Math. Ann. 310, 395–409 (1998)

Mathematische
Annalen
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Deformation of compact Clifford-Klein forms of indefinite-Riemannian homogeneous manifolds

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Received: 30 September 1996 / Revised version: 20 January 1997

Mathematics Subject Classification (1991): 22E46; 32G07; 53C50; 57S30; 58H15

1 Introduction and formulation of local rigidity

1.1. Let G be a Lie group, and H a closed subgroup of G. If a discrete subgroup Γ of G acts properly discontinuously and freely on G/H, then the double coset space $\Gamma\backslash G/H$ carries naturally a manifold structure such that the quotient map $G/H \to \Gamma\backslash G/H$ is locally diffeomorphic. The manifold $\Gamma\backslash G/H$ is said to be a *Clifford-Klein form of* G/H. If it is compact, then Γ is said to be a *uniform lattice for* G/H. A typical example is a compact Riemann surface M_g with genus $g \geq 2$, which is biholomorphic to a compact Clifford-Klein form of the Poincaré disk $G/H \simeq SL(2,\mathbb{R})/SO(2)$ by the uniformization theorem. It is important from geometric view point that a Clifford-Klein form $\Gamma\backslash G/H$ inherits any G-invariant local geometric structure on G/H such as (indefinite)-Riemannian metric, complex structure, symplectic structure, causal structure and so on.

1.2. Our interest is in the indefinite-Riemannian Clifford-Klein forms. But, we start with a brief review of classical results on Riemannian Clifford-Klein forms.

Let G be a real reductive linear Lie group and H a maximal compact subgroup of G. The homogeneous manifold G/H carries a G-invariant Riemannian metric and is called a *Riemannian symmetric space*. Then, G/H always admits a compact Clifford-Klein form by a theorem of Borel, Harish-Chandra, Mostow, and Tamagawa ([4, 5, 27]). The local rigidity theorem for Riemannian symmetric spaces due to Selberg and Weil ([30, 34]), later extended by Mostow, Margulis and some others, asserts that a compact Clifford-Klein form $\Gamma\backslash G/H$ is locally rigid (see Sect. 1.5 for definition) except for the Poincaré disk in the irreducible case. In other words, non-trivial deformation of $\Gamma\backslash G/H$ exists in this case only if $\dim G/H = 2$, namely, only if $\Gamma\backslash G/H \simeq M_g$ ($g \geq 2$). The study of the corresponding deformation theory is nothing but the *Teichmüller theory*.

1.3. More generally, let G be a real reductive linear Lie group, and H a subgroup that is reductive in G. We say G/H is a *homogeneous manifold of reductive type*. Semisimple symmetric spaces such as $SL(n,\mathbb{R})/SO(p,n-p)$ are typical examples (see [3, 7] and references therein).

If G/H is of reductive type, then there exists a natural G-invariant indefinite-Riemannian metric on G/H. On the other hand, G/H does not always admit compact Clifford-Klein forms. In fact, it can happen that only finite discrete subgroups of G can act properly discontinuously on G/H. This is so called the *Calabi-Markus phenomenon* named after their first discovery in the Lorentzian manifold $SO(n,1)/SO(n-1,1)$ ([6, 13, 21, 35]). The existence problem of compact Clifford-Klein forms of indefinite-Riemannian homogeneous manifolds has been actively studied in the last decade by various methods, such as the criterion of the Calabi-Markus phenomenon, characteristic classes, cohomology of discrete groups, symplectic geometry, ergodic actions, decay of matrix coefficients, and so on (cf. [1, 2, 13–15, 18, 19, 21, 23, 32, 36]). But the classification of homogeneous manifolds having compact Clifford-Klein forms is still unsolved even for semisimple symmetric spaces (we recall that the classification of semisimple symmetric spaces was done by Berger [3] about 40 years ago).

1.4. We recall the known construction of a compact Clifford-Klein form of a homogeneous manifold of reductive type. Assume that there exist subgroups Γ and L of G such that the following three conditions are satisfied:

i) L acts properly on G/H.

ii) The double coset space $L\backslash G/H$ is compact.

iii) Γ is a torsion free, cocompact discrete subgroup of L.

Then, $\Gamma\backslash G/H$ is a compact Clifford-Klein form of G/H. If L is a reductive subgroup, then simple criteria for (i) and (ii) are obtained in [13] (see Sect. 2.2) and there always exists Γ satisfying (iii). A list of homogeneous manifolds G/H admitting compact Clifford-Klein forms by this method is presented in [18]. Conversely, it is conjectured that there exists a reductive subgroup L satisfying (i) and (ii) if G/H (of reductive type) admits a compact Clifford-Klein form. The conjecture is true for all examples known so far (including Riemannian cases and group manifold cases).

1.5. Let us introduce a rigorous definition of the *local rigidity* for the homogeneous manifold G/H. Let G be a Lie group and Γ a finitely generated group. We denote by $\mathscr{H}(\Gamma,G)$ the set of all homomorphisms of Γ to G. We equip $\mathscr{H}(\Gamma,G)$ with the topology of pointwise convergence. $\mathscr{H}(\Gamma,G)$ is a real analytic variety if Γ is finitely presented. Let H be a closed subgroup of G. We define

$$\begin{aligned} R(\Gamma,G,H) \;:=\; \{u \in \mathscr{H}(\Gamma,G) :\ & u \text{ is injective, and} \\ & u(\Gamma) \text{ acts properly discontinuously on } G/H\}. \end{aligned}$$

Then the double coset space $u(\Gamma)\backslash G/H$ forms a family of Clifford-Klein forms parametrized by $u \in R(\Gamma, G, H)$, provided Γ is torsion free.

There is a natural action of G on $\mathscr{A}(\Gamma, G)$ by inner automorphisms:

$$(g \cdot u)(\gamma) = gu(\gamma)g^{-1}, \; g \in G, \gamma \in \Gamma, u \in \mathscr{A}(\Gamma, G).$$

This action stabilizes $R(\Gamma, G, H)$. We say that a homomorphism $u \in R(\Gamma, G, H)$ is *locally rigid as a discontinuous group* acting on G/H if the G-orbit through $u \in R(\Gamma, G, H)$ is open in $R(\Gamma, G, H)$. This terminology coincides with the standard one if H is compact (e.g. [29, 34]).

