\delta$  is a tiny length. The angular MM current  $\vec{j}_{m\omega}$  is equivalent to two magnetic charge currents: one is  $\vec{j}_{+q} = \hat{n}_j q \delta |\vec{\omega}| \sin \theta$  at the location  $\delta\hat{n}_m$ , the other is  $\vec{j}_{-q} = -\hat{n}_j q \delta |\vec{\omega}| \sin \theta$  at  $-\delta\hat{n}_m$  [see Fig. 2(e)], with  $\hat{n}_j$  being the unit vector of  $\vec{j}_{m\omega}$  and  $\theta$  the angle between  $\vec{\omega}$  and  $\vec{m}$ . In our previous work,<sup>20</sup> by using the relativistic theory, we have arrived at the formulae for the induced electric field by a magnetic charge current. The electric field induced by  $\vec{j}_{m\omega} dV$  can be calculated by adding the contributions from the two magnetic charge currents. Let

$\delta \rightarrow 0$ , and note that  $2q\delta \rightarrow |\vec{m}|$  and  $|\vec{\omega}||\vec{m}| \sin \theta = |\vec{j}_{m\omega}|$ , we obtain the electric field  $\vec{E}_\omega$  generated by an element of the angular spin current  $\vec{j}_\omega dV$ :

$$\vec{E}_\omega = \frac{-\mu_0}{4\pi} \int \frac{\vec{j}_{m\omega} dV \times \mathbf{r}}{r^3} = \frac{-\mu_0 g \mu_B}{h} \int \vec{j}_\omega dV \times \frac{\mathbf{r}}{r^3}. \quad (22)$$

We also rewrite the electric field  $\vec{E}_s$  generated by an element of the linear spin current using the tensor  $\mathbf{j}_s$ :<sup>20</sup>

$$\vec{E}_s = \frac{-\mu_0 g \mu_B}{h} \nabla \times \int \mathbf{j}_s dV \cdot \frac{\mathbf{r}}{r^3}. \quad (23)$$

Below we emphasize three points. (i) In the large  $r$  case, the electric field  $\vec{E}_\omega$  decays as  $1/r^2$ . Note that the field from a linear spin current  $\vec{E}_s$  goes as  $1/r^3$ . In fact, in terms of generating an electric field, the angular spin current is as effective as a magnetic charge current. (ii) In the steady-state case, the total electric field  $\vec{E}_T = \vec{E}_\omega + \vec{E}_s$  contains the property  $\oint_C \vec{E}_T \cdot d\vec{l} = 0$ , where  $C$  is an arbitrary close contour not passing through the region of spin current. However, for each  $\vec{E}_\omega$  or  $\vec{E}_s$ ,  $\oint_C \vec{E}_\omega \cdot d\vec{l}$  or  $\oint_C \vec{E}_s \cdot d\vec{l}$  can be nonzero. (iii) As mentioned above, an angular spin current  $\vec{j}_\omega$  may consist of different  $\vec{\omega}$  and  $\vec{s}$  [see Figs. 2(b) and 2(c)]. However, the resulting electric field only depends on  $\vec{j}_\omega = \vec{\omega} \times \vec{s}$ . This is because a spin vector contains only a direction and a magnitude, but not a spatial size (i.e., the distance  $\delta$  approaches to zero). In the limit  $\delta \rightarrow 0$ , both magnetic charge currents  $\vec{j}_{\pm q}$  reduce to  $\vec{\omega} \times \vec{m}/2$  at the origin. Therefore, the overall effect of the rotational motion is only related to  $\vec{\omega} \times \vec{m}$ , not separately on  $\vec{\omega}$  and  $\vec{m}$ . Hence it is enough to describe the spin rotational motion by using a vector  $\vec{\omega} \times \vec{s}$ , instead of a tensor  $\vec{\omega} \vec{s}$ .

