Definitions of Equilibrium in Network Formation Games

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Abstract

We examine a variety of stability and equilibrium definitions that have been used to study the formation of social networks among a group of players. In particular we compare variations on three types of definitions: those based on a pairwise stability notion, those based on the Nash equilibria of a link formation game, and those based on equilibria of a link formation game where transfers are possible.

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1 Introduction

Following a long tradition in sociology, a literature has recently emerged in economics that addresses social and economic networks. One contribution of the economics literature has been the study of the endogenous formation of social networks by self-interested economic agents. Different models have been proposed to analyze the formation of bilateral links in small societies where agents are fully aware of the shape of the social network they belong to and the impact of the network on their well-being. Jackson and Wolinsky (1996) proposed a

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stability test for social networks – *pairwise stability* – which is a notion that applies directly to the network and players' payoffs from networks. In contrast, Myerson (1991) suggests (informally) a *noncooperative linking game* in which agents independently announce which bilateral links they would like to see formed and then standard game-theoretic equilibrium concepts can be used to make predictions about which networks will form. Recently, Bloch and Jackson (2005) proposed another linking game where players can offer or demand transfers along with the links they suggest, which allows players to subsidize the formation of particular links.

The fast-growing literature on strategic network formation (both in theory and in applications) has borrowed from these models, often defining new solution concepts which are refinements or variants on pairwise stability and/or equilibria of a linking game.[1] While every solution concept may have its own merit, and be based on an insightful intuition, the global picture is hard to decipher. Simple examples show that the networks which are deemed stable according to one solution concept are not selected by another, and the connections between various solution concepts are rarely explicitly stated and studied.

The objective of this paper is to clarify the relations between some of the definitions of equilibrium proposed in the literature on strategic network formation. We consider most (but not all) equilibrium definitions based on pairwise stability, the linking game, and the linking game with transfers. We draw general relations between solution concepts (when they exist) and provide simple counterexamples to show that solution concepts may be unrelated, selecting different equilibrium networks for the same profile of utility functions.

Our findings can be summarized as follows.

- The set of Nash equilibrium outcomes of the linking game as well as of the game with transfers can be completely disjoint from the set of pairwise stable networks. However, one useful refinement of both the set of pairwise stable networks and Nash equilibrium of the linking game (called pairwise Nash equilibrium[2]) is exactly the intersection of the set of Nash equilibrium outcomes of the linking game and the set of pairwise stable networks, when such networks exist.

- The networks that are supported by a Nash equilibrium in the linking game are a subset of networks which are supported in a Nash equilibrium of the game with transfers. However, this inclusion fails to hold once one considers the refinement of allowing pairs of players to coordinate on adding a link (pairwise Nash equilibrium), as the ability to make transfers simultaneously enlarges the set of networks which are achievable and the set of deviations that subcoalitions could propose.

- Finally, we make comparisons where we vary both the presence of transfers and the type of stability notion. That is, we relate equilibria of the linking game to pairwise

[1] We discuss the relevant literature below, as we introduce the different models and solution concepts.

[2] The terms pairwise Nash equilibrium refers to the strategies, and the term pairwise Nash stable refers to the resulting network.

stability with transfers; and we relate equilibria of the game with transfers to pairwise stability without transfers.

The rest of the paper is organized as follows. Section 2 contains basic notations on networks and utilities. Section 3 introduces the different models of network formation, and the various equilibrium definitions. Section 4 contains our discussion of the relation between the different models and equilibrium notions.

2 Modeling Networks

Players and Networks

$N = \{1, \ldots, n\}$ is the set of players who may be involved in a network relationship.[3]

A network g is a list of pairs of players who are linked to each other. For simplicity, we denote the link between i and j by ij, so $ij \in g$ indicates that i and j are linked in the network g. Let g^N be the set of all subsets of N of size 2. The network g^N is referred to as the complete network. The set $G = \{g \subset g^N\}$ denotes the set of all possible networks on N.

We let $g + ij$ denote the network found by adding the link ij to the network g, and $g - ij$ denote the network found by deleting the link ij from the network g.

Utility Functions

The *utility* of a network to player i is given by a function $u_i : G \to \mathbb{R}_+$.[4] Let u denote the vector of utility functions $u = (u_1, \ldots, u_n)$. We normalize payoffs so that $u_i(\emptyset) = 0$.

A utility function tells us what payoff accrues to any given player as a function of the network. This might include all sorts of costs, benefits, and externalities.

