J. reine angew. Math. **634** (2009), 51—106
DOI 10.1515/CRELLE.2009.069

Journal für die reine und
angewandte Mathematik


# Deformation theory of representations of prop(erad)s I

By *Sergei Merkulov* at Stockholm and *Bruno Vallette* at Nice

**Abstract.** In this paper and its follow-up [32], we study the deformation theory of morphisms of properads and props thereby extending Quillen's deformation theory for commutative rings to a non-linear framework. The associated chain complex is endowed with an $L_\infty$-algebra structure. Its Maurer-Cartan elements correspond to deformed structures, which allows us to give a geometric interpretation of these results.

To do so, we endow the category of prop(erad)s with a model category structure. We provide a complete study of models for prop(erad)s. A new effective method to make minimal models explicit, that extends the Koszul duality theory, is introduced and the associated notion is called *homotopy Koszul*.

As a corollary, we obtain the (co)homology theories of (al)gebras over a prop(erad) and of homotopy (al)gebras as well. Their underlying chain complex is endowed with an $L_\infty$-algebra structure in general and a Lie algebra structure only in the Koszul case. In particular, we make the deformation complex of morphisms from the properad of associative bialgebras explicit. For any minimal model of this properad, the boundary map of this chain complex is shown to be the one defined by Gerstenhaber and Schack. As a corollary, this paper provides a complete proof of the existence of an $L_\infty$-algebra structure on the Gerstenhaber-Schack bicomplex associated to the deformations of associative bialgebras.

## Introduction

The theory of props and properads, which generalizes the theory of operads, provides us with a universal language to describe many algebraic, topological and differential geometric structures. Our main purpose in this paper is to establish a new and surprisingly strong link between the theory of prop(erad)s and the theory of $L_\infty$-algebras.

We introduce several families of $L_\infty$-algebras canonically associated with prop(erad)s, moreover, we develop new methods which explicitly compute the associated $L_\infty$-brackets in terms of prop(erad)ic compositions and differentials. Many important dg Lie algebras in algebra and geometry (such as Hochschild, Poisson, Schouten, Frölicher-Nijenhuis and many others) are proven to be of this prop(erad)ic origin.

The $L_\infty$-algebras we construct in our paper out of dg prop(erad)s encode many important properties of the latter. The most important one controls the deformation theory of morphisms of prop(erad)s and, in particular, the deformation theory of (al)gebras over prop(erad)s. Applications of our results to the deformation theory of many algebraic and geometric structures become therefore another major theme of our paper.

On the technical side, the story develops (roughly speaking) as follows: first we associate canonically to a pair, $(\mathscr{F}(V), \partial)$ and $(\mathscr{Q}, d)$, consisting of a differential graded (dg, for short) quasi-free prop(erad) $\mathscr{F}(V)$ on an $\mathbb{S}$-bimodule $V$ and an arbitrary dg prop(erad) $\mathscr{Q}$, a structure of $L_\infty$-algebra on the (shifted) graded vector space, $s^{-1}\,\mathrm{Hom}_\bullet^{\mathbb{S}}(V, \mathscr{Q})$, of morphisms of $\mathbb{S}$-bimodules; then we prove the Maurer-Cartan elements of this $L_\infty$-algebra are in *one-to-one* correspondence with the set of all dg morphisms,

$$\{(\mathscr{F}(V), \partial) \to (\mathscr{Q}, d)\},$$

of dg prop(erad)s. This canonical $L_\infty$-algebra is used then to define, for any particular morphism $\gamma : (\mathscr{F}(V), \partial) \to (\mathscr{Q}, d)$, another twisted $L_\infty$-algebra which controls deformation theory of the morphism $\gamma$. In the special case when $(\mathscr{Q}, d)$ is the endomorphism prop(erad), $(\mathrm{End}_X, d_X)$, of some dg vector space $X$, our theory gives $L_\infty$-algebras which control deformation theory of many classical algebraic and geometric structures on $X$, for example, associative algebra structure, Lie algebra structure, commutative algebra structure, Lie bialgebra structure, associative bialgebra structure, formal Poisson structure, Nijenhuis structure etc. As the case of associative bialgebras has never been rigorously treated in the literature before, we discuss this example in full details; we prove, in particular, that the first term of the canonical $L_\infty$-structure controlling deformation theory of bialgebras is precisely the Gerstenhaber-Schack differential.

We derive and study the deformation complex and its $L_\infty$-structure from several different perspectives. One of them can be viewed as a nontrivial generalization to the case of prop(erad)s of Van der Laan's approach [43] to the deformation theory of algebras over operads, while others are completely new and provide us with, perhaps, a conceptual explanation of the observed (long ago) phenomenon that deformation theories are controlled by dg Lie and, more generally, $L_\infty$ structures.

First, we define the deformation complex of a morphism of prop(erad)s $\mathscr{P} \to \mathscr{Q}$ generalizing Quillen's definition of the deformation complex of a morphism of commutative rings. When $(\mathscr{F}(\mathscr{C}), \partial)$ is a quasi-free resolution of $\mathscr{P}$, we prove that this chain complex is isomorphic, up to a shift of degree, to the space of morphisms of $\mathbb{S}$-bimodule $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{Q})$, where $\mathscr{C}(\simeq s^{-1}V)$ is a homotopy coprop(erad), that is the dual notion of prop(erad) with relations up to homotopy. Since $\mathscr{Q}$ is a (strict) prop(erad), we prove that the space $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{Q})$ has a rich algebraic structure, namely it is an $L_\infty$-algebra. We also obtain higher operations with $m + n$ inputs acting on $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{Q})$ which are important in applications. In the case of, for example, the non-symmetric operad, $\mathscr{A}ss$, of associative algebras the deformation complex is the Hochschild cochain complex of an associative algebra, and the higher homotopy operations are shown to be non-symmetric braces operations which play a fundamental role in the proof of Deligne's conjecture (see [37], [45], [20], [33], [4], [19]).

Recall that M. Markl proved in [27] a first interesting partial result, that is for a given minimal model $(\mathscr{F}(\mathscr{C}), \partial)$ a prop(erad) $\mathscr{P}$ concentrated in degree 0, there exists an $L_\infty$-structure on the space of derivations from $\mathscr{F}(\mathscr{C})$ to $\mathrm{End}_X$, where $X$ is a $\mathscr{P}$-(al)gebra. By using a different conceptual method based on the notions of homotopy (co)prop(erad)s and convolution prop(erad)s, we prove here that for any representation $\mathscr{Q}$ of any prop(erad) $\mathscr{P}$, there exists a homotopy prop(erad) structure on the space of derivations from any quasi-free resolution of $\mathscr{P}$ to $\mathscr{Q}$. Moreover this construction is shown to be functorial, that is does not depend on the model chosen. From this we derive functorially the general $L_\infty$-structure.

Another approach of deriving the deformation complex and its $L_\infty$-structure is based on a canonical enlargement of the category of dg prop(erad)s via an extension of the notion of *morphism*; the set of morphisms, $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}_1, \mathscr{P}_2)$, in this enlarged category is identified with a certain *dg affine scheme* naturally associated with both $\mathscr{P}_1$ and $\mathscr{P}_2$; moreover, when the dg prop(erad) $\mathscr{P}_1$ is quasi-free, the dg affine scheme $\mathrm{Mor}(\mathscr{P}_1, \mathscr{P}_2)$ is proven to be a *smooth* dg manifold for any $\mathscr{P}_2$ and hence gives canonically rise to an $L_\infty$-structure.

The third major theme of our work is the theory of models and minimal models. To make the deformation complex explicit, we need models, that is quasi-free resolutions of prop(erad)s. We extend the bar and cobar construction to prop(erad)s and show that the bar-cobar construction provides a canonical cofibrant resolution of a properad. Since this construction is not very convenient to work with because it is too big, we would rather use minimal models. We give a complete account to the theory of minimal models for prop(erad)s. We prove that minimal models for prop(erad)s are not in general determined by resolutions of their genus 0 parts, namely dioperads, giving thereby a negative answer to a question raised by M. Markl and A. A. Voronov [29], that is we prove that the free functor from dioperads to prop(erad)s is not exact. We provide an explicit example of a Koszul dioperad which does not induce the prop(erad)ic resolution of the associated prop(erad).