Due to the fact that the direction of  $\vec{s}$  can change during the particle motion, the linear spin current density  $\mathbf{j}_s$  is not a conserved quantity. It is always interesting to uncover a conserved physical quantity from both theoretical and experimental points of view. Let us apply  $\nabla \cdot$  acting on two sides of Eq. (10), we have

$$\frac{d}{dt} (\nabla \cdot \vec{s}) + \nabla \cdot (\nabla' \cdot \mathbf{j}_s - \vec{j}_\omega) = 0, \quad (24)$$

where  $\nabla' \cdot \mathbf{j}_s$  means that  $\nabla'$  acts on the second index of  $\mathbf{j}_s$ , i.e.,  $(\nabla' \cdot \mathbf{j}_s)_i = \sum_j (d/dj) j_{s,ij}$  with  $i, j \in (x, y, z)$ . To introduce  $\vec{j}_{\nabla \cdot \vec{s}} = \nabla' \cdot \mathbf{j}_s - \vec{j}_\omega$ . Note it is different from  $\nabla \cdot \mathbf{j} - \vec{j}_\omega$ . In a steady state,  $\nabla \cdot \mathbf{j} - \vec{j}_\omega = 0$ , however  $\vec{j}_{\nabla \cdot \vec{s}} = \nabla' \cdot \mathbf{j} - \vec{j}_\omega$  is usually nonzero. By using  $\vec{j}_{\nabla \cdot \vec{s}}$ , the above equation reduces to

$$\frac{d}{dt} (\nabla \cdot \vec{s}) + \nabla \cdot \vec{j}_{\nabla \cdot \vec{s}} = 0. \quad (25)$$

This means that the current  $\vec{j}_{\nabla \cdot \vec{s}}$  of the spin divergence is a conserved quantity in the steady state case. In fact,  $-\nabla \cdot \vec{s}(\mathbf{r}, t)$  represents an equivalent magnetic charge, so  $\vec{j}_{\nabla \cdot \vec{s}}$  can also be named the magnetic charge current density.

Moreover, the total electric field produced by  $\mathbf{j}_s$  and  $\vec{j}_\omega$  can be rewritten as

$$\begin{aligned}\vec{E}_T &= \vec{E}_s + \vec{E}_\omega \\ &= \frac{\mu_0 g \mu_B}{h} \int (\nabla' \cdot \mathbf{j}_s - \vec{j}_\omega) dV \times \frac{\mathbf{r}}{r^3} \\ &= \frac{\mu_0 g \mu_B}{h} \int \vec{j}_{\nabla \cdot \vec{s}} dV \times \frac{\mathbf{r}}{r^3}.\end{aligned}\quad (26)$$

So the total electric field  $\vec{E}_T$  only depends on the current  $\vec{j}_{\nabla \cdot \vec{s}}$  of the spin divergence. Note that  $\vec{E}_T$  can be measured experimentally in principle. Through the measurement of  $\vec{E}_T(\mathbf{r})$ ,  $\vec{j}_{\nabla \cdot \vec{s}}$  can be uniquely obtained.

In the following, let us calculate the induced electric fields at the location  $\mathbf{r}=(x, y, z)$  by the spin currents in the example of Fig. 3(a). Substituting the spin currents of Eqs. (18)–(21) into Eqs. (22) and (23) and assuming the transverse sizes of the 1D wire are much smaller than  $\sqrt{y^2+z^2}$ , the induced fields  $\vec{E}_\omega$  and  $\vec{E}_s$  can be obtained straightforwardly. Then the total electric field  $\vec{E}_T = \vec{E}_\omega + \vec{E}_s$  is

$$\begin{aligned}\vec{E}_T &= a \frac{\hbar k}{m} \nabla \int \frac{z \sin 2\phi(x')}{[(x-x')^2 + y^2 + z^2]^{3/2}} dx' \\ &= a \nabla \int (\vec{\nabla} \times \vec{S}) \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dx',\end{aligned}\quad (27)$$