3 Network Formation Games

We now provide definitions of stability and equilibria emerging from three different models of network formation: the pairwise stability (and its variants) introduced by Jackson and Wolinsky (1996), the equilibria of the network formation game without transfers initially proposed by Myerson (1991), and the equilibria of the network formation game with transfers introduced by Bloch and Jackson (2005).

3.1 Pairwise stable networks

Jackson and Wolinsky (1996)'s pairwise stability is based on two considerations. A network is deemed to be stable if (i) no individual agent has an incentive to sever a link, and (ii) no

[3] For background and discussion of the model of networks discussed here, see Jackson (2004).

[4] In contrast with Jackson and Wolinsky (1996), we do not distinguish between a value function and an allocation rule. Instead, our primitive is the set of individual utility payoffs for every network.

pair of agents have an incentive to form a new link. Formally, these requirements are written as follows.

A network g is *pairwise stable* (PS) with respect to a profile of utility functions u if

(i) for all i and $ij \in g$, $u_i(g) \geq u_i(g - ij)$, and

(ii) for all $ij \notin g$, if $u_i(g + ij) > u_i(g)$ then $u_j(g + ij) < u_j(g)$.

As is clear from the definitions above, pairwise stability is not based on an explicit noncooperative game of network formation. Instead, it is a direct stability check which rules out networks which can intuitively be considered as unstable. The idea is that if some player could gain by deleting a link, or two players could gain from adding a link, then the network would not be stable. Pairwise stability has nice computational properties, and therefore is an easy concept to use in applications. Its main shortcoming is that it only considers very simple deviations, and may be too permissive in labeling a network as being stable. Nevertheless, it often produces sharp predictions.

In their discussion of pairwise stability, Jackson and Wolinsky (1996, Section 5) propose different directions in which stability can be strengthened. In particular, they allow for *transfers* among the players. We consider the variation on that concept which is defined as follows.[5]

A network g is *pairwise stable with transfers* (PS^t) with respect to a profile of functions u if

(i) $ij \in g \Rightarrow u_i(g) + u_j(g) \geq u_i(g - ij) + u_j(g - ij)$, and

(ii) $ij \notin g \Rightarrow u_i(g) + u_j(g) \geq u_i(g + ij) + u_j(g + ij)$.

3.2 A Linking Game

As an alternative pairwise stability, one may explicitly describe the process by which agents form bilateral links. Myerson (1991, p. 448) suggests a noncooperative game of network formation. For every player i, the strategy set is an $n-$tuple of 0 and 1, $S_i = \{0, 1\}^n$. Let s_{ij} denote the jth coordinate of s_i. If $s_{ij} = 1$, player i indicates her willingness to form a link with player j. By convention, we suppose that $s_{ii} = 0$. Given the strategy profile s, an undirected network $g(s)$ is formed by letting link ij form if and only if $s_{ij}s_{ji} = 1$. In other words, the formation of a link requires the consent of both players.

[5] Their definition is that a network g is pairwise stable with transfers if there exists no other network g' for which either $g' = g \backslash ij$ and $u_i(g') > u_i(g)$ or $u_j(g') > u_j(g)$ or $g' = g + ij$ and $u_i(g') + u_j(g') > u_i(g) + u_j(g)$. This definition treats transfers in deviations and transfers in the original network asymmetrically. (See footnote 17 in Jackson and Wolinsky (1996)). Players are allowed to make transfers to add new links but not to sustain existing links. We work with an alternative definition of pairwise stability with transfers, which treats transfers in deviations and in the original network symmetrically.

A strategy profile s is a *Nash equilibrium* of Myerson's game if and only if, for all player i, all strategies s_i' in $S_i, u_i(g(s)) \geq u_i(g(s_i', s_{-i}))$. A network g is *Nash stable* (NS) with respect to a profile of utility functions u if there exists a (pure strategy) Nash equilibrium s such that $g = g(s)$.

It is easy to see that the concept of Nash stability is too weak as a concept for modeling network formation when links are bilateral, as it allows for too many equilibrium networks. For instance, the empty network is always a Nash network, regardless of the payoff structure. We need to allow agents to coordinate on their decision to form new links in order to refine the set of equilibrium networks. In line with pairwise stability, this can be done by considering the following definition.