A properad is Koszul if and only if it admits a quadratic model. In this case, Koszul duality theory of properad [42] provides an effective method to compute this special minimal model. Unfortunately, not all properads are Koszul. For instance, the properad encoding associative bialgebras is not. We include this example in a new notion, called *homotopy Koszul*. A homotopy Koszul properad is shown to have a minimal model that can be explicitly computed. Its space of generators is equal to the Koszul dual of a quadratic properad associated to it. And the differential is made explicit by use of the (dual) formulae of J. Granåker [16] of transfer of homotopy coproperad structure, that is by perturbing the differential. We apply this notion to show that morphisms of homotopy $\mathscr{P}$-algebras are in one-to-one correspondence with Maurer-Cartan elements of a convolution $L_\infty$-algebra.

In the appendix of [32], we endow the category of dg prop(erad)s with a model category structure which is used throughout the text.

The paper is organized as follows. In §1 we remind key facts about properads and props and we define the notion of *non-symmetric prop*(*erad*). In §2 we introduce and study the convolution prop(erad) canonically associated with a pair, $(\mathscr{C}, \mathscr{P})$, consisting of an arbitrary coprop(erad) $\mathscr{C}$ and an arbitrary prop(erad) $\mathscr{P}$; our main result is the construction of a Lie algebra structure on this convolution properad, as well as higher operations. In §3 we discuss bar and cobar constructions for (co)prop(erad)s. We introduce the notion of *twisting morphism* (*cochain*) for prop(erads) and prove Theorem 19 on bar-cobar resolutions extending thereby earlier results of [41] from weight-graded dg properads to arbitrary dg properads. In §4 we recall to the notion and properties of homotopy properads which were first introduced in [16] and we define the notions of *homotopy* (*co*)*prop*(*erad*). We apply these notions to convolution prop(erad)s. In §5, we give a complete study of minimal models for properads. In §6 we define the relaxed notion of *homotopy $\mathscr{P}$-gebra* and interpret it in terms of Maurer-Cartan elements.

**Contents**

In this paper, we will always work over a field $\mathbb{K}$ of characteristic 0.

## 1. (Co)properads and (co)props

In this section, we recall briefly the definitions of (co)properad and (co)prop. For the reader who does not know what a properad or what a prop is, we strongly advise to read the first sections of [41] before reading the current article since we will make use of the notions everywhere in the sequel. Generalizing the notion of non-symmetric operads to prop(erad), we introduce the notions of *non-symmetric properad* and *non-symmetric prop*.

**1.1. $\mathbb{S}$-bimodules, graphs, composition products.** A (*dg*) $\mathbb{S}$-*bimodule* is a collection $\mathscr{P} = \{\mathscr{P}(m,n)\}_{m,n\in\mathbb{N}}$ of dg modules over the symmetric groups $\mathbb{S}_n$ on the right and $\mathbb{S}_m$ on the left. These two actions are supposed to commute. In the sequel, we will mainly consider *reduced* $\mathbb{S}$-bimodules, that is $\mathbb{S}$-bimodules $\mathscr{P}$ such that $\mathscr{P}(m,n) = 0$ when $n = 0$ or $m = 0$. We use the homological convention, that is the degree of differentials is $-1$. An $\mathbb{S}$-bimodule $\mathscr{P}$ is *augmented* when it naturally splits as $\mathscr{P} = \bar{\mathscr{P}} \oplus I$ where $I = \{I(m,n)\}$ is an $\mathbb{S}$-bimodule with all components $I(m,n)$ vanishing except for $I(1,1)$ which equals $\mathbb{K}$. We denote the module of morphisms of $\mathbb{S}$-bimodules by $\mathrm{Hom}(\mathscr{P},\mathscr{Q})$ and the module of equivariant morphisms, with respect to the action of the symmetric groups, by $\mathrm{Hom}^{\mathbb{S}}(\mathscr{P},\mathscr{Q})$.

A graph is given by two sets, the set $V$ of vertices and the set $E$ of edges, and relations among which say when an edge is attached to one or two vertices (see [15], (2.5)). The egdes of the graph considered in the sequel will always be directed by a global flow (*directed graphs*). The edges can be divided into two parts: the ones with one vertex at each end, called *internal edges*, and the ones with just one vertex on one end, called *external edges*. The *genus* of a graph is the first Betti number of the underlying topological space of a graph. A 2-*levelled directed graph* is a directed graph such that the vertices are divided into two parts, the ones belonging to a bottom level and the ones belonging to a top level. In the category of $\mathbb{S}$-bimodule, we define two *composition* products, $\boxtimes$ based on the composition of operations indexing the vertices of a 2-levelled directed graph, and $\boxtimes_c$ based on the composition of operations indexing the vertices of a 2-levelled directed connected graph (see Figure 1 for an example). Let $\mathscr{G}$ be such a graph with $N$ internal edges between vertices of the two levels. This set of edges between vertices of the first level and vertices of the second level induces a permutation of $\mathbb{S}_N$.

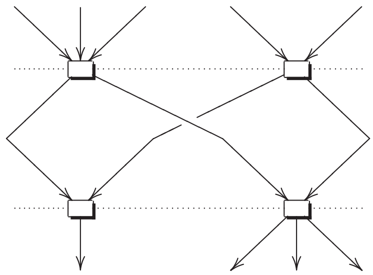



Figure 1. Example of a 2-level graph.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two $\mathbb{S}$-bimodules, their composition product is given by the explicit formula

$$\mathscr{P} \boxtimes \mathscr{Q}(m,n) := \bigoplus_{N \in \mathbb{N}^*} \left( \bigoplus_{\bar{l},\bar{k},\bar{j},\bar{\imath}} \mathbb{K}[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} \mathscr{P}(\bar{l},\bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} \mathbb{K}[\mathbb{S}_N] \otimes_{\mathbb{S}_{\bar{j}}} \mathscr{Q}(\bar{j},\bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} \mathbb{K}[\mathbb{S}_n] \right)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a},$$

where the second direct sum runs over the $b$-tuples $\bar{l}$, $\bar{k}$ and the $a$-tuples $\bar{j}$, $\bar{\imath}$ such that $|\bar{l}| = m$, $|\bar{k}| = |\bar{j}| = N$, $|\bar{\imath}| = n$ and where the coinvariants correspond to the following action of $\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a$:

$$\begin{aligned} &\theta \otimes p_1 \otimes \cdots \otimes p_b \otimes \sigma \otimes q_1 \otimes \cdots \otimes q_a \otimes \omega \\ &\quad \sim \theta\tau_{\bar{l}}^{-1} \otimes p_{\tau^{-1}(1)} \otimes \cdots \otimes p_{\tau^{-1}(b)} \otimes \tau_{\bar{k}}\sigma\nu_{\bar{j}} \otimes q_{\nu(1)} \otimes \cdots \otimes q_{\nu(a)} \otimes \nu_{\bar{\imath}}^{-1}\omega, \end{aligned}$$

for $\theta \in \mathbb{S}_m$, $\omega \in \mathbb{S}_n$, $\sigma \in \mathbb{S}_N$ and for $\tau \in \mathbb{S}_b$ with $\tau_{\bar{k}}$ the corresponding block permutation, $\nu \in \mathbb{S}_a$ and $\nu_{\bar{j}}$ the corresponding block permutation. This product is associative but has no unit. To fix this issue, we restrict to connected graphs.

The permutations of $\mathbb{S}_N$ associated to connected graphs are called *connected* (for more details see [41], Section 1.3). We denote the set of connected permutations by $\mathbb{S}^{\mathrm{c}}$. We define the *connected composition product* by the following formula

$$\mathscr{P} \boxtimes_{\mathrm{c}} \mathscr{Q}(m,n) := \bigoplus_{N \in \mathbb{N}^*} \left( \bigoplus_{\bar{l},\bar{k},\bar{j},\bar{\imath}} \mathbb{K}[\mathbb{S}_m] \otimes_{\mathbb{S}_{\bar{l}}} \mathscr{P}(\bar{l},\bar{k}) \otimes_{\mathbb{S}_{\bar{k}}} \mathbb{K}[\mathbb{S}^{\mathrm{c}}_{\bar{k},\bar{j}}] \otimes_{\mathbb{S}_{\bar{j}}} \mathscr{Q}(\bar{j},\bar{\imath}) \otimes_{\mathbb{S}_{\bar{\imath}}} \mathbb{K}[\mathbb{S}_n] \right)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a}.$$

The unit $I$ for this monoidal product is given by

$$\begin{cases} I(1,1) := \mathbb{K}, & \text{and} \\ I(m,n) := 0 & \text{otherwise.} \end{cases}$$

We denote by $(\mathbb{S}\text{-biMod}, \boxtimes_{\mathrm{c}}, I)$ this monoidal category.