1.6. Our object of study is the local rigidity of a compact Clifford-Klein form. The failure of local rigidity leads to a theory of the moduli space of specific geometric structures that model on a homogeneous manifold G/H. Previous to this, a few examples where local rigidity fails were studied in low dimensions:
1) The Poincaré disk $G/H = SL(2, \mathbb{R})/SO(2)$.
2) $G/H = G' \times G'/\operatorname{diag} G'$ with $G' = SL(2, \mathbb{R})$ ([9, 22]).
3) $G/H = G' \times G'/\operatorname{diag} G'$ with $G' = SL(2, \mathbb{C})$ ([8]).

These cases concern with the deformation of complex structures of a closed Riemann surface with genus ≥ 2 (the Teichmüller space), 3 dimensional Lorentz structures, and 3 dimensional complex structures, respectively.

The failure of local rigidity might be also of interest in connection with spectral geometry for indefinite-Riemannian manifolds (e.g. an indefinite-Riemannian analog of the Phillips-Sarnak conjecture [28]).

1.7. This paper proves that there exists non-trivial deformation of a uniform lattice even in higher dimensional compact Clifford-Klein forms $\Gamma\backslash G/H$ constructed in Sect. 1.4.

Our main results are Theorem 2.4 and Corollary 2.6. With minimal notation, we just illustrate here by typical cases in the following two theorems:

Theorem A. *Suppose $G/H = G' \times G'/\operatorname{diag} G'$ with G' a simple linear Lie group. Then the following two conditions are equivalent.*
i) *There exists a uniform lattice Γ of G' such that $\Gamma \times 1 \in R(\Gamma, G, H)$ is not locally rigid as a discontinuous group acting on G/H.*
ii) *G' is locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$.*

Theorem B. *The following homogeneous manifolds admit uniform lattices that are not locally rigid ($n \geq 1$):*
$SO(2n, 2)/SO(2n, 1)$, $SU(2n, 2)/Sp(n, 1)$, $SO(4, 3)/G_2(\mathbb{R})$, $SO(4, 4)/Spin(4, 3)$.

All of the above homogeneous manifolds carry G-invariant indefinite-Riemannian metric. It is in sharp contrast to the Selberg-Weil local rigidity theorem in the Riemannian case.

We note that known examples in (2) and (3) in Sect. 1.6 deal with $G' = SL(2, \mathbb{R}) \approx SO(2, 1) \approx SU(1, 1)$, $G' = SL(2, \mathbb{C}) \approx SO(3, 1)$, respectively (namely, n = 1, 2 or 3 in the condition (ii) in Theorem A). Much more than Theorems A and B, we shall give a quantitative estimate of the deformation parameter that allows Γ to deform without destroying properly discontinuous actions. The quantitative estimate will be described in terms of the diameter of a certain compact Riemannian manifold and the “angle” between H and L with notation in Sect. 1.4. This is stated in Theorem 2.4 and Sect. 3.7. The proof of our main results contains the affirmative solution of a generalization of a conjecture of Goldman [9] (see Remark 2.5), which was originally posed for $\widetilde{SL(2,\mathbb{R})}$. In particular, there are at least $b_1(\Gamma)\operatorname{rank} G'$ parameters for the codimension of the G-orbit through $\Gamma \times 1 \in R(\Gamma, G, H)$, namely, parameters for non-trivial deformations preserving properly discontinuous actions. Our proof is based on the recent progress on properly discontinuous actions ([1, 17]) and on classical results of Milnor about the fundamental group of a negatively curved manifold ([26]).

We note that the deformation given in this paper produces new compact Clifford-Klein forms $\Gamma\backslash G/H$, namely, the manifolds that enjoy the same local properties and the same fundamental groups. A distinguished feature is that Γ does not necessarily have the property:

(1.7) the Zariski closure of Γ acts properly on G/H,

whereas previous examples in [13, 21] (see Sect. 1.4) have the property (1.7).

It should be noted that the property (1.7) is also important in the non-reductive case such as the Auslander Conjecture where $G/H = GL(n,\mathbb{R}) \ltimes \mathbb{R}^n/GL(n,\mathbb{R})$.

1.8. We remark that there also exist compact Clifford-Klein forms with indefinite-Riemannian metric where local rigidity holds.

Proposition. *The following homogeneous manifolds admit uniform lattices that are locally rigid ($n \geq 1$, $m \geq 2$):*

$$SU(2n,2)/U(2n,1),\ SO(4m,4)/SO(4m,3),\ SO(4n,4)/Sp(n,1).$$

2 Deformation of a uniform lattice for G/H

2.1. Let G be a real reductive linear Lie group, K a maximal compact subgroup of G, and θ the corresponding Cartan involution. Then we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

of the Lie algebra $\mathfrak{g}$ of G. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. We put

$$\mathbb{R}\text{-}\operatorname{rank} G := \dim_{\mathbb{R}} \mathfrak{a}, \qquad \mathrm{d}(G) := \dim_{\mathbb{R}} \mathfrak{p}.$$

Let A be the analytic subgroup of G corresponding to $\mathfrak{a}$ and we write

$$\log : A \to \mathfrak{a},$$

for the inverse map of the diffeomorphism $\exp : \mathfrak{a} \to A$. Let

$$\begin{aligned} W_G &:= N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \\ &= \{g \in K : \operatorname{Ad}(g)\mathfrak{a} = \mathfrak{a}\} \,/\, \{g \in K : \operatorname{Ad}(g)Y = Y \text{ for } Y \in \mathfrak{a}\}. \end{aligned}$$

Then W_G acts on $\mathfrak{a}$ effectively. The finite group W_G is isomorphic to the Weyl group for the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ if G is connected. Associated to a subset C of G, we define a W_G-invariant subset of $\mathfrak{a}$ by

$$\mathfrak{a}(C) := \log(A \cap KCK). \tag{2.1}$$

2.2. We recall how we find a uniform lattice for an indefinite-Riemannian homogeneous manifold G/H where H is non-compact.