A strategy profile s is a *pairwise Nash equilibrium* of the linking game if and only if, for all player i, all strategies s_i' in $S_i, u_i(g(s)) \geq u_i(g(s_i', s_{-i}))$ and there does not exist a pair of agents (i, j) such that $u_i(g(s) + ij) \geq u_i(g(s)), u_j(g(s) + ij) \geq u_j(g(s))$ with strict inequality for one of the two agents. A network g is *pairwise Nash stable* (PNS) with respect to a profile of utility functions u if there exists a pairwise Nash equilibrium s of the linking game such that $g = g(s)$.

Pairwise Nash stability is a refinement of both pairwise stability and Nash stability, where one requires that a network be immune to the formation of a new link by any two agents, and the deletion of *any number of links* by any individual agent. This was one of the refinements mentioned by Jackson and Wolinsky (1996, Section 5), and it has been used in applications (Goyal and Joshi (2003), Belleflamme and Bloch (2004)), before being formally studied by Calvo-Armengol and Ilkilic (2004), Ilkilic (2004) and Gilles and Sarangi (2004).[6,7]

3.3 A Linking Game with Transfers

Bloch and Jackson (2005) propose several network formation games where transfers are negotiated at the time of the formation of the network. We focus on the simplest of those that they define, which we simply call the linking game with transfers.

In the linking game with transfers, every player $i \in N$ announces a vector of transfers $t^i \in I\!R^{n-1}$. We denote the entries in this vector by t^i_{ij}, representing the transfer that player

[6] Calvo-Armengol and Ilkilic (2004) provide a necessary and sufficient condition on utilities for pairwise Nash stability and pairwise stability to coincide (see also Gilles and Sarangi (2005)). Ilkilic (2004) studies the existence of pairwise stable networks for a class of utility functions. Gilles and Sarangi (2004) also define pairwise Nash stable networks that they term "strongly pairwise stable."

[7] An anonymous referee suggests a useful taxonomy for classifying network solutions (which we mention here with a changed terminology, but retained spirit). First, there are notions, like pairwise stability, that work directly with the networks themselves and see if some sets of players could gain from some change in the links that they are involved with. We might call these "network-based" stability concepts. Next, there are notions, like Nash stability, such that one instead considers a game where networks result as the outcome of explicitly modeled strategic interaction of the players and then uses an equilibrium concept to solve the game and predict networks. One might called this "game-based" stability concepts. Finally, there are "hybrid" stability concepts, like Pairwise Nash stability, for which these approaches are combined.

i proposes on link ij. Announcements are simultaneous.[8]

Link ij is formed if and only if $t^i_{ij}+t^j_{ij} \geq 0$. Formally, the network that forms as a function of the profile of announced vectors of transfers $t = (t^1, \ldots, t^n)$ is

$$g(t) = \{ij \mid t^i_{ij} + t^j_{ij} \geq 0\}$$

In this game, player i's payoff is given by

$$\pi_i(t) = u_i(g(t)) - \sum_{ij \in g(t)} t^i_{ij}.$$

The linking game with transfers is easily interpreted. Players simultaneously announce a transfer for each possible link that they might form. If the transfer is positive, it represents the offer that the player makes to form the link. If the transfer is negative, it represents the demand that a player requests to form the link. Note that the offer may exceed the demand, $t^i_{ij} + t^j_{ij} > 0$. In that case, we hold both players to their promises. If for instance $t^i_{ij} > -t^j_{ij} > 0$, player i ends up making a bigger payment than player j demanded. Player j only gets his demand, and the excess payment is wasted.[9]

Note that wasted transfers never occur in equilibrium. Alternative specifications of the game (for instance, letting player i only pay player j's demand or player j receive the total offer of player i) would not change the structure of the equilibria.

A vector t forms *a Nash equilibrium* of the linking game with transfers if

$$\pi(t) \geq \pi(t_{-i}, \widehat{t^i}),$$

for all i and $\widehat{t^i}$. A network g is *Nash stable* $(NS^t)$with respect to a profile of utility functions u if there exists a Nash equilibrium t such that $g = g(t)$.

[8] For variations on games where there are demands for payoffs and the are made sequentially, see Aumann and Myerson (1988), Currarini and Morelli (2000), and Mutuswami and Winter (2002). Other uses of bargaining in network formation appear in Slikker and van den Nouweland (2001) and Matsubayashi and Yamakawa (2004).