We define the *concatenation product* of two bimodules $\mathscr{P}$ and $\mathscr{Q}$ by

$$\mathscr{P} \otimes \mathscr{Q}(m,n) := \bigoplus_{\substack{m'+m''=m \\ n'+n''=n}} \mathbb{K}[\mathbb{S}_{m'+m''}] \otimes_{\mathbb{S}_{m'} \times \mathbb{S}_{m''}} \mathscr{P}(m',n') \otimes_{\mathbb{K}} \mathscr{Q}(m'',n'') \otimes_{\mathbb{S}_{n'} \times \mathbb{S}_{n''}} \mathbb{K}[\mathbb{S}_{n'+n''}].$$

This product corresponds to taking the (horizontal) tensor product of the elements of $\mathscr{P}$ and $\mathscr{Q}$ (see [41], Figure 3, for an example). It is symmetric, associative and unital. On the contrary to the two previous products, it is linear on the left and on the right. We denote by $\mathscr{S}_\otimes(\mathscr{P})$ the free symmetric monoid generated by an $\mathbb{S}$-bimodule $\mathscr{P}$ for the concatenation product (and $\bar{\mathscr{S}}_\otimes(\mathscr{P})$ its augmentation ideal). There is a natural embedding $\mathscr{P} \boxtimes_c \mathscr{Q} \rightarrowtail \mathscr{P} \boxtimes \mathscr{Q}$. And we obtain the composition product from the connected composition product by concatenation, that is $\bar{\mathscr{S}}_\otimes(\mathscr{P} \boxtimes_c \mathscr{Q}) \cong \mathscr{P} \boxtimes \mathscr{Q}$. (From this relation, we can see that $I \boxtimes \mathscr{P} = \bar{\mathscr{S}}_\otimes(\mathscr{P})$ and not $\mathscr{P}$.)

**1.2. Properad.** A *properad* is a monoid in the monoidal category $(\mathbb{S}\text{-biMod}, \boxtimes_c, I)$. We denote the set of morphisms of properads by $\mathrm{Mor}(\mathscr{P}, \mathscr{Q})$. A properad $\mathscr{P}$ is *augmented* if there exists a morphism of properads $\varepsilon : \mathscr{P} \to I$. We denote by $\bar{\mathscr{P}}$ the kernel of the augmentation $\varepsilon$ and call it the *augmentation ideal*. When $(\mathscr{P}, \mu, \eta, \varepsilon)$ is an augmented properad, $\mathscr{P}$ is canonically isomorphic to $I \oplus \bar{\mathscr{P}}$. We denote by $(I \oplus \underbrace{\bar{\mathscr{P}}}_{r}) \boxtimes_c (I \oplus \underbrace{\bar{\mathscr{P}}}_{s})$ the sub-$\mathbb{S}$-bimodule of $\mathscr{P} \boxtimes_c \mathscr{P}$ generated by compositions of $s$ non-trivial elements of $\mathscr{P}$ on the first level with $r$ non-trivial elements of $\mathscr{P}$ on the second level. The corresponding restriction of the composition product $\mu$ on this sub-$\mathbb{S}$-bimodule is denoted $\mu_{(r,s)}$. The bilinear part of $\mathscr{P} \boxtimes_c \mathscr{P}$ is the $\mathbb{S}$-bimodule $(I \oplus \underbrace{\bar{\mathscr{P}}}_{1}) \boxtimes_c (I \oplus \underbrace{\bar{\mathscr{P}}}_{1})$. It corresponds to the compositions of only 2 non-trivial operations of $\mathscr{P}$. We denote it by $\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}$. The composition of two elements $p_1$ and $p_2$ of $\bar{\mathscr{P}}$ is written $p_1 \boxtimes_{(1,1)} p_2$ to lighten the notations. The restriction $\mu_{(1,1)}$ of the composition product $\mu$ of a properad $\mathscr{P}$ on $\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}$ is called the *partial product*.

A properad is called *reduced* when the underlying $\mathbb{S}$-bimodule is reduced, that is when $\mathscr{P}(m,n) = 0$ for $n = 0$ or $m = 0$.

**1.3. Connected coproperad.** Dually, we defined the notion of *coproperad*, which is a comonoid in $(\mathbb{S}\text{-biMod}, \boxtimes_c)$. Recall that the partial coproduct $\Delta_{(1,1)}$ of a coproperad $\mathscr{C}$ is the projection of the coproduct $\Delta$ on $\mathscr{C} \boxtimes_{(1,1)} \mathscr{C} := (I \oplus \underbrace{\mathscr{C}}_{1}) \boxtimes_c (I \oplus \underbrace{\mathscr{C}}_{1})$. More generally, one can define the $(r,s)$-part of the coproduct by the projection of the image of $\Delta$ on $(I \oplus \underbrace{\mathscr{C}}_{r}) \boxtimes_c (I \oplus \underbrace{\mathscr{C}}_{s})$.

Since the dual of the notion of coproduct is the notion of product, we have to be careful with coproperad. For instance, the target space of a morphism of coproperads is a direct sum of modules and not a product. (The same problem appears at the level of algebras and coalgebras.) We generalize here the notion of *connected* coalgebra, which is the dual notion of Artin rings, introduced by D. Quillen in [35], Appendix B, Section 3 (see also J.-L. Loday and M. Ronco [24], Section 1).

Let $(\mathscr{C}, \Delta, \varepsilon, u)$ be an coaugmented (dg) coproperad. Denote by $\bar{\mathscr{C}} := \mathrm{Ker}(\mathscr{C} \xrightarrow{\varepsilon} I)$ its *augmentation*. We have $\mathscr{C} = \bar{\mathscr{C}} \oplus I$ and $\Delta(I) = I \boxtimes_c I$. For $X \in \bar{\mathscr{C}}$, denote by $\bar{\Delta}$ the non-primitive part of the coproduct, that is $\Delta(X) = I \boxtimes_c X + X \boxtimes_c I + \bar{\Delta}(X)$. The *coradical filtration* of $\mathscr{C}$ is defined inductively as follows:

$$F_0 := \mathbb{K}I,$$

$$F_r := \{X \in \mathscr{C} \mid \bar{\Delta}(X) \in F_{r-1} \boxtimes_c F_{r-1}\}.$$

An augmented coproperad is *connected* if the coradical filtration is exhaustive $\mathscr{C} = \bigcup_{r \geqq 0} F_r$. This condition implies that $\mathscr{C}$ is *conilpotent* which means that for every $X \in \mathscr{C}$, there is an integer $n$ such that $\overline{\Delta}^n(X) = 0$. This assumption is always required to construct morphisms or coderivations between coproperads (see next paragraph and Lemma 15 for instance).

For the same reason, we will sometimes work with the invariant version of the composition product denoted $\mathscr{P} \boxtimes_c^{\mathbb{S}} \mathscr{Q}$ when working with coproperads. It is defined by the same formula as for $\boxtimes_c$ but where we consider the invariant elements under the actions of the symmetric groups instead of the coinvariants (see Lemma 2 for instance). When we want to emphasize the difference between invariants and coinvariants, we use the notations $\boxtimes^{\mathbb{S}}$ and $\boxtimes_{\mathbb{S}}$. Otherwise, we use only $\boxtimes$ since the two are isomorphic in characteristic 0.

**1.4. Free properad and cofree connected coproperad.** Recall from [38] the construction of the free properad. Let $V$ be an $\mathbb{S}$-bimodule. Denote by $V^+ := V \oplus I$ its augmentation and by $V_n := (V^+)^{\boxtimes_c n}$ the $n$-fold "tensor" power of $V^+$. This last module can be thought of as $n$-levelled graphs with vertices indexed by $V$ and $I$. We define on $V_n$ an equivalence relation $\sim$ by identifying two graphs when one is obtained from the other by moving an operation from a level to an upper or lower level. (Note that this permutation of the place of the operations induces signs.) We consider the quotient $\tilde{V}_n := V_n/\sim$ by this relation. Finally, the free properad $\mathscr{F}(V)$ is given by a particular colimit of the $\tilde{V}_n$. The dg $\mathbb{S}$-bimodule $\mathscr{F}(V)$ is generated by graphs without levels with vertices indexed by elements of $V$. We denote such graphs by $\mathscr{G}(v_1, \ldots, v_n)$, with $v_1, \ldots, v_n \in V$. Since $\mathscr{G}(v_1, \ldots, v_n)$ represents an equivalence class of levelled graphs, we can chose, up to signs, an order for the vertices. (Any graph $\mathscr{G}$ with $n$ vertices is the quotient by the relation $\sim$ of a graph with $n$ levels and only one non-trivial vertex on each level.) The composition product of $\mathscr{F}(V)$ is given by the grafting. It is naturally graded by the number of vertices. This grading is called the *weight*. The part of weight $n$ is denoted by $\mathscr{F}(V)^{(n)}$.