Suppose that H is a θ-stable closed subgroup of a real reductive linear Lie group G with finitely many connected components. Then H is also a real reductive linear Lie group with Cartan involution $\theta|_H$. The corresponding Cartan decomposition of the Lie algebra $\mathfrak{h}$ of H is given by

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p}).$$

We have $\mathrm{d}(H) = \dim(\mathfrak{h} \cap \mathfrak{p})$. We take a maximal abelian subspace $\mathfrak{b}$ in $\mathfrak{h} \cap \mathfrak{p}$. Then $\mathfrak{b}$ is not necessarily contained in $\mathfrak{a}$, but there exists an element g of K such that $\mathrm{Ad}(g)\mathfrak{b} \subset \mathfrak{a}$. We fix such g and put $\mathfrak{a}_H := \mathrm{Ad}(g)\mathfrak{b}$. Then $\mathfrak{a}_H$ is a subspace of $\mathfrak{a}$ that is unique up to conjugation by W_G. The definition (2.1) amounts to

$$\mathfrak{a}(H) = W_G \cdot \mathfrak{a}_H .$$

Analogous notation is used for a θ-stable closed subgroup L of G.

Now, we consider the following conditions:

(2.2.1) H, L are θ-stable closed subgroups of a real reductive linear Lie group G with finitely many connected components.
(2.2.2) $\mathfrak{a}(H) \cap \mathfrak{a}(L) = \{0\}$.
(2.2.3) $\mathrm{d}(H) + \mathrm{d}(L) = \mathrm{d}(G)$.
(2.2.4) Γ is a cocompact discrete subgroup of L without torsion.

Under the above four conditions, Γ acts properly discontinuously and freely on G/H such that $\Gamma \backslash G/H$ is a compact Clifford-Klein form of G/H (see [13], Theorem 4.1 and Theorem 4.7). The assumption (2.2.1) assures that there exists a G-invariant (indefinite-)Riemannian metric on G/H with signature $(\dim \mathfrak{p} - \dim(\mathfrak{p} \cap \mathfrak{h}), \dim \mathfrak{k} - \dim(\mathfrak{k} \cap \mathfrak{h}))$. For instance, $SO(2n, 2)/U(n, 1)$ carries an $SO(2n, 2)$-invariant indefinite-Riemannian metric of signature $(2n, n^2 - n)$. If H is a maximal compact subgroup of G (i.e. $\mathfrak{k} = \mathfrak{h}$) so that the metric is Riemannian, then we can take $L := G$ satisfying (2.2.1)–(2.2.3), which explains the well-known result ([4, 5, 27]) on the existence of compact Clifford-Klein forms of Riemannian symmetric spaces in the framework here. A typical indefinite-Riemannian example satisfying (2.2.1)–(2.2.3) is given by $G/H = SO(2n, 2)/U(n, 1)$ and $L = SO(2n, 1)$. We refer to [18], Corollary 4.7 for a list of (L, G, H) satisfying the above assumptions (cf. [13, 21]).

2.3. In the setting (2.2), we put $L' := Z_G(L)$, the centralizer of L in G. For $\rho \in \mathrm{Hom}(\Gamma, L')$, we form a subgroup of G by

$$\Gamma_\rho := \{\gamma\rho(\gamma) \in G : \gamma \in \Gamma\} .$$

Denoting by **1** the trivial representation of Γ, we have obviously $\Gamma_{\mathbf{1}} = \Gamma$.

Example. Suppose G' is a semisimple Lie group having no center, and we set $G := G' \times G'$, $H := \operatorname{diag} G'$, and $L := G' \times 1$. Then the conditions (2.2.1)–(2.2.3) are satisfied. Suppose Γ is a countable subgroup of G'. We note that $L' = 1 \times G'$ and $\mathscr{A}(\Gamma \times 1, L') \simeq \operatorname{Hom}(\Gamma, G')$. Any torsion free discontinuous group ($\subset G' \times G'$) acting on a group manifold $G' \simeq G/H$ is of the form Γ_ρ up to switch of factor if and only if $\mathbb{R}\text{-}\operatorname{rank} G = 1$ (see [22] for $SL(2, \mathbb{R})$; [16], Corollary 3.4 for the general case).

2.4. Here is our main theorem:

Theorem. *Suppose we are in the setting (2.2) and retain the notation as above. There exists an open neighbourhood $W \subset \mathscr{A}(\Gamma, L')$ of the trivial representation* **1** *such that Γ_ρ acts properly discontinuously and freely on G/H and that $\Gamma_\rho \backslash G/H$ is a compact Clifford-Klein form of G/H for any $\rho \in W$.*

The quantitative estimate of W will be given in (3.7) and an easiest case of W is illustrated in Sect. 3.8.

2.5. Remark. W. Goldman constructed "non-standard Lorentz space forms" by proving a similar result to Theorem 2.4 in the special case where

$$G/H = G' \times G' / \operatorname{diag} G' \qquad \text{with } G' = \widetilde{SL(2, \mathbb{R})}$$

under the assumption that the image of ρ is contained in a one dimensional abelian group ([9]). He conjectured that this assumption on $\operatorname{Image} \rho$ could be removed (see Remarks (i) in loc. cit.). Theorem 2.4 affirms his conjecture (for any linear real reductive group G'), with an explicit estimate of the open set W (see (3.7)).

2.6. Corollary. *Suppose we are in the setting (2.2.1), (2.2.2) and (2.2.3). Assume that $\mathfrak{l}$ contains $\mathfrak{so}(n, 1)$ ($n \geq 2$) or $\mathfrak{su}(n, 1)$ ($n \geq 1$) as a normal factor and that $\mathfrak{l}' \neq 0$. Then there exists a discrete subgroup Γ of G which is not locally rigid as a cocompact discontinuous group acting on G/H.*