[9] Bloch and Jackson (2005) also define version of the transfer game where players can also make indirect transfers (subsidize the formation of links they are not directly involved in), and also where they can make the transfers contingent on which network is formed. In the indirect transfer game, every player i announces a vector of transfers $t^i \in I\!R^{n(n-1)/2}$. The entries in the vector t^i are given by t^i_{jk}, denoting the transfer that player i puts on the link jk. If $i \notin jk$, $t^i_{jk} \geq 0$. Player i can make demands on the links that he or she involved with (it is permissible to have $t^i_{ij} < 0$), but can only make offers on the other links. Link jk is formed if and only if $\sum_{i \in N} t^i_{jk} \geq 0$. In the contingent game, every player announces a vector of contingent transfers $t^i(g)$ contingent on g forming, for each conceivable nonempty $g \in G$. In the direct transfer game, $t^i(g) \in I\!R^{n-1}$ for each i, while in the indirect transfer game, $t^i(g) \in I\!R^{n(n-1)/2}$. With contingent transfers, there are many possible ways to determine which network forms given a set of contingent announcements. See Bloch and Jackson (2005) for a specification.

The definition of pairwise Nash equilibrium can be transposed to the linking game with transfers by allowing pairs of players who are not linked to modify jointly their transfers in order to form a link.

Given t, let t_{-ij} indicate the vector of transfers found simply by deleting t^i_{ij} and t^j_{ij}.

A vector t is a *pairwise Nash equilibrium with transfers* if it is an equilibrium of the linking game with transfers, and there does not exist any $ij \notin g(t)$, and $\widehat{t}$ such that

(1) $\pi_i(t_{-ij}, \widehat{t}^i_{ij}, \widehat{t}^j_{ji}) \geq \pi_i(t)$,

(2) $\pi_j(t_{-ij}, \widehat{t}^i_{ij}, \widehat{t}^j_{ji}) \geq \pi_j(t)$, and

(3) at least one of (1) or (2) holds strictly.[10]

A network g is *pairwise Nash stable with transfers* (PNS^t) with respect to a profile of utility functions u if there exists a pairwise Nash equilibrium t such that $g = g(t)$.

4 A Comparison of Stability Concepts

We now discuss the relation between the different stability definitions. Given that the definitions belong to three broad classes of models (pairwise stability, equilibria of the linking game, and equilibria of the linking game with transfers), we organize our discussion around three questions: (1) What are the connections between pairwise stable networks and the networks supported in the linking game? (2) What are the connections between pairwise stable networks with transfers and the networks supported in the linking game with transfers? (3) What are the connections between the equilibria supported in the linking game, and the equilibria supported in games with transfers?

4.1 Comparing Stability and Equilibrium Notions without Transfers

The connection between pairwise stable networks and equilibrium networks of the linking game is straightforward and summarized in the following Remark.[11]

Remark 1 *For any N and profile of utility functions u,*

$$PNS(u) = PS(u) \cap NS(u).$$

[10] Given the continuity of transfers, this is easily seen to be equivalent to requiring that both (1) and (2) hold strictly.

[11] Jackson and van den Nouweland (2005), Chakrabarti (2005), Slikker and van den Nouweland (2005), Gilles and Sarangi (2004, 2005) and Gilles, Chakrabarti, and Sarangi (2005) all discuss the relation between different equilibrium notions in games without transfers.

There exist profiles of utility functions u for which $NS(u) \cap PS(u) = \emptyset$, even though $NS(u) \neq \emptyset$ and $PS(u) \neq \emptyset$.[12]

The characterization of Pairwise Nash Equilibria as the intersection of pairwise stable and Nash equilibria of the linking game is an immediate consequence of the definition. The second statement in the remark may be more surprising, and is based on the following example:

EXAMPLE **1** *Pairwise stable and Nash stable networks can be disjoint*

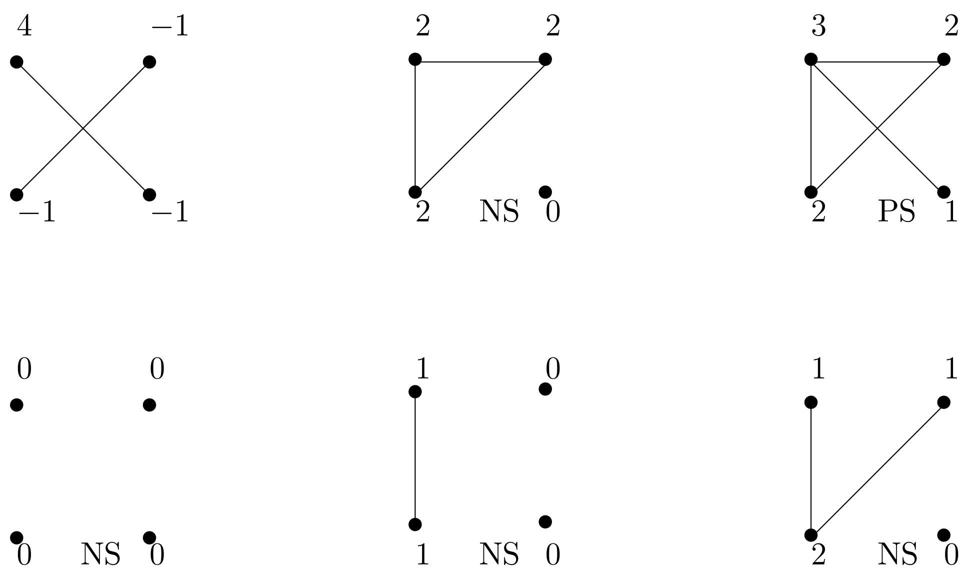

Figure 2

All other networks generate a value of zero to disconnected players and a value of minus the number of links that a player has to a player who has at least one link. The Nash stable and pairwise stables are labeled, and the non-pictured networks are neither Nash nor pairwise stable.

Finally, before considering transfers, we note that while Nash stable networks always exist (e.g., the empty network is always Nash stable), pairwise stable networks and hence pairwise Nash equilibria can fail to exist. Examples of nonexistence and sufficient conditions for existence of pairwise stable networks appear in Jackson and Watts (2001). Sufficient conditions for existence of pairwise Nash equilibria appear in Chakrabarti and Gilles (2005).

[12] One can also consider stronger refinements where coalitions of up to some size are able to jointly coordinate on the addition of links between members of the coalition, and deletion of any links where at least one of them is involved. These refine the set of Pairwise Nash stable networks, and can strictly do so. For various definitions in this direction, see Dutta and Mutuswami (1998), Jackson and van den Nouweland (2005), Chakrabarti (2005), and Slikker and van den Nouweland (2005).

4.2 Comparing Stability and Equilibrium Notions, with Transfers

Our next result shows that the notion of pairwise stability with transfers captures some of the spirit of the equilibria of the linking game with transfers. In fact, the statements of Proposition 1 have an exact equivalent for the transfer game.

PROPOSITION **1** *In the linking game with transfers, for any utility functions u,*

$$PNS^t(u) = PS^t(u) \ \cap NS^t(u).$$

There exist profiles of utility functions u for which $NS^t(u) \cap PS^t(u) = \emptyset$, even though $NS^t(u) \neq \emptyset$ and $PS^t(u) \neq \emptyset$.

Proof of Proposition 1: We first show that any pairwise Nash stable network with transfers is also pairwise stable with transfers. Consider a pairwise Nash equilibrium with transfers $\widehat{t}$. For any link $ij \in g$, player i prefers to announce $\widehat{t^i_{ij}}$ than any transfer X such that $X + \widehat{t^j_{ij}} < 0$. Hence, $u_i(g) - \widehat{t^i_{ij}} \geq u_i(g - ij)$. Similarly, $u_j(g) - \widehat{t^j_{ij}} \geq u_i(g - ij)$. Summing up the two inequalities, $u_i(g) + u_j(g) - (\widehat{t^i_{ij}} + \widehat{t^j_{ij}}) \geq u_i(g - ij) + u_j(g - ij)$ and as $(\widehat{t^i_{ij}} + \widehat{t^j_{ij}}) \geq 0$, $u_i(g) + u_j(g) \geq u_i(g - ij) + u_j(g - ij)$. Conversely, suppose that $ij \notin g$. If $u_i(g) + u_j(g) > u_i(g - ij) + u_j(g - ij)$, define a new transfer vector $\widetilde{t}$ where $\widetilde{t}^h_{kl} = \widehat{t^h_{kl}}$ for all $kl \neq ij$ and $\widetilde{t}^i_{ij} = u_i(g) - u_i(g - ij) - \varepsilon, \widetilde{t}^j_{ij} = u_j(g) - u_j(g - ij) - \varepsilon$ where ε is chosen so that $\widetilde{t}^i_{ij} + \widetilde{t}^j_{ij} \geq 0$. It follows that $u_i(g(\widetilde{t})) - \sum_{k, ik \in g(\widetilde{t})} \widetilde{t}^i_{ik} = u_i(g - ij) - \sum_{k \neq j, ik \in g(\widehat{t})} \widehat{t^i_{ik}} + \varepsilon > u_i(g(\widehat{t})) - \sum_{k, ik \in g(\widehat{t})} \widehat{t^i_{ik}}$ and similarly, $u_j(g(\widetilde{t})) - \sum_{k, jk \in g(\widetilde{t})} \widetilde{t}^j_{jk} > u_j(g(\widehat{t})) - \sum_{k, jk \in g(\widehat{t})} \widehat{t^j_{jk}}$, contradicting the definition of pairwise Nash equilibrium.