Since we are working over a field of characteristic 0, the cofree connected coproperad on a dg $\mathbb{S}$-bimodule $V$ has the same underlying space as the free properad, that is $\mathscr{F}^c(V) = \mathscr{F}(V)$. The coproduct is given by pruning the graphs into two parts. This coproperad verifies the universal property only among connected coproperads (see [41], Proposition 2.7).

**1.5. Props.** We would like to define the notion of *prop* as a monoid in the category of $\mathbb{S}$-bimodules with the composition product $\boxtimes$. Since this last one has no unit and is not a monoidal product, strictly speaking, we have to make this definition explicit.

**Definition** (prop). A *prop* $\mathscr{P}$ is an $\mathbb{S}$-bimodule endowed with two maps $\mathscr{P} \boxtimes \mathscr{P} \xrightarrow{\mu} \mathscr{P}$ and $I \xrightarrow{\eta} \mathscr{P}$ such that the first is associative and the second one verifies

$$\begin{array}{ccccccccc} I\boxtimes_c\mathscr{P} & \rightarrowtail & I\boxtimes\mathscr{P} & \xrightarrow{\eta\boxtimes\mathscr{P}} & \mathscr{P}\boxtimes\mathscr{P} & \xleftarrow{\mathscr{P}\boxtimes\eta} & \mathscr{P}\boxtimes I & \leftarrowtail & \mathscr{P}\boxtimes_c I \\ & \searrow^{\sim} & & & \downarrow{\scriptstyle\mu} & & & \swarrow^{\sim} & \\ & & & & \mathscr{P}. & & & & \end{array}$$

This definition is equivalent to the original definition of Adams and MacLane [1], [25]. The original definition consists of two coherent bilinear products, the vertical and horizontal compositions of operations. The definition of the composition product given here includes these two previous compositions at the same time. The partial product $\mu_{(1,1)} : \mathscr{P} \boxtimes_{(1,1)} \mathscr{P} \to \mathscr{P}$ composes two operations. If they are connected by at least one edge, this composition is the vertical composition, otherwise this composition can be seen as the horizontal composition of operations. This presentation will allow us later to define the bar construction, resolutions and minimal models for props.

It is straightforward to extend the results of the preceding subsections to props. There exist notions of augmented props, free prop, coprop and connected cofree coprop. We refer the reader to [41], Section 2 for a complete treatment.

**1.6. (Co)triple interpretation.** The free prop(erad) functor induces a triple $\mathscr{F} : \mathbb{S} - \mathrm{biMod} \to \mathbb{S} - \mathrm{biMod}$ such that an algebra over it is a prop(erad) (see D. Borisov and Y. I. Manin [6]). When $(\mathscr{P}, \mu)$ is a prop(erad), we will denote by $\tilde{\mu}_{\mathscr{P}} : \mathscr{F}(\mathscr{P})^{(\geqq 2)} \to \mathscr{P}$ the associated map. Dually, the notion of coprop(erad) is equivalent to the notion of coalgebra over the cotriple $\mathscr{F}^{\mathrm{c}} : \mathbb{S} - \mathrm{biMod} \to \mathbb{S} - \mathrm{biMod}$. When $(\mathscr{C}, \Delta)$ is a coprop(erad), we will denote by $\tilde{\Delta}_{\mathscr{C}} : \mathscr{C} \to \mathscr{F}^{\mathrm{c}}(\mathscr{C})^{(\geqq 2)}$ the associated map.

**1.7. Non-symmetric prop(erad).** In the sequel, we will have to work with algebraic structures endowed with operations having no symmetries. One can model them with properads but the action of the symmetric group gives no real information. Therefore, we introduce the notion of *non-symmetric properad* which will be enough. Since this structure is the direct generalization of the notion of non-symmetric operad, we call it *non-symmetric properad*. All the definitions and propositions of this section can be generalized directly to props. For simplicity, we only make them explicit in the case of properads.

**Definition.** A (dg) $\mathbb{N}$-*bimodule* is a collection $\{\mathsf{P}(m,n)\}_{m,n \in \mathbb{N}^*}$ of dg modules.

**Definition** (non-symmetric connected composition product). Let $\mathsf{P}$ and $\mathsf{Q}$ be two $\mathbb{N}$-bimodules, we define their *non-symmetric connected composition product* by the formula

$$\mathsf{P} \boxtimes_{\mathrm{c}} \mathsf{Q}(m,n) := \bigoplus_{N \in \mathbb{N}^*} \left( \bigoplus_{\bar{l}, \bar{k}, \bar{\jmath}, \bar{\imath}} \mathsf{P}(\bar{l}, \bar{k}) \otimes \mathbb{K}[\mathbb{S}^{\mathrm{c}}_{\bar{k}, \bar{\jmath}}] \otimes \mathsf{Q}(\bar{\jmath}, \bar{\imath}) \right)_{\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a},$$

where the second direct sum runs over the $b$-tuples $\bar{l}$, $\bar{k}$ and the $a$-tuples $\bar{\jmath}$, $\bar{\imath}$ such that $|\bar{l}| = m$, $|\bar{k}| = |\bar{\jmath}| = N$, $|\bar{\imath}| = n$ and where the coinvariants correspond to the following action of $\mathbb{S}_b^{\mathrm{op}} \times \mathbb{S}_a$:

$$p_1 \otimes \cdots \otimes p_b \otimes \sigma \otimes q_1 \otimes \cdots \otimes q_a \sim p_{\tau^{-1}(1)} \otimes \cdots \otimes p_{\tau^{-1}(b)} \otimes \tau_{\bar{k}} \sigma \nu_{\bar{\jmath}} \otimes q_{\nu(1)} \otimes \cdots \otimes q_{\nu(a)},$$

for $\sigma \in \mathbb{S}^{\mathrm{c}}_{\bar{k}, \bar{\jmath}}$ and for $\tau \in \mathbb{S}_b$ with $\tau_{\bar{k}}$ the corresponding block permutation, $\nu \in \mathbb{S}_a$ and $\nu_{\bar{\jmath}}$ the corresponding block permutation. Since the context is obvious, we still denote it by $\boxtimes_{\mathrm{c}}$.

The definition of the monoidal product for $\mathbb{S}$-bimodule is based on 2-levelled graphs with leaves, inputs and outputs labelled by integers. This definition is based on non-labelled 2-levelled graphs. We define the *non-symmetric composition product* $\boxtimes$ by the same formula with the set of all permutations of $\mathbb{S}_N$ instead of connected permutations.

**Proposition 1.** *The category* $(\mathbb{N}\text{-biMod}, \boxtimes_c, I)$ *of* $\mathbb{N}$*-bimodules with the product* $\boxtimes_c$ *and the unit* $I$ *is a monoidal category.*

*Proof.* The proof is similar to the one for $\mathbb{S}$-bimodules (see [41], Proposition 1.6). □

**Definition** (non-symmetric properad). A *non-symmetric properad* $(\mathsf{P}, \mu, \eta)$ is a monoid in the monoidal category $(\mathbb{N}\text{-biMod}, \boxtimes_c, I)$.

**Example.** A non-symmetric properad $\mathsf{P}$ concentrated in arity $(1, n)$, with $n \geqq 1$, is the same as a non-symmetric operad.

**1.8. Representations of prop(erad)s, gebras.** Let $\mathscr{P}$ and $\mathscr{Q}$ be two prop(erad)s. A morphism $\mathscr{P} \xrightarrow{\Phi} \mathscr{Q}$ of $\mathbb{S}$-bimodules is a *morphism of prop*(*erad*)*s* if it commutes with the products and the units of $\mathscr{P}$ and $\mathscr{Q}$. In this case, we say that $\mathscr{Q}$ is a *representation* of $\mathscr{P}$.