3 Proof of Theorems

3.1. Suppose we are in the setting of Sects. 2.2 and 2.3. In particular, we recall $\theta L = L$ and $L' = Z_G(L)$. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $\mathfrak{a}_L := \mathfrak{a} \cap \mathfrak{l}$ and $\mathfrak{a}_{L'} := \mathfrak{a} \cap \mathfrak{l}'$ are maximal abelian subspaces of $\mathfrak{p} \cap \mathfrak{l}$ and $\mathfrak{p} \cap \mathfrak{l}'$, respectively. Note that $\mathfrak{a}(L) = W_G \cdot \mathfrak{a}_L$. We fix an $\operatorname{Ad}(G)$-invariant non-degenerate bilinear form $\langle\, , \,\rangle$ on $\mathfrak{g}$, which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$ such that $\mathfrak{k}$ is orthogonal to $\mathfrak{p}$ (e.g. the Killing form if G is semisimple). We define the norm on $\mathfrak{p}$ by

$$|Y| := \langle Y, Y \rangle^{\frac{1}{2}}, \text{ for } Y \in \mathfrak{p}.$$

We identify the tangent space at $o = eK \in G/K$ with $\mathfrak{p}$. Then $\langle\ ,\ \rangle|_{\mathfrak{p}\times\mathfrak{p}}$ induces a G-invariant Riemannian metric on the homogeneous space G/K, which makes G/K into a Riemannian symmetric space. We write $d(x,y)$ for the distance between two points x, y in G/K, and $\overline{d}(A,B)$ for the distance between two compact subsets $A, B \subset G/K$. Let $X := L/L \cap K$, which is a totally geodesic submanifold of G/K. We set

$$B(o;R) := \{x \in X : d(x,o) \le R\}.$$

Suppose Γ is a cocompact discrete subgroup of L without torsion. Let δ be the diameter of the compact Clifford-Klein form $\Gamma\backslash X \simeq \Gamma\backslash L/K \cap L$, on which the Riemannian metric is induced from X. We define:

$$F := \{\gamma \in \Gamma : \gamma B(o;\delta) \cap B(o;\delta) \neq \emptyset\}, \tag{3.1.1}$$

$$\nu_\Gamma := \min_{\gamma \in \Gamma\backslash F} \overline{d}(\gamma B(o;\delta), B(o;\delta)). \tag{3.1.2}$$

We note that F is a finite set of generators of Γ. We write

$$l \equiv l_F : \Gamma \to \mathbb{N}$$

for the word length with respect to the generating set F.

3.2. We define a function on G by

$$\varphi : G \to \mathbb{R},\ g = k_1 \exp(X) k_2 \mapsto |X|,$$

where $k_1, k_2 \in K$ and $X \in \mathfrak{a}$. Here are some elementary facts about φ:

Lemma.
1) *φ is well-defined and $\varphi(g) = d(g \cdot o, o)$ for $g \in G$.*
2) *$\varphi(gg') \le \varphi(g) + \varphi(g')$ for $g, g' \in G$.*
3) *$\varphi(g) \ge 0$ for any $g \in G$. Furthermore, $\varphi(g) = 0$ if and only if $g \in K$.*

Proof. Suppose $g = k_1 \exp(X) k_2$ with $k_1, k_2 \in K$ and $X \in \mathfrak{a}$. Then we have

$$\begin{aligned} d(g \cdot o, o) &= d(k_1 \exp(X) k_2 \cdot o, o) = d(\exp(X) k_2 \cdot o, (k_1)^{-1} \cdot o) \\ &= d(\exp(X) \cdot o, o) = |X|, \end{aligned}$$

proving (1). Next, we have

$$\begin{aligned} \varphi(gg') &= d(gg' \cdot o, o) \le d(gg' \cdot o, g \cdot o) + d(g \cdot o, o) \\ &= d(g' \cdot o, o) + d(g \cdot o, o) = \varphi(g') + \varphi(g), \end{aligned}$$

showing (2). The statement (3) is clear. □

3.3. Lemma (cf. [26]). *For $R \ge 0$, there exists a constant $T_R \in \mathbb{N}$ such that*

$$d(x_1, g \cdot x_2) \ge \nu_\Gamma (l(g) - T_R)$$

for any $x_1, x_2 \in B(o;R)$ and for any $g \in \Gamma$. Furthermore, if $R = 0$, we can take $T_R = 1$.

Proof. We take a finite subset $\Gamma_R \subset \Gamma$ such that

$$B(o; R) \subset \bigcup_{\gamma \in \Gamma_R} \gamma \cdot B(o; \delta)$$

and define

$$T_R := 2K_R + 1,\ K_R := \max_{\gamma \in \Gamma_R} l(\gamma) \in \mathbb{N}.$$

Let $x_1, x_2 \in B(o; R)$ and $g \in \Gamma$. We put $k := [\frac{d(x_1, g \cdot x_2)}{\nu_\Gamma}] + 1$. Then $d(x_1, g \cdot x_2) < \nu_\Gamma k$. We choose points $y_1, \dots, y_{k+1}$ along the minimal geodesic from x_1 to x_2 such that $y_1 = x_1$, $y_{k+1} = g \cdot x_2$ and that $d(y_i, y_{i+1}) < \nu_\Gamma$ for $1 \le i \le k$. We take $\gamma_i \in \Gamma$ with $y_i \in \gamma_i \cdot B(o; \delta)$ $(1 \le i \le k+1)$. We may assume $\gamma_1, g^{-1}\gamma_{k+1} \in \Gamma_R$. Because

$$\bar{d}(B(o; \delta), \gamma_i{}^{-1}\gamma_{i+1} \cdot B(o; \delta)) = \bar{d}(\gamma_i \cdot B(o; \delta), \gamma_{i+1} \cdot B(o; \delta)) \le d(y_i, y_{i+1}) < \nu_\Gamma,$$

we have $\gamma_i{}^{-1}\gamma_{i+1} \in F$ from the definition of ν_Γ. In view of

$$g = \gamma_1(\gamma_1{}^{-1}\gamma_2) \dots (\gamma_k{}^{-1}\gamma_{k+1})(g^{-1}\gamma_{k+1})^{-1}$$

we have $l(g) \le K_R + k + K_R = k + T_R - 1 \le \frac{d(x_1, g \cdot x_2)}{\nu_\Gamma} + T_R$. □

3.4. For $Y \in \mathfrak{a}$ and $r \ge 0$, we define a ball by $B'(Y, r) := \{Z \in \mathfrak{a} : |Z - Y| \le r\}$, and a closed cone containing the subspace $\mathfrak{a}_L$ in $\mathfrak{a}$ by

$$\mathfrak{a}_L(r) := \bigcup_{Y \in \mathfrak{a}_L} B'(Y, r|Y|).$$

With notation in Sect. 3.1, we put

$$M_\rho := \max_{f \in F} d(\rho(f) \cdot o, o),$$

for $\rho \in \mathrm{Hom}(\Gamma, L')$. Here is an upper estimate of the Cartan projection of Γ_ρ:

Lemma. *With the notation as above, if* $\rho \in \mathrm{Hom}(\Gamma, L')$*, then we have*

$$\mathfrak{a}(\Gamma_\rho) \subset W_G \cdot \left(\mathfrak{a}_L(\frac{M_\rho}{\nu_\Gamma}) + B'(0; M_\rho) \right).$$

Proof. Suppose $\gamma\rho(\gamma) \in \Gamma_\rho$. We write $\gamma = k_1 \exp(Y) k_2$ $(k_1, k_2 \in L \cap K,\ Y \in \mathfrak{a}_L)$ and $\rho(\gamma) = k_1' \exp(Z) k_2'$ $(k_1', k_2' \in L' \cap K,\ Z \in \mathfrak{a}_{L'})$. Then

$$\gamma\rho(\gamma) = k_1 k_1' \exp(Y + Z) k_2 k_2'$$

because L and L' commute. It follows from Lemma 3.2 and Lemma 3.3 that

$$|Y| = d(\gamma \cdot o, o) \ge \nu_\Gamma(l(\gamma) - 1). \tag{3.4.1}$$

It follows from Lemma 3.2 (1) and (2) that

$$|Z| = d(\rho(\gamma)\cdot o, o) \leq l(\gamma)M_\rho. \tag{3.4.2}$$

Hence we have $|Z| \leq (1 + \frac{|Y|}{\nu_\Gamma})M_\rho = \frac{M_\rho|Y|}{\nu_\Gamma} + M_\rho$, namely,

$$Y + Z \in B'(Y; \frac{M_\rho|Y|}{\nu_\Gamma}) + B'(0; M_\rho) \subset \mathfrak{a}_L(\frac{M_\rho}{\nu_\Gamma}) + B'(0; M_\rho).$$

This completes the proof. □

3.5. Lemma. *In the setting (2.2), if $M_\rho < \nu_\Gamma$, then Γ is isomorphic to Γ_ρ.*

Proof. It suffices to show the injectivity of the map $\tilde{\rho} : \Gamma \to \Gamma_\rho$, $\gamma \mapsto \gamma\rho(\gamma)$, provided $M_\rho < \nu_\Gamma$. Suppose $\gamma \in \operatorname{Ker}\tilde{\rho}$. By (3.4.1) and (3.4.2), we have

$$\nu_\Gamma(l(\gamma) - 1) \leq d(\gamma\cdot o, o) = d(o, \gamma^{-1}\cdot o) = d(o, \rho(\gamma)\cdot o) \leq l(\gamma)M_\rho.$$

Hence, if $M_\rho < \nu_\Gamma$, then we have

$$\operatorname{Ker}\tilde{\rho} \subset \left\{\gamma \in \Gamma : l(\gamma) \leq \frac{\nu_\Gamma}{\nu_\Gamma - M_\rho}\right\}.$$

Since the right side is a finite set and since a torsion free group Γ does not contain a finite subgroup except for $\{e\}$, $\operatorname{Ker}\tilde{\rho} = \{e\}$. □

3.6. We recall the criterion for the proper actions:

Fact ([17], Corollary 1.2; see also [1]). *Let G be a real reductive group, H a closed subgroup, and Γ a discrete subgroup. Then Γ acts properly discontinuously on G/H if and only if $\mathfrak{a}(\Gamma) \cap (\mathfrak{a}(H) + V)$ is relatively compact for any compact subset V of $\mathfrak{a}$.*

3.7. Proof of Theorem 2.4. First we note that $\mathfrak{a}(L)$ (resp. $\mathfrak{a}(H)$) is a finite union of the W_G-orbit of the subspace in $\mathfrak{a}_L$ (resp. $\mathfrak{a}_H$) (see Sect. 2.2). Let $\psi \equiv \psi(H, L)$ be the minimum of the angle between $w\cdot\mathfrak{a}_L$ and $\mathfrak{a}_H$ where w runs over the Weyl group W_G. The assumption (2.2.2) implies $\psi > 0$. If $\sin\psi > r > 0$, then $\mathfrak{a}_L(r) \cap \mathfrak{a}(H) = \{0\}$ and $(\mathfrak{a}_L(r) + V_1) \cap (\mathfrak{a}(H) + V_2)$ is relatively compact for any compact subsets V_1, V_2 of $\mathfrak{a}$. With notation in Sect. 3.1, we define an open set in $\mathscr{A}(\Gamma, L')$ by

$$W := \{\rho \in \mathscr{A}(\Gamma, L') : d(o, \rho(f)\cdot o) < \nu_\Gamma \sin\psi \text{ for any } f \in F\}. \tag{3.7}$$

Clearly the trivial representation $\mathbf{1} \in W$. If $\rho \in W$, then $M_\rho < \nu_\Gamma \sin\psi \leq \nu_\Gamma$. Then

$$\mathfrak{a}(\Gamma_\rho) \cap (\mathfrak{a}(H) + V)$$

is relatively compact for any compact subset V of $\mathfrak{a}$ by Lemma 3.4. Therefore the action of Γ_ρ on G/H is properly discontinuous by Fact 3.6.

Because Γ is isomorphic to Γ_ρ by Lemma 3.5, Γ_ρ is torsion free. Hence the action of Γ_ρ on G/H is free because it is properly discontinuous.