Finally, let us argue that any network g that is Nash stable with transfers and is also pairwise stable with transfers is supportable as a pairwise Nash equilibrium outcome with transfers. Consider an equilibrium $\widehat{t}$ that supports g. We argue that $\widehat{t}$ must also be a pairwise Nash equilibrium with transfers. Suppose to the contrary that there exists some $ij \notin g$ such that

$$u_i(g + ij) - \sum_{ik \in g} t^i_{ik} - \widehat{t}^i_{ij} \geq u_i(g) - \sum_{ik \in g} t^i_{ik}$$

and

$$u_j(g + ij) - \sum_{jk \in g} t^j_{jk} - \widehat{t}^j_{ij} \geq u_j(g) - \sum_{jk \in g} t^j_{jk},$$

with one inequality holding strictly, and where $\widehat{t}^i_{ij} + \widehat{t}^j_{ij} \geq 0$ (as otherwise the link ij does not form and the payoffs could not have changed). Thus,

$$u_i(g + ij) - \widehat{t}^i_{ij} + u_j(g + ij) - \widehat{t}^j_{ij} > u_i(g) + u_j(g).$$

Since $\widehat{t}^i_{ij} + \widehat{t}^j_{ij} \geq 0$ it follows that

$$u_i(g + ij) + u_j(g + ij) > u_i(g) + u_j(g),$$

which contradicts the fact that g is pairwise stable with transfers. ▮

Example 1 can again be used to show that the sets of Nash networks and pairwise stable networks with transfers may be disjoint. However, change the payoff of 4 to a payoff of 10 (to ensure that the unique network that is Pairwise stable with transfers is not Nash stable with transfers). With that change, the Pairwise stable networks and Pairwise stable networks with transfers are the same; and the Nash stable networks and the Nash stable networks with transfers are the same.

4.3 Comparing Stability with and without Transfers

We now see the role of transfers, by comparing solutions based on notions without transfers to solutions when transfers are considered. This discussion highlights the role played by transfers in the formation of social networks. Not surprisingly, our results show that any network which is a Nash outcome of the linking game is also a Nash outcome of the game with transfers. However, the set of pairwise Nash equilibrium outcomes in the linking game and transfer game may be disjoint, as the ability to make transfers simultaneously enlarges the set of networks which are achievable by individual agents, and the deviations by pairs of players.

PROPOSITION **2** *For any profile of utility functions u, $NS(u) \subseteq NS^t(u)$. There exist profiles of utility functions u for which $NS(u) \cap PNS^t(u) = \emptyset$.*

Proof of Proposition 2: To prove the first statement, consider a Nash equilibrium s supporting a network g. Define strategies in the transfer game by $t^i_{ij} = t^j_{ij} = 0$ if $s_{ij}s_{ji} = 1$ and $t^i_{ij} = t^j_{ij} = X$ where X is a negative number such that $X + u_i(g) < 0$ for all i and all g otherwise. Clearly, no player has an incentive to choose $t^j_{ij} \neq X$ when the other player chooses $t^j_{ij} = X$. Furthermore, no player has an incentive to increase her transfer above 0 in the case where $t^i_{ij} = t^j_{ij} = 0$. Choosing a transfer $t^i_{ij} < 0$ when the other player chooses $t^j_{ij} = 0$ results in the link not being formed, resulting in a utility $u_i(g - ij)$. However, because s is a Nash equilibrium profile of the linking game, $u_i(g - ij) \leq u_i(g)$, and this deviation cannot be profitable. Hence, t is a Nash equilibrium of the game with transfers.

To prove the second statement, consider the following Example.

EXAMPLE **2** *Nash stable and Pairwise Nash Stable with transfers may have an empty intersection.*

Figure 3

In this two-player example, the only Nash stable equilibrium of the linking game is the empty network. However the one-link network is pairwise Nash stable with transfers $t^1_{12} = -1.5, t^{12}_2 = 1.5$. Furthermore, it is the only pairwise Nash stable network in the transfer game. The empty network is not pairwise Nash stable with transfers, as the two players have an incentive to jointly deviate to the transfers $t^1_{12} = -1.5, t^{12}_2 = 1.5$. ∎

Some remarks are in order. First, Example 2 also shows that the inclusion between NS and NS^t can be strict. There are networks which are achievable with transfers but not without transfers. Second, in Example 2 , the empty network is also a Nash pariwise network in the linking game. Hence, this example shows that the sets PNS, PNS^t may have an empty intersection.