We will be mainly interested in representations of the following form. Let $X$ be a dg vector space. We consider the $\mathbb{S}$-bimodule $\mathrm{End}_X$ defined by

$$\mathrm{End}_X(m, n) := \mathrm{Hom}_{\mathbb{K}}(X^{\otimes n}, X^{\otimes m}).$$

The natural composition of maps provides this $\mathbb{S}$-bimodule with a structure of prop and properad. It is called the *endomorphism prop*(*erad*) of the space $X$.

Props and properads are meant to model the operations acting on types of algebras or bialgebras in a generalized sense. When $\mathscr{P}$ is a prop(erad), we call $\mathscr{P}$*-gebra* a dg vector space $X$ with a morphism of prop(erad)s $\mathscr{P} \to \mathrm{End}_X$, that is a representation of $\mathscr{P}$ of the form $\mathrm{End}_X$. When $\mathscr{P}$ is an operad, a $\mathscr{P}$-gebra is an algebra over $\mathscr{P}$. To encode operations with multiple inputs and multiple outputs acting on an algebraic structure, we cannot use operads anymore and we need to use prop(erad)s. The categories of (involutive) Lie bialgebras and (involutive) Frobenius bialgebras are categories of gebras over a properad (see Section 5). The categories of (classical) associative bialgebras and infinitesimal Hopf algebras (see [2]) are governed by non-symmetric properads. In these cases, the associated prop is freely obtained from a properad. Therefore, the prop does not model more relations than the properad and the two categories of gebras over the prop and the properad are equal. For more details, see the beginning of Section 5.5.

## 2. Convolution prop(erad)

When $A$ is an associative algebra and $C$ a coassociative coalgebra, the space of morphisms $\mathrm{Hom}_{\mathbb{K}}(C, A)$ from $C$ to $A$ is naturally an associative algebra with the convolution product. We generalize this property to prop(erad)s, that is the space of morphisms $\mathrm{Hom}(\mathscr{C}, \mathscr{P})$ from a coprop(erad) $\mathscr{C}$ and a prop(erad) $\mathscr{P}$ is a prop(erad). From this rich structure, we get general operations, the main one being the *intrinsic* Lie bracket used to study the deformation theory of algebraic structures later in [32], Sections 2 and 3.

**2.1. Convolution prop(erad).** For two $\mathbb{S}$-bimodules

$$M = \{M(m, n)\}_{m,n} \quad \text{and} \quad N = \{N(m, n)\}_{m,n},$$

we denote by $\mathrm{Hom}(M,N)$ the collection $\{\mathrm{Hom}_{\mathbb{K}}(M(m,n),N(m,n))\}_{m,n}$ of morphisms of $\mathbb{K}$-modules. It is an $\mathbb{S}$-bimodule with the action by conjugation, that is

$$(\sigma.f.\tau)(x) := \sigma.\big(f(\sigma^{-1}.x.\tau^{-1})\big).\tau,$$

for $\sigma \in \mathbb{S}_m$, $\tau \in \mathbb{S}_n$ and $f \in \mathrm{Hom}(M,N)(m,n)$. An invariant element under this action is an equivariant map from $M$ to $N$, that is $\mathrm{Hom}(M,N)^{\mathbb{S}} = \mathrm{Hom}^{\mathbb{S}}(M,N)$.

When $\mathscr{C}$ is a coassociative coalgebra and $\mathscr{P}$ is an associative algebra, $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ is an associative algebra known as the *convolution algebra*. When $\mathscr{C}$ is a cooperad and $\mathscr{P}$ is an operad, $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ is an operad called the *convolution operad* by C. Berger and I. Moerdijk in [5], Section 1. We extend this construction to properads and props.

**Lemma 2.** *Let $\mathscr{C}$ be a coprop(erad) and $\mathscr{P}$ be a prop(erad). The space of morphisms $\mathrm{Hom}(\mathscr{C},\mathscr{P}) = \mathscr{P}^{\mathscr{C}}$ is a prop(erad).*

*Proof.* We use the notations of Section 1.1 (see also [41], Section 1.2). We define an associative and unital map $\mu_{\mathscr{P}^{\mathscr{C}}} : \mathscr{P}^{\mathscr{C}} \boxtimes_{\mathbb{S}} \mathscr{P}^{\mathscr{C}} \to \mathscr{P}^{\mathscr{C}}$ as follows. Let

$$\mathscr{G}^2(f_1,\dots,f_r;g_1,\dots,g_s) \in \mathscr{P}^{\mathscr{C}} \boxtimes \mathscr{P}^{\mathscr{C}}(m,n)$$

be a 2-levelled graph whose vertices of the first level are labelled by $f_1,\dots,f_r$ and whose vertices of the second level are labelled by $g_1,\dots,g_s$. The image of $\mathscr{G}^2(f_1,\dots,f_r;g_1,\dots,g_s)$ under $\mu_{\mathscr{P}^{\mathscr{C}}}$ is the composite

$$\mathscr{C} \xrightarrow{\Delta_{\mathscr{C}}} \mathscr{C} \boxtimes^{\mathbb{S}} \mathscr{C} \rightarrowtail \mathscr{C} \boxtimes \mathscr{C} \xrightarrow{\tilde{\mathscr{G}}^2(f_1,\dots,f_r;g_1,\dots,g_s)} \mathscr{P} \boxtimes \mathscr{P} \twoheadrightarrow \mathscr{P} \boxtimes_{\mathbb{S}} \mathscr{P} \xrightarrow{\mu_{\mathscr{P}}} \mathscr{P},$$

where $\tilde{\mathscr{G}}^2(f_1,\dots,f_r;g_1,\dots,g_s)$ applies $f_i$ on the element of $\mathscr{C}$ indexing the $i^{\text{th}}$ vertex of the first level and $g_j$ on the element of $\mathscr{C}$ indexing the $j^{\text{th}}$ vertex of the second level of an element of $\mathscr{C} \boxtimes \mathscr{C}$. Since the action of the symmetric groups on $\mathscr{P}^{\mathscr{C}}$ is defined by conjugation and since the image of the coproduct lives in the space of invariants, this map factors through the coinvariants, that is $\mathscr{P}^{\mathscr{C}} \boxtimes_{\mathbb{S}} \mathscr{P}^{\mathscr{C}} \to \mathscr{P}^{\mathscr{C}}$.

The unit is given by the map $\mathscr{C} \xrightarrow{\varepsilon} I \xrightarrow{\eta} \mathscr{P}$. The associativity of $\mu_{\mathscr{P}^{\mathscr{C}}}$ comes directly from the coassociativity of $\Delta_{\mathscr{C}}$ and the associativity of $\mu_{\mathscr{P}}$. □

**Definition.** The properad $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ is called the *convolution prop(erad)* and is denoted by $\mathscr{P}^{\mathscr{C}}$.

Assume now that $(\mathscr{C},d_{\mathscr{C}})$ is a dg coprop(erad) and $(\mathscr{P},d_{\mathscr{P}})$ is a dg prop(erad). The *derivative* of a graded linear map $f$ from $\mathscr{C}$ to $\mathscr{P}$ is defined as follows:

$$D(f) := d_{\mathscr{P}} \circ f - (-1)^{|f|} f \circ d_{\mathscr{C}}.$$

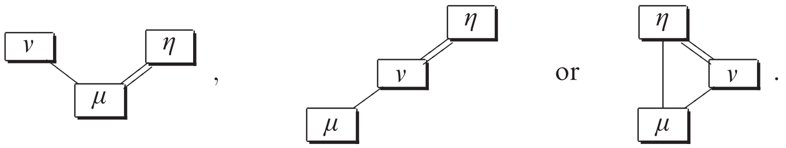
A 0-cycle for this differential is a morphism of chain complexes, that is it commutes with the differentials. In [32], Section 1, we give a geometric interpretation of this derivative. The derivative is a derivation for the product of the prop(erad) $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ that verifies $D^2 = 0$. We sum up these relations in the following proposition. The same result holds in the non-symmetric case.