We recall that the cohomological dimension of Γ over $\mathbb{R}$ denoted by $\operatorname{cd}_{\mathbb{R}}(\Gamma)$ is the projective dimension of $\mathbb{R}$ as a left $\mathbb{R}[\Gamma]$-module. Equivalently,

$$\text{cd}_{\mathbb{R}}(\varGamma) = \sup\{n \in \mathbb{N} : H^n(\varGamma; A) \neq 0 \text{ for some left } \mathbb{R}[\varGamma]-\text{module } A\}.$$

Because $\varGamma$ is a uniform lattice of L, we have $\text{cd}_{\mathbb{R}}\,\varGamma = \text{d}(L)$ ([31]). As $\varGamma \simeq \varGamma_\rho$ as an abstract group, we have $\text{cd}_{\mathbb{R}}\,\varGamma_\rho = \text{cd}_{\mathbb{R}}\,\varGamma$. Hence, we have

$$\text{cd}_{\mathbb{R}}\,\varGamma_\rho = \text{d}(L) = \text{d}(G) - \text{d}(H)$$

by (2.2.3). Because $\varGamma_\rho$ acts properly discontinuously on G/H, $\varGamma_\rho\backslash G/H$ is compact because of Corollary 5.5 in [13]. Hence we have completed the proof of the Theorem. $\square$

3.8. Although our concern here is with non-abelian case, it is illustrative to see how the general proper discontinuity condition (3.7) works in the easiest (trivial) case, namely, in the abelian case. Let $G := \mathbb{R}^2$ equipped with standard Riemannian metric and $H := \mathbb{R}\overrightarrow{e_1} = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix}$. We fix $\overrightarrow{a} := \begin{pmatrix}a_1\\a_2\end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$, and define subgroups of G by

$$\varGamma := \mathbb{Z}\overrightarrow{a} \ \subset\ L := \mathbb{R}\overrightarrow{a}.$$

In view of $\mathfrak{a}(L) = \mathbb{R}\overrightarrow{a}$ and $\mathfrak{a}(H) = \mathbb{R}\overrightarrow{e_1}$, the condition (2.2.2) is satisfied if and only if $a_2 \neq 0$, which we shall assume from now on. Then the conditions (2.2.1) – (2.2.4) are satisfied since $\text{d}(H) = \text{d}(L) = 1$ and $\text{d}(G) = 2$. The angle $\psi \equiv \psi(H, L)$ (see Sect. 3.7 for definition) satisfies $|\overrightarrow{a}|\sin\psi = |a_2|$.

With notation in Sect. 3.1, the diameter δ of $\varGamma\backslash L/(L\cap K) = \mathbb{Z}\overrightarrow{a}\backslash\mathbb{R}\overrightarrow{a}/\{0\}$ equals $|\overrightarrow{a}|$, a generating subset F of $\varGamma$ is given by $\{0, \pm\overrightarrow{a}\}$, and $\nu_\varGamma = |\overrightarrow{a}|$. For each $\overrightarrow{b} \in \mathbb{R}^2$, we define a homomorphism

$$\rho_{\overrightarrow{b}} : \varGamma \to G, \quad n\overrightarrow{a} \mapsto n\overrightarrow{b} \ (n \in \mathbb{Z}).$$

Since $L' = Z_G(L) = G$ because $G \simeq \mathbb{R}^2$ is abelian, we have a diffeomorphism

$$\mathbb{R}^2 \xrightarrow{\sim} \mathscr{A}(\varGamma, L') = \mathscr{A}(\mathbb{Z}\overrightarrow{a}, \mathbb{R}^2), \quad \overrightarrow{b} \mapsto \rho_{\overrightarrow{b}}.$$

Now, the open set W of $\mathscr{A}(\varGamma, L')$ given in (3.7) has the form

$$\begin{aligned} W &= \left\{\rho_{\overrightarrow{b}} \in \mathscr{A}(\varGamma, L') : d(o, \rho_{\overrightarrow{b}}(f)\cdot o) < |\overrightarrow{a}|\sin\psi, \text{ for any } f \in F\right\} \\ &= \left\{\rho_{\overrightarrow{b}} \in \mathscr{A}(\varGamma, L') : d(o, \rho_{\overrightarrow{b}}(\overrightarrow{a})\cdot o) < |a_2|\right\} \\ &\simeq \left\{\overrightarrow{b} \in \mathbb{R}^2 : |\overrightarrow{b}| < |a_2|\right\}. \end{aligned}$$

For $\rho_{\overrightarrow{b}} \in \mathscr{A}(\varGamma, L')$, the subgroup $\varGamma_{\rho_{\overrightarrow{b}}}$ of G (Sect. 2.3) is given by

$$\varGamma_{\rho_{\overrightarrow{b}}} = \left\{\gamma\rho_{\overrightarrow{b}}(\gamma) \in G : \gamma \in \varGamma = \mathbb{Z}\overrightarrow{a}\right\} = \left\{n(\overrightarrow{a} + \overrightarrow{b}) : n \in \mathbb{Z}\right\}.$$

Theorem 2.4 asserts that $\varGamma_{\rho_{\overrightarrow{b}}}\backslash G/H = \varGamma_{\rho_{\overrightarrow{b}}}\backslash\mathbb{R}^2/\mathbb{R}\overrightarrow{e_1}$ is a compact Clifford-Klein form if $\rho_{\overrightarrow{b}} \in W$, namely, if $|\overrightarrow{b}| = \sqrt{{b_1}^2 + {b_2}^2} < |a_2|$. One can observe that the action of $\Gamma_{\rho_{\vec{b}}}$ on $G/H = \mathbb{R}^2/\mathbb{R}\vec{e_1}$ is properly discontinuous because $\vec{a} + \vec{b}$ and $\vec{e_1}$ are linearly independent if $|\vec{b}| < |a_2|$ (i.e. if $\rho_{\vec{b}} \in W$).

However, this is not always the case if $\rho_{\vec{b}} \notin W$. In fact, let us put $\vec{b} := \begin{pmatrix} \varepsilon \\ -a_2 \end{pmatrix}$. Then, $\rho_{\vec{b}} \notin W$ because $|\vec{b}| \geq |a_2|$. We have:

i) $\Gamma_{\rho_{\vec{b}}} \backslash G/H$ is non-compact if $\varepsilon + a_1$ is rational.
ii) $\Gamma_{\rho_{\vec{b}}} \backslash G/H$ is not Hausdorff if $\varepsilon + a_1$ is irrational.

In either case, $\Gamma_{\rho_{\vec{b}}} \backslash G/H$ fails to be a compact Clifford-Klein form. This shows that the condition $\rho_{\vec{b}} \in W$ is critical for Theorem 2.4 in this abelian example.