where  $\vec{\nabla} = (\hbar k/m, 0, 0)$ ,  $\vec{S} = (\cos 2\phi(x'), \sin 2\phi(x'), 0)$ ,  $\mathbf{r}' = (x', 0, 0)$ , the constant  $a = \mu_0 g \mu_B \rho_s / 4\pi$ , and  $\rho_s$  is the linear density of moving electrons under the bias of an external voltage. The total electric field  $\vec{E}_T$  represents the one generated by a 1D wire of electric dipole moment  $\vec{p}_e = (0, 0, c \sin 2\phi(x))$  at the  $x$  axis [see Fig. 3(b)], where  $c$  is a constant. It is obvious that  $\nabla \times \vec{E}_T = 0$ , i.e.,  $\oint_C \vec{E}_T \cdot d\vec{l} = 0$ . However, in general  $\oint_C \vec{E}_\omega \cdot d\vec{l}$  and  $\oint_C \vec{E}_s \cdot d\vec{l}$  are separately nonzero.

Finally, we estimate the magnitude of  $\vec{E}_T$ . We use parameters consistent with realistic experimental samples. Take the Rashba parameter  $\alpha = 3 \times 10^{-11}$  eV m (corresponding to  $k_R = 1/100$  nm for  $m = 0.036m_e$ ),  $\rho_s = 10^6/\text{m}$  (i.e., one moving electron per 1000 nm in length), and  $k = k_F = 10^8/\text{m}$ . The electric potential difference between the two points A and B [see Fig. 3(b)] is about  $0.01 \mu\text{V}$ , where the positions of A and B are  $(1/2k_R)(\pi/2, 0, 0.01)$  and  $(1/2k_R)(\pi/2, 0, -0.01)$ . This value of the potential is measurable with today's

technology.<sup>19</sup> Furthermore, with the above parameters the electric field  $\vec{E}_T$  at A or B is about 5 V/m which is rather large.

## V. HEAT PRODUCED BY SPIN CURRENTS

Here we consider another physical effect, the heat produced by the spin currents. Assume a uniform isotropic conductor having a linear spin current  $\mathbf{j}_s$  and a charge current  $\vec{j}_e$ , and considering the simple case that there exists no spin flip process (i.e.,  $\vec{s}$  is conserved) so that  $\vec{j}_\omega = 0$ . Then the produced Joule heat  $Q$  in unit volume and in unit time is

$$\begin{aligned}Q &= \sum_{i=x,y,z} \frac{2\rho}{4} [(|\vec{j}_{si}| + j_{ei})^2 + (|\vec{j}_{si}| - j_{ei})^2] \\ &= \rho \left( \sum_{ij} j_{s,ij}^2 + \sum_i j_{ei}^2 \right) \equiv \rho (\mathbf{j}_s^2 + \vec{j}_e^2),\end{aligned}$$

where  $\rho$  is the resistivity. The term  $\rho \vec{j}_e^2$  is the Joule heat from the charge current which is well known, and the other term  $\rho \mathbf{j}_s^2$  is the heat produced by the linear spin current. So the produced heat can indeed be expressed by  $\mathbf{j}_s$  (for the case of  $\vec{j}_\omega = 0$ ). Also note the produced heat  $Q$  depends on  $\mathbf{j}_s^2$ , whereas the induced electric field depends on  $\nabla' \cdot \mathbf{j}_s - \vec{j}_\omega$ .

## VI. CONCLUSION

In summary, we find that in order to completely describe the spin flow (including both classic and quantum flows), apart from the conventional spin current (or linear spin current), one has to introduce another quantity, the angular spin current. The angular spin current describes the rotational motion of the spin, and it plays a parallel role in comparison with the conventional linear spin current for the spin translational motion. In particular, we point out that the angular spin current (or the spin torque as being called in other works) can also induce an electric field. The formula for the generated electric field  $\vec{E}_\omega$  is derived and  $\vec{E}_\omega$  scales as  $1/r^2$  at large  $r$ . In addition, a conserved quantity, the current  $\vec{j}_{\nabla \cdot \vec{s}}$  of the spin divergence, is discovered, and the total electric field only depends on  $\vec{j}_{\nabla \cdot \vec{s}}$ .

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