**Proposition 3.** *When* $(\mathscr{C}, d_{\mathscr{C}})$ *is a* dg *coprop(erad) and* $(\mathscr{P}, d_{\mathscr{P}})$ *is a dg prop(erad),* $\big(\mathrm{Hom}(\mathscr{C}, \mathscr{P}), D\big)$ *is a* dg *prop(erad).*

*When* $(\mathscr{C}, d_{\mathscr{C}})$ *is a* dg *non-symmetric coprop(erad) and* $(\mathscr{P}, d_{\mathscr{P}})$ *is a* dg *non-symmetric prop(erad),* $\big(\mathrm{Hom}(\mathscr{C}, \mathscr{P}), D\big)$ *is a* dg *non-symmetric prop(erad).*

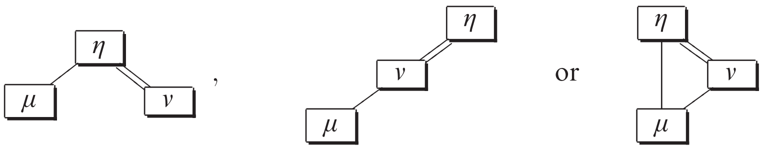
**2.2. Lie-admissible products and Lie brackets associated to a properad.** In [18], the authors proved that the total space $\bigoplus_n \mathscr{P}(n)$, as well as the space of coinvariants $\bigoplus_n \mathscr{P}(n)_{\mathbb{S}_n}$, of an operad is endowed with a natural Lie bracket. This Lie bracket is the anti-symmetrization of the preLie product $p \circ q = \sum_i p \circ_i q$ defined by the sum on all possible ways of composing two operations $p$ and $q$. Notice that this result was implicitly stated by Gerstenhaber in [13]. We generalize this results to properads.

For any pair of elements, $\mu$ and $\nu$, in a (non-symmetric) properad $\mathscr{P}$, we denote by $\mu \circ \nu$ the sum of all the possible compositions of $\mu$ by $\nu$ along any 2-levelled graph with two vertices in $\mathscr{P}$. For another element $\eta$ in $\mathscr{P}$, the composition $(\mu \circ \nu) \circ \eta$ is spanned by graphs of the form



Let us denote by $\mu \circ (\nu, \eta)$ the summand spanned by graphs of the first type.

In the same way, $\mu \circ (\nu \circ \eta)$ is spanned by graphs of the form



and we denote by $(\mu, \nu) \circ \eta$ the summand of $\mu \circ (\nu \circ \eta)$ spanned by graphs of the first (from the left) type. With these notations, we have in $\mathscr{P}$ the formula

$$(\mu \circ \nu) \circ \eta - \mu \circ (\nu \circ \eta) = \mu \circ (\nu, \eta) - (\mu, \nu) \circ \eta.$$

When $\mathscr{P} = A$ is concentrated in arity $(1,1)$, it is an associative algebra. In this case, the product $\circ$ is the associative product of $A$. When $\mathscr{P}$ is an operad, the operation $(\mu, \nu) \circ \eta$ vanishes and the product $\mu \circ \nu$ is right symmetric, that is

$$(\mu \circ \nu) \circ \eta - \mu \circ (\nu \circ \eta) = (\mu \circ \eta) \circ \nu - \mu \circ (\eta \circ \nu).$$

Such a product is called *preLie*. In the general case of properads, this product verifies a weaker relation called *Lie-admissible* because its anti-symmetrized bracket verifies the Jacobi identity. Denote by $\mathrm{As}(\mu, \nu, \eta) := (\mu \circ \nu) \circ \eta - \mu \circ (\nu \circ \eta)$ the associator of $\mu$, $\nu$ and $\eta$.

**Definition** (Lie-admissible algebra). A graded vector space $A$ with a binary product $\circ$ is called a (graded) *Lie-admissible* algebra if one has $\sum_{\sigma \in \mathbb{S}_3} \operatorname{sgn}(\sigma)\, \mathrm{As}(-,-,-)^{\sigma} = 0$, where, for instance, $\mathrm{As}(-,-,-)^{(23)}$ applied to $\mu$, $\nu$ and $\eta$ is equal to

$$(-1)^{|\nu|\,|\eta|}\big((\mu \circ \eta) \circ \nu - \mu \circ (\eta \circ \nu)\big).$$

A *differential graded Lie-admissible algebra* (or dg Lie-admissible algebra for short) is a dg module $(A, d_A)$ endowed with a Lie-admissible product $\circ$ such that the $d_A$ is a derivation.

**Proposition 4.** *Let $\mathscr{P}$ be a* dg *properad or a non-symmetric* dg *properad, the space $\bigoplus_{m,n} \mathscr{P}(m,n)$, endowed with the product $\circ$, is a* dg *Lie-admissible algebra.*

*Proof.* Let $H = \{\mathrm{id}, (23)\}$ and $K = \{\mathrm{id}, (12)\}$ be two subgroups of $\mathbb{S}_3$. We have

$$\begin{aligned}\sum_{\sigma \in \mathbb{S}_3} \operatorname{sgn}(\sigma)\, \mathrm{As}(-,-,-)^{\sigma} &= \sum_{\sigma \in \mathbb{S}_3} \operatorname{sgn}(\sigma)\big((- \circ (- \circ -))^{\sigma} - ((- \circ -) \circ -)^{\sigma}\big)\\ &= \sum_{\tau H \in \mathbb{S}_3 \backslash H} \operatorname{sgn}(\tau)\, \underbrace{\big((- \circ (-,-))^{\tau} - (- \circ (-,-))^{\tau(23)}\big)}_{=0}\\ &\quad - \sum_{\omega K \in \mathbb{S}_3 \backslash K} \operatorname{sgn}(\omega)\, \underbrace{\big(((-,-) \circ -)^{\omega} - ((-,-) \circ -)^{\omega(12)}\big)}_{=0}\\ &= 0. \qquad \square\end{aligned}$$

Actually on the direct sum $\bigoplus_{m,n} \mathscr{P}(m,n)$ of the components of a properad, there are higher operations with $r+s$ inputs which turns it into a "non-differential $B_\infty$-algebra". We refer to the next section for more details.

For a prop $\mathscr{P}$, we still define the product $p \circ q$ on $\bigoplus_{m,n} \mathscr{P}(m,n)$ by all the possible ways of composing the operations $p$ and $q$, that is all vertical composites and the horizontal one.

**Proposition 5.**[1)] *Let $\mathscr{P}$ be a* dg *prop or a non-symmetric* dg *prop, the space $\bigoplus_{m,n} \mathscr{P}(m,n)$, endowed with the product $\circ$, is a* dg *associative algebra.*

*Proof.* We denote by $p \circ_v q$ the sum of all vertical (connected) composites of $p$ and $q$ and by $p \circ_h q$ the horizontal composite. We continue to use the notation $p \circ_v (q, r)$ to represent the composite of an operation $p$ connected to two operations $q$ and $r$ above. We have (in degree 0)

$$\begin{aligned}(p \circ q) \circ r &= (p \circ_v q + p \circ_h q) \circ r\\ &= p \circ_v q \circ_v r + p \circ_v (q, r) + (p \circ_v q) \circ_h r + (p \circ_v r) \circ_h q\\ &\quad + p \circ_h (q \circ_v r) + (p, q) \circ_v r + p \circ_h q \circ_h r,\end{aligned}$$

1) This result was mentioned to the second author by M. M. Kapranov (long time ago).

and

$$\begin{aligned} p \circ (q \circ r) &= p \circ (q \circ_v r + q \circ_h r) \\ &= p \circ_v q \circ_v r + (p, q) \circ_v r + p \circ_h (q \circ_v r) + (p \circ_v q) \circ_h r \\ &\quad + q \circ_h (p \circ_v r) + p \circ_v (q, r) + p \circ_h q \circ_h r. \end{aligned}$$

Since the horizontal product is commutative, $(p \circ_v r) \circ_h q$ is equal to $q \circ_h (p \circ_v r)$, which finally implies $(p \circ q) \circ r = p \circ (q \circ r)$. □

These structures pass to coinvariants $\bigoplus \mathscr{P}_{\mathbb{S}} := \bigoplus_{m,n} \mathscr{P}(m,n)_{\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n}$ as follows.