3.9. *Proof of Corollary 2.6.* Let L_{sn} be the maximal semisimple normal subgroup of L. There exists a unique connected normal subgroup L_n of L with the following properties: L/L_n is compact and L_n is the direct product of L_{sn} and $\mathbb{R}^d$ for some d. We note that $\mathfrak{l}_{sn} \cap \mathfrak{l}' \neq 0$. There is a finite covering

$$\varpi : \widetilde{L_n} \to L_n$$

such that the analytic subgroup $\widetilde{L_{sn}}$ with Lie algebra $\mathfrak{l}_{sn}$ is a direct product of non-compact simple linear Lie groups, say, $G_1 \times \cdots \times G_k$. We note that at least one of the factors G_j is locally isomorphic to $SO(n,1)$ or $SU(n,1)$ from our assumption. We take a cocompact, torsion free subgroup Γ_j of G_j for each j. Here, we can and do choose Γ_j such that $b_1(\Gamma_j) := \dim_{\mathbb{R}} H^1(\Gamma_j; \mathbb{R}) \neq 0$ if G_j is locally isomorphic to $SO(n,1)$ or $SU(n,1)$ by a theorem of Millson and Kazhdan [25, 12]. We define

$$\begin{aligned} \widetilde{\Gamma} &:= \Gamma_1 \times \cdots \times \Gamma_k \\ \Gamma &:= \varpi(\widetilde{\Gamma} \times \mathbb{Z}^d). \end{aligned}$$

We note that $b_1(\widetilde{\Gamma}) \neq 0$. Then Γ is a cocompact torsion free subgroup of L_n. We take an open neighbourhood $W \subset \mathscr{A}(\Gamma, L')$ of the trivial representation $\mathbf{1}$ as in Theorem 2.4. We fix an arbitrary Cartan subgroup J' of L', and write the inclusion $\iota : J' \hookrightarrow L'$ and the projection $\pi : \Gamma \to \Gamma/[\Gamma, \Gamma]$. The dimension of J' is denoted by $\operatorname{rank} L'$ as usual. We note that the abelianization $\Gamma/[\Gamma, \Gamma]$ is isomorphic to $\mathbb{Z}^{b_1(\Gamma)}$ modulo torsion. In view of the inclusion

$$i : \operatorname{Hom}(\Gamma/[\Gamma, \Gamma], J') \hookrightarrow \mathscr{A}(\Gamma, L'), \quad \tau \mapsto \iota \circ \tau \circ \pi,$$

there exists a neighbourhood W' of $\mathbf{1}$ in $\operatorname{Hom}(\Gamma/[\Gamma, \Gamma], J')$ with $i(W') \subset W$, where W' is homeomorphic to a Euclidean ball of dimension $b_1(\Gamma) \operatorname{rank} L' (> 0)$.

Suppose $g\Gamma g^{-1} = \Gamma_\rho$ for some $\rho \in i(W')$ and $g \in G$. The Zariski closure of Γ, denoted by $\overline{\Gamma}$, equals L_n, while $\overline{\Gamma_\rho}$ is contained in $L_n J'$. Therefore, we have $gL_n g^{-1} \subset L_n J'$, which leads to

$$gL_{sn} g^{-1} = L_{sn}$$

because L_{sn} is the unique maximal connected semisimple subgroup of $L_n J'$. If $\rho \in i(W')$ is generic, then

$$\operatorname{rank}\left(\Gamma_\rho L_{sn}/L_{sn}\right) = b_1(\widetilde{\Gamma}) + d\,.$$

On the other hand,

$$\Gamma L_{sn}/L_{sn} \simeq g\Gamma L_{sn}g^{-1}/gL_{sn}g^{-1} = (g\Gamma g^{-1})(gL_{sn}g^{-1})/gL_{sn}g^{-1} = \Gamma_\rho L_{sn}/L_{sn}\,.$$

The left side is isomorphic to $\mathbb{Z}^d$. Hence the equality holds only if $b_1(\widetilde{\Gamma}) = 0$, which contradicts to our choice of $\widetilde{\Gamma}$. Thus, Γ is not conjugate to Γ_ρ. This shows that Γ is not locally rigid as a discontinuous group acting on G/H by Theorem 2.4. □

3.10. Before proving Theorem A, we make some remarks on local rigidity in Sect. 1. Assume $H' \subset H$. Then the following is immediate from definition.
(3.10.1) $R(\Gamma, G, H) \subset R(\Gamma, G, H')$.
(3.10.2) If $u \in R(\Gamma, G, H)$ is locally rigid for G/H', then it is locally rigid for G/H.

We denote by $H^1(\Gamma, \operatorname{Ad}\circ u)$ the cohomology of Γ with coefficients in the Lie algebra $\mathfrak{g}$ regarded as a Γ module under $\operatorname{Ad}\circ u$. $u \in R(\Gamma, G, \{e\})$ is said to be *infinitesimally rigid* if $H^1(\Gamma, \operatorname{Ad}\circ u) = 0$. A theorem of Weil ([34]) asserts that
(3.10.3) if u is infinitesimally rigid, then u is locally rigid for $G/\{e\}$.

Proof of Theorem A.

(2) ⇒ (1) As in Example 2.3, we put $L = G' \times 1$. We apply Corollary 2.6.

(1) ⇒ (2) Suppose Γ is a torsion free, cocompact discrete subgroup of a simple Lie group G', which is not locally isomorphic to $SO(n,1)$ or $SU(n,1)$. We write $\iota := \operatorname{id}\times\mathbf{1} : \Gamma \to G' \times G'$ so that $\iota(\Gamma) = \Gamma \times \{e\}$. Then $\iota \in R(\Gamma, G' \times G', \operatorname{diag} G')$. By using the local rigidity theorem due to Weil:

$$H^1(\Gamma, \mathfrak{g}') = 0 \quad \text{ if } \mathfrak{g}' \neq \mathfrak{sl}(2,\mathbb{R}),$$

and the vanishing theorem of the first Betti number due to Matsushima, Kaneyuki-Nagano, Kazhdan, Wang and Kostant ([10, 11, 20, 24, 33]):

$$H^1(\Gamma, \mathbb{R}) = 0 \quad \text{ if } \mathfrak{g}' \neq \mathfrak{so}(n,1), \mathfrak{su}(n,1),$$

we have

$$H^1(\Gamma, \operatorname{Ad}\circ\iota) = H^1(\Gamma, \mathfrak{g}') \oplus \dim\mathfrak{g}' H^1(\Gamma, \mathbb{R}) = 0$$

(note that $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1) \simeq \mathfrak{su}(1,1)$). Hence $\iota \in R(\Gamma, G' \times G', \{e\})$ is infinitesimally rigid, and therefore locally rigid as a discontinuous group acting on $G' \times G'/\operatorname{diag} G'$ by (3.10.2) and (3.10.3). □

3.11. *Proof of Theorem B.* We define a subgroup L of G as follows:
Case 1) $L := SU(n,1)$ if $G/H = SO(2n,2)/SO(2n,1)$, $(n \geq 1)$,
Case 2) $L := SU(2n,1)$ if $G/H = SU(2n,2)/Sp(n,1)$, $(n \geq 1)$,
Case 3) $L := SO(4,1)$ if $G/H = SO(4,3)/G_2(\mathbb{R})$,
Case 4) $L := SO(4,1)$ if $G/H = SO(4,4)/Spin(4,3)$.