We now provide further comparisons of stability notions when transfers are and are not considered, now relating the set of pairwise stable networks to the equilibrium networks of the game with transfers, and the set of pairwise stable networks with transfers to the equilibrium networks of the linking game. We consider first the basic results of Remark 1 and Proposition 1 relating the intersection of Nash stable networks and pairwise stable networks to pairwise Nash stable networks. We show that we can compare these sets in three cases, and that the sets are incomparable in the last case. We also investigate the relation between pairwise stability and Nash stability, and show that these sets may be disjoint.

PROPOSITION **3** *For all profiles of utility functions* u,

(i) $NS^t(u) \cap PS(u) \supseteq PNS(u)$

(ii) $NS(u) \cap PS^t(u) \subseteq PNS(u)$

(iii) $NS(u) \cap PS^t(u) \subseteq PNS^t(u)$.

For each of the following pairs of sets, there exist profiles of utility function u *for which the sets are disjoint even though neither is empty:*

(iv) $PS(u)$ *and* $NS^t(u)$

(v) $PS^t(u)$ *and* $NS(u)$

(vi) $NS^t(u) \cap PS(u)$ *and* $PNS^t(u)$

Proof of Proposition 3 Statement (i) stems from the fact that $PNS(u) = PS(u) \cap NS(u) \subseteq PS(u) \cap NS^t(u)$.To prove statement (ii) consider a network which is supported by a Nash equilibrium of the linking game, and is pairwise stable with transfers. Suppose by contradiction that it is not pairwise Nash stable. there exists then a pair of agents ij for whom $u_i(g+ij) > u_j(g+ij)$ and $u_j(g+ij) \geq u_j(g)$. Summing up, we obtain $u_i(g+ij)+u_j(g+ij) > u_i(g) + u_j(g)$, contradicting the fact that g is pairwise stable with transfers. Statement (iii) comes from the fact that $PNS^t(u) = NS^t(u) \cap PS^t(u) \supseteq NS(u) \cap PS^t(u)$. Statements (iv) and (v) are based on Example 1 (where the payoff of 4 is changed to 10), as there PS and PS^t coincide, as do NS and NS^t. Statement (vi) derives from Example 2. In that Example, the empty network is the only pairwise stable network (and is also Nash stable hence Nash stable with transfers), and the one-link network is the only pairwise Nash stable network with transfers. ▮

The inclusions of Proposition 3 may be strict. In Example 2, the set $NS(u) \cap PS^t(u)$ is empty. However, the empty network belongs to the set $PNS(u)$ and the one-link network to the set $PNS^t(u)$. The next example shows that there are networks in $NS^t(u) \cap PS(u)$ which do not belong to $PNS(u)$.

Example **3** *Network supportable in the transfer game and pairwise stable, but not pairwise Nash stable.*

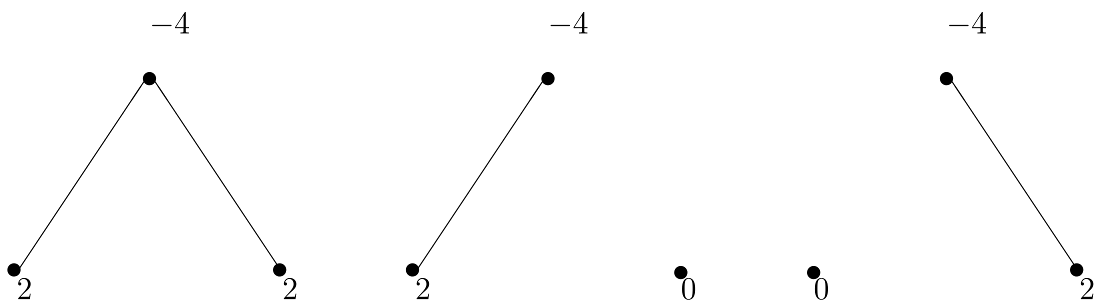

Figure 4

All other networks generate a value of zero. In this example, the two-link network is Nash stable with transfers (for example with transfers $t_2^{12} = t_3^{13} = +1.5$ and $t_1^{12} = t_1^{13} = -1.5$) but not Nash stable. It is also pairwise stable (player 1 has no incentive to cut a single link). However, this network is not pairwise Nash stable as player 1 has an incentive to cut both links at once when transfers are not allowed.

5 Concluding Remarks

We have analyzed the relationships between some of the stability concepts for modeling network formation. The differences between the emerging wide variety of equilibrium definitions can make it difficult to know which one is most compelling and when. More work is needed (especially in applications) to identify further properties of the solution concepts and match them up with the environments to which they are each best suited.

One of the main findings here (e.g., Proposition 3) is that the introduction of transfers can lead to differences in the set of stable networks from any stability notions that do not incorporate transfers. That is, the availability of transfers neither refines nor enlarges the set of stable networks, but changes them in noncomparable ways. This stems from the fact that while transfers enhance the ability to support certain networks, they also enhance the deviation possibilities relative to others. Given that that in many, if not most, economic settings some sort of transfer is possible (which might be as simple as an implicit agreement of how the cost of a link is to be split), it is important for us to have a deeper understanding of when and how transfers matter.[13]

6 References

Aumann, R. and Myerson, R. (1988) "Endogenous Formation of Links Between Players and Coalitions: An Application of the Shapley Value," In: Roth, A. (ed.) *The Shapley Value*, Cambridge University Press, 175–191.

Belleflamme, P. and Bloch, F. (2004) "Market Sharing Agreements and Stable Collusive Networks", *International Economic Review*, 45, 387-411.

Bloch, F. and M.O. Jackson (2004) "The Formation of Networks with Transfers among Players," forthcoming, *Journal of Economic Theory*.

Calvó-Armengol, A. and R. Ilkilic (2004) "Pairwise Stability and Nash Equilibria in Network Formation," mimeo: Universitat Autonoma de Barcelona.

Chakrabarti, S. (2005) "Stable and efficient networks under constrained coalition formation," presentation at SAET, Vigo 2005.

Chakrabarti, S. and R.P. Gilles (2005) " Network Potentials," mimeo: Queens University (Belfast) and Virginia Tech.

Currarini, S. and M. Morelli (2000) "Network Formation with Sequential Demands," *Review of Economic Design*, 5, 229–250.

Gilles. R.P. and S. Sarangi (2004) "The Role of Trust in Costly Network Formation" mimeo: Virginia Tech.

Gilles. R.P. and S. Sarangi (2005) "Stable Networks and Convex Payoffs," mimeo: Virginia Tech.

Gilles. R.P., Chakrabarti, S. and S. Sarangi (2005) "Social Network Formation with Consent: Nash Equilibria and Pairwise Refinements," mimeo: Virginia Tech.

Goyal, S. and S. Joshi (2003) "Networks of Collaboration in Oligopoly," *Games and Economic Behavior*, 43, 57-85.

[13] Bloch and Jackson (2005) examine when transfers can help sustain efficient networks in equilibrium, but understanding the role of transfers more generally is an important area for further research.

Ilkilic, R. (2004) "Pairwise Stability: Externalities and Existence," mimeo: Universitat Autonoma de Barcelona.

Jackson, M.O. (2004). "A Survey of Models of Network Formation: Stability and Efficiency," in *Group Formation in Economics: Networks, Clubs, and Coalitions*, edited by G. Demange and M. Wooders, Cambridge University Press: Cambridge.

Jackson, M.O. and van den Nouweland, A. (2005) "Strongly Stable Networks," *Games and Economic Behavior*, 51: 420-444.

Jackson, M.O. and Watts, A. (2001) "The Existence of Pairwise Stable Networks," *Seoul Journal of Economics*, vol. 14, no. 3, pp 299-321.

Jackson, M.O. and A. Wolinsky (1996) "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory*, 71, 44–74.

Matsubayashi, N. and S. Yamakawa (2004) "A Network Formation Game with an Endogenous Cost Allocation Rule," mimeo: NTT Communications.

Mutuswami, S. and E. Winter (2002) "Subscription Mechanisms for Network Formation," *Journal of Economic Theory*, 106, 242-264.

Myerson, R. (1991) *Game Theory: Analysis of Conflict*, Harvard University Press: Cambridge, MA.

Slikker, M. and A. van den Nouweland (2001) "A One-Stage Model of Link Formation and Payoff Division," *Games and Economic Behavior*, 34, 153-175.

Slikker, M. and A. van den Nouweland (2005) "Pair Stable Networks," presentation at SAET Vigo.