**Proposition 6.** *Let $\mathscr{P}$ be a* dg *properad* (*respectively* dg *prop*), *the* dg *Lie-admissible* (*associative*) *product $\circ$ on $\bigoplus \mathscr{P}$ induces a* dg *Lie-admissible* (*associative*) *product on the space of coinvariants $\bigoplus \mathscr{P}_{\mathbb{S}}$.*

*Proof.* It is enough to prove that the space

$$C := \{p - \tau.p.\nu;\, p \in \mathscr{P}(m,n), \tau \in \mathbb{S}_m, \nu \in \mathbb{S}_n\}$$

is a two-sided ideal for the Lie-admissible product $\circ$. Let us denote $p \circ q$ by $\sum_{\sigma} \mu(p, \sigma, q)$, where $\mu$ is the composition map of the properad $\mathscr{P}$ and where $\sigma$ runs through connected permutations. For any $\tau \in \mathbb{S}_m$, we have

$$(p - \tau.p) \circ q = \sum_{\sigma} \big(\mu(p, \sigma, q) - \mu(\tau.p, \sigma, q)\big) = \sum_{\sigma} \big(\mu(p, \sigma, q) - \tau_{\sigma}.\mu(p, \sigma, q)\big) \in C,$$

where $\tau_{\sigma}$ is a permutation which depends on $\sigma$. For any $\nu \in \mathbb{S}_n$, we have

$$\begin{aligned} (p - p.\nu) \circ q &= \sum_{\sigma} \mu(p, \sigma, q) - \sum_{\sigma} \mu(p, \nu.\sigma, q) = \sum_{\sigma} \mu(p, \sigma, q) - \sum_{\sigma} \mu(p, \sigma', q).\nu_{\sigma} \\ &= \sum_{\sigma} \big(\mu(p, \sigma, q) - \mu(p, \sigma, q).\nu_{\sigma''}\big) \in C, \end{aligned}$$

since the connected permutations $\sigma'$ obtained runs through the same set of connected permutations as $\sigma$. Therefore, $C$ is a right ideal. The same arguments prove that $C$ is a left ideal. □

In the sequel, we will have to work with the space of invariants

$$\bigoplus \mathscr{P}^{\mathbb{S}} := \bigoplus_{m,n} \mathscr{P}(m,n)^{\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n},$$

and not coinvariants, of a properad. Since we work over a field of characteristic zero, both are canonically isomorphic. Let $V$ be a vector space with an action of a finite group $G$. The subspace of invariants is defined by $V^G := \{v \in V \mid v.g = v, \forall g \in G\}$ and the quotient space of coinvariants is defined by $V_G := V / \langle v - v.g, \forall (v, g) \in V \times G \rangle$. The map from $V^G$ to $V_G$ is the composite of the inclusion $V^G \rightarrowtail V$ followed by the projection $V \twoheadrightarrow V_G$. The inverse map $V_G \to V^G$ is given by $[v] \mapsto \frac{1}{|G|} \sum_{g \in G} v.g$, where $[v]$ denotes the class of $v$ in $V_G$.

**Corollary 7.** *Let $\mathscr{P}$ be a* dg *properad* (*respectively* dg *prop*), *its total space of invariant elements* $\bigoplus \mathscr{P}^{\mathbb{S}}$ *is a* dg *Lie-admissible algebra* (dg *associative algebra*).

The Lie-admissible relation of a product $\circ$ is equivalent to the Jacobi identity $[[-,-],-] + [[-,-],-]^{(123)} + [[-,-],-]^{(132)} = 0$ for its induced bracket

$$[\mu, \nu] := \mu \circ \nu - (-1)^{|\mu|\,|\nu|} \nu \circ \mu.$$

**Theorem 8.** *Let $\mathscr{P}$ be a* dg *properad* (*respectively* dg *prop*), *its total space* $\bigoplus \mathscr{P}$, *the total space of coinvariant elements* $\bigoplus \mathscr{P}_{\mathbb{S}}$ *and the total space of invariant elements* $\bigoplus \mathscr{P}^{\mathbb{S}}$ *are* dg *Lie algebras.*

The first of this statement is also true for non-symmetric dg prop(erad)s.

**2.3. LR-algebra associated to a properad.** On the total space of a properad, we have constructed a binary product $\circ$ in the previous section. We now define more general operations with multiple inputs.

**Definition** (LR-operations). Let $(\mathscr{P}, \mu)$ be a properad and $p_1, \dots, p_r$ and $q_1, \dots, q_s$ be elements of $\mathscr{P}$. Their LR-*operation* $\{p_1, \dots, p_r\}\{q_1, \dots, q_s\}$ is defined by

$$\sum_{\sigma} \mu(p_1, \dots, p_r; \sigma; q_1, \dots, q_s),$$

where $\sigma$ runs through connected permutations.

In other words, the LR-product is the sum over all possible ways to compose the elements of $\mathscr{P}$.

These operations are obviously graded symmetric with respect to Koszul-Quillen sign convention, that is

$$\{p_1, \dots, p_r\}\{q_1, \dots, q_s\} = \varepsilon(\sigma, p_1, \dots, p_r).\varepsilon(\tau, q_1, \dots, q_s)\{p_{\sigma(1)}, \dots, p_{\sigma(r)}\}\{q_{\tau(1)}, \dots, q_{\tau(s)}\},$$

for $\sigma \in \mathbb{S}_r$ and $\tau \in \mathbb{S}_s$. The element $\varepsilon(\sigma, p_1, \dots, p_r) \in \{+1, -1\}$ stands for the Koszul-Quillen signs induced by the permutations of the graded elements $p_1, \dots, p_r$ under $\sigma$. Notice that the Lie-admissible product is equal to $p \circ q := \{p\}\{q\}$. By convention, we set $\{\ \}\{\ \} = 0$, $\{\ \}\{q\} = q$, $\{p\}\{\ \} = p$ and $\{\ \}\{q_1, \dots, q_s\} = 0$ for $s > 1$, $\{p_1, \dots, p_r\}\{\ \} = 0$ for $r > 1$. The name LR-*operations* stands for Left-Right operations as well as for Loday-Ronco operations since such operations are used in [24] to extend the Cartier-Milnor-Moore Theorem to non-cocommutative Hopf algebras.

**Proposition 9.** *The* LR-*operations satisfy the relations of a "non-differential $B_\infty$-algebra", that is, for all $o_1, \dots, o_r$, $p_1, \dots, p_s$, $q_1, \dots, q_t$ in $\mathscr{P}$.*

$$\sum_{\Theta} \varepsilon\{\{o_1, \dots, o_{i_1}\}\{p_1, \dots, p_{j_1}\}, \dots, \{o_{i_1+\dots+i_{a-1}+1}, \dots, o_r\}\{p_{j_1+\dots+j_{a-1}+1}, \dots, p_r\}\}\{q_1, \dots, q_t\}$$

$$= \sum_{\Theta'} \varepsilon'\{o_1, \dots, o_s\}\{\{p_1, \dots, p_{k_1}\}\{q_1, \dots, q_{l_1}\}, \dots, \{p_{k_1+\dots+k_{b-1}+1}, \dots, p_s\}$$

$$\times \{q_{l_1+\dots+l_{b-1}+1}, \dots, q_t\}\},$$

*where* $\Theta$ *runs over* $1 \leqq a \leqq \mathrm{Max}(r,s)$, $i_1, \dots, i_a \geqq 0$ *such that* $i_1 + \cdots + i_a = r$, $j_1, \dots, j_a \geqq 0$ *such that* $j_1 + \cdots + j_a = s$ *and where* $\Theta'$ *runs over* $1 \leqq b \leqq \mathrm{Max}(s,t)$, $k_1, \dots, k_b \geqq 0$ *such that* $k_1 + \cdots + k_b = s$, $l_1, \dots, l_b \geqq 0$ *such that* $l_1 + \cdots + l_b = t$. *The sign* $\varepsilon$ *comes from the permutations of the* $o$ *and the* $p$ *and the sign* $\varepsilon'$ *comes from the permutations of the* $p$ *and the* $q$.

*Proof.* It is a direct translation to LR-operations of the associativity of the operad $\mathscr{P}$. See also, [24], Example 1.7 (d), and Lemma 2. □

Therefore, the total space $\bigoplus \mathscr{P}$ of a properad $\mathscr{P}$, with the LR-operations, forms a "non-differential $B_\infty$", structure that we call an LR-*algebra*. The same result also holds for non-symmetric prop(erad)s.

**Proposition 10.** *Let* $\mathscr{P}$ *be a* dg *prop(erad), its total space* $\bigoplus \mathscr{P}$, *the total space of coinvariant elements* $\bigoplus \mathscr{P}_{\mathbb{S}}$ *and the total space of invariant elements* $\bigoplus \mathscr{P}^{\mathbb{S}}$ *form an* LR-*algebra.*

*Proof.* The structure of LR-algebra of $\bigoplus \mathscr{P}$ factors through the coinvariant elements $\bigoplus \mathscr{P}_{\mathbb{S}}$ by the same arguments as in Proposition 6. Since the space of coinvariant and invariant elements are isomorphic, we can transfer this structure to invariant elements. □

**2.4. Lie-admissible bracket and LR-algebra of a convolution properad.** Since $\mathrm{Hom}(\mathscr{C}, \mathscr{P})$ is a properad, it has a Lie-admissible bracket and more generally it enjoys a structure of LR-algebra by the preceding sections. We make these structures explicit here. We will use them later on in our study of deformation theory (see [32], Sections 2 and 3).

**Definition** (convolution product). Let $f$ and $g$ be two elements of $\mathrm{Hom}(\mathscr{C}, \mathscr{P})$. Their *convolution product* $f \star g$ is defined by the following composite

$$\mathscr{C} \xrightarrow{\Delta_{(1,1)}} \mathscr{C} \boxtimes_{(1,1)} \mathscr{C} \xrightarrow{f \boxtimes_{(1,1)} g} \mathscr{P} \boxtimes_{(1,1)} \mathscr{P} \xrightarrow{\mu} \mathscr{P}.$$

Since the partial coproduct of a coproperad (or a cooperad) is not coassociative in general, the convolution product is not associative.

**Proposition 11.** *Let* $\mathscr{P}$ *be a* dg *prop(erad) and* $\mathscr{C}$ *be a* dg *coprop(erad). The convolution product* $\star$ *on* $\bigoplus \mathrm{Hom}(\mathscr{C}, \mathscr{P})$ *is equal to the product* $\circ$ *associated to the convolution* dg *prop(erad). In the case of* dg *(co)properads, it is* dg *Lie-admissible and for* dg *(co)props, it is* dg *associative.*

*This convolution product is stable on the space of invariant elements* $\bigoplus \mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, \mathscr{P})$ *with respect to the action of the symmetric groups.*

*Proof.* The image of the map $\Delta_{(1,1)}$ is a sum over all possible 2-levelled graphs with two vertices indexed by some elements of $\mathscr{C}$. Therefore, the map $\star$ is equal to the sum of all possible compositions of $f$ and $g$.

Saying that $f$ and $g$ are invariant elements in $\mathrm{Hom}(\mathscr{C}, \mathscr{P})$ means that they are equivariant maps. Since the composition map $\mu$ of $\mathscr{P}$ and the partial coproduct $\Delta_{(1,1)}$ are also equivariant maps, we have

$$(\sigma.f \star g.\tau)(c) = \sigma.\big(f \star g(\sigma^{-1}.c.\tau^{-1})\big).\tau = \sigma.\big(\mu \circ (f \otimes g) \circ \Delta_{(1,1)}(\sigma^{-1}.c.\tau^{-1})\big).\tau$$
$$= \sigma.\sigma^{-1}.(f \star g)(c).\tau^{-1}.\tau = f \star g(c). \quad \square$$

Using the projections $\Delta_{(r,s)}$ of the coproduct of $\mathscr{C}$, we make explicit the LR-operations with $r$ and $s$ inputs of $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ as follows.

**Proposition 12.** *Let $f_1,\dots,f_r$ and $g_1,\dots,g_s$ be elements of* $\mathrm{Hom}(\mathscr{C},\mathscr{P})$. *Their* LR-*operation* $\{f_1,\dots,f_r\}\{g_1,\dots,g_s\}$ *is equal to*

$$\mathscr{C} \xrightarrow{\Delta_{(r,s)}} (I \oplus \underbrace{\mathscr{C}}_{r}) \boxtimes^{\mathbb{S}} (I \oplus \underbrace{\mathscr{C}}_{s})$$
$$\rightarrowtail \underbrace{\mathscr{C} \otimes \cdots \otimes \mathscr{C}}_{r} \boxtimes \underbrace{\mathscr{C} \otimes \cdots \otimes \mathscr{C}}_{s} \xrightarrow{\{f_1,\dots,f_r\}\boxtimes\{g_1,\dots,g_r\}} \mathscr{P} \boxtimes \mathscr{P} \twoheadrightarrow \mathscr{P} \boxtimes_{\mathbb{S}} \mathscr{P} \xrightarrow{\mu} \mathscr{P},$$

*where* $\{f_1,\dots,f_r\} = \sum_{\sigma \in \mathbb{S}_r} \varepsilon(\sigma, f_1,\dots,f_r) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$. *The element*

$$\varepsilon(\sigma, f_1,\dots,f_r) \in \{+1,-1\}$$

*stands for the Koszul-Quillen signs induced by the permutations of the graded elements $f_1,\dots,f_r$ under $\sigma$. This means that we apply $\{f_1,\dots,f_r\}$ and $\{g_1,\dots,g_s\}$ everywhere we can.*

*Proof.* The proof is similar to the previous one. $\square$

**Theorem 13.** *Let $\mathscr{C}$ be a* dg *coprop(erad) and $\mathscr{P}$ be a* dg *prop(erad), the space* $\bigoplus \mathrm{Hom}(\mathscr{C},\mathscr{P})$ *is a* dg LR-*algebra and thus a* dg *Lie algebra, structures that are stable on the space of equivariant maps* $\bigoplus \mathrm{Hom}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$.

*Proof.* Since the $\Delta_{(r,s)}$ and $\mu$ are equivariant maps, the LR-operations are stable on the space of equivariant maps $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$ by their explicit form given in the previous proposition. $\square$

**Remark.** In the case of the convolution properad, we do not have to transfer the structure of LR-algebra or Lie algebra from $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ to $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$ through the coinvariant-invariant isomorphism. These structures are directly stable on the space of invariant elements.

When $\mathscr{C} = C$ is a coassociative coalgebra and $\mathscr{P} = A$ an associative algebra, the product $\boxtimes$ is equal to $\otimes$ and is bilinear. In this case, the partial coproduct of $C$ is equal to the coproduct of $C$ and is coassociative. (All the $\Delta_{(r,s)}$ are null for $r > 1$ or $s > 1$.) In this case, the product $\star$ is the classical convolution product on $\mathrm{Hom}(C, A)$, which is associative.

When $\mathscr{C}$ is a cooperad and $\mathscr{P}$ is an operad we have $\Delta_{(r,s)} = 0$ for $r > 1$ as the operations $\{f_1,\dots,f_r\}\{g_1,\dots,g_s\}$ are null unless $r = 1$. The remaining operations $\{f\}\{g_1,\dots,g_s\}$ are graded symmetric *brace* operations coming from the brace-type relations verified by the operadic product (see [17], [21]). Remark that when $\mathscr{C}$ is a non-symmetric cooperad and $\mathscr{P}$ a non-symmetric operad, we can define non-symmetric braces on $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ without the sum over all permutations. In this case, we find the classical non-symmetric braces of [13], see also [14], [40]. The convolution product verifies the relation $(f\star g)\star h - f\star(g\star h) = \{f\}\{g,h\}$. Therefore, in the operadic case, the (graded) symmetry of the brace products implies that the associator $(f\star g)\star h - f\star(g\star h)$ is symmetric in $g$ and $h$. Hence the convolution product $\star$ on $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ is a graded preLie product. For an interpretation of the LR-operations (or brace operations) on cohomology theories, we refer the reader to [32], Section 2.

## 3. Bar and cobar constructions

In this section, we recall the definitions of the bar and cobar constructions for (co)properads and extend it to (co)props. We prove adjunction between these two constructions using the notion of *twisting morphism*, that is Maurer-Cartan elements in the convolution prop(erad). Finally, we show that the bar-cobar construction provides us with a canonical cofibrant resolution.

**3.1. Infinitesimal bimodule over a prop(erad).** The notion of bimodule $M$ over a prop(erad) $\mathscr{P}$, defined in a categorical way, is given by two compatible actions, left $\mathscr{P}\boxtimes M\to M$ and right $M\boxtimes\mathscr{P}\to M$. For our purposes, we need a *linearized or infinitesimal* version of bimodules. Such a phenomenon cannot be seen on the level of associative algebras since the monoidal product $\otimes$ defining them is bilinear.

The $\mathbb{S}$-bimodule $(M\oplus N)\boxtimes O$ is the sum over 2-levelled graphs with vertices on the top level labelled by elements of $O$ and with vertices on the bottom level labelled by elements of $M$ or $N$. We denote by $(\underbrace{M}_{r}\oplus N)\boxtimes O$ the sub-$\mathbb{S}$-module of $(M\oplus N)\boxtimes O$ with exactly $r$ elements of $M$ on the bottom level.

**Definition** (infinitesimal bimodule). Let $(\mathscr{P},\mu)$ be a prop(erad). An *infinitesimal $\mathscr{P}$-bimodule* is an $\mathbb{S}$-bimodule $M$ endowed with two action maps of degree 0

$$\lambda:\