The triple (L, G, H) satisfies the conditions (2.2.1), (2.2.2) and (2.2.3) (see Corollary 4.7 in [18]). Then the Lie algebra of the centralizer $L' = Z_G(L)$ is given by

$$\mathfrak{l}' \simeq \begin{cases} \mathfrak{sl}(2,\mathbb{R}) & (n = 1 \text{ in Case 1}), \\ \mathbb{R} & (n \geq 2 \text{ in Case 1, Case 2, Case 3}), \\ \mathfrak{so}(3) & (\text{Case 4}). \end{cases}$$

Therefore all the assumptions of Corollary 2.6 are satisfied. Hence Theorem B is proved. ☐

3.12. Proof of Proposition 1.8. We define a subgroup L of G as follows:
Case 1) $L := Sp(n,1)$ if $G/H = SU(2n,2)/U(2n,1)$, $(n \geq 1)$,
Case 2) $L := Sp(n,1)$ if $G/H = SO(4n,4)/SO(4n,3)$, $(n \geq 2)$,
Case 3) $L := SO(4n,3)$ if $G/H = SO(4n,4)/Sp(n,1)$, $(n \geq 1)$.

Then the triple (L,G,H) satisfies the conditions (2.2.1), (2.2.2) and (2.2.3). We take a torsion free cocompact discrete subgroup Γ of L. As we remarked in 3.10, it suffices to show that Γ is infinitesimally rigid in G, that is,

Lemma. *Suppose (L,G) is one of the above. If $\Gamma \subset L$ is a torsion free cocompact discrete subgroup, then $H^1(\Gamma,\mathfrak{g}) = 0$.*

Proof. We take a fundamental Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$, and denote by $F(L,\lambda)$ by the irreducible finite dimensional representation of L with extremal weight $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$.
Case 1 and Case 2) $L = Sp(n,1)$. We fix a suitable base $\{f_1,\dots,f_{n+1}\}$ in $\sqrt{-1}\mathfrak{h}^*$, and fix a positive system $\Delta^+(\mathfrak{l},\mathfrak{h})$ such that $\Sigma_2 := \Delta^+(\mathfrak{l},\mathfrak{h}) \cap \Delta(\mathfrak{p},\mathfrak{h}) = \{f_i \pm f_{n+1} : 1 \leq i \leq n\}$.
Case 3) Suppose $L = SO(4n,3)$ and $G = SO(4n,4)$. We fix a suitable base $\{f_1,\dots,f_{2n+1}\}$ in $\sqrt{-1}\mathfrak{h}^*$, and fix a positive system $\Delta^+(\mathfrak{l},\mathfrak{h})$ such that $\Sigma_2 := \Delta^+(\mathfrak{l},\mathfrak{h}) \cap \Delta(\mathfrak{p},\mathfrak{h}) = \{f_{2n+1}\} \cup \{f_i \pm f_{2n+1} : 1 \leq i \leq 2n\}$.

The adjoint representation of L on $\mathfrak{g}_{\mathbb{C}}$ is decomposed as follows:

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &\simeq F(L,f_1) \oplus F(L,f_1+f_2), && (\text{Case 1}), \\ \mathfrak{g}_{\mathbb{C}} &\simeq F(L,f_1) \oplus 3F(L,f_1+f_2) \oplus 3F(L,0). && (\text{Case 2}), \\ \mathfrak{g}_{\mathbb{C}} &\simeq F(L,f_1+f_2) \oplus F(L,f_1), && (\text{Case 3}). \end{aligned}$$

In any of the above cases, we have

$$\begin{aligned} \sharp\{\alpha \in \Sigma_2 : \langle \alpha, f_1 \rangle \neq 0\} &= \sharp\{f_1 \pm f_{n+1}\} &&= 2 > 1 \\ \sharp\{\alpha \in \Sigma_2 : \langle \alpha, f_1+f_2 \rangle \neq 0\} &= \sharp\{f_1 \pm f_{n+1}, f_2 \pm f_{n+1}\} &&= 4 > 1, \end{aligned}$$

and then

$$H^1(\Gamma, F(L,f_1)) = 0, \quad H^1(\Gamma, F(L,f_1+f_2)) = 0,$$

by the vanishing theorem of Raghunathan ([29], Theorem 1). On the other hand, we have

$$H^1(\Gamma,\mathbb{C}) = H^1(\Gamma, F(L,0)) = 0$$

in the case (2), namely, where Γ is a uniform lattice of $Sp(n,1)$ with $n \geq 2$ ([20]). Thus, $H^1(\Gamma,\mathfrak{g}_{\mathbb{C}}) = 0$. Hence, $H^1(\Gamma,\mathfrak{g}) = 0$. ☐

Acknowledgements. I thank Professors A. Borel, K. Fukaya, W. Goldman, S. Kaneyuki, J. S. Li, G. Tomanov, and S. Yeung, for helpful discussions. The idea of this paper was obtained during my stay in the Institute for Advanced Study at Princeton, and the final manuscript was written at the Institut Mittag-Leffler of the Royal Swedish Academy of Sciences. I am grateful to both of the institutes for the wonderful atmosphere of research and the financial support. Partial results were announced at the Geometry Colloquium at University of Tokyo, November 1992, and at at the conferences at RIMS Kyoto in Japan, July 1993 and at Institut Mittag-Leffler in Sweden, May 1996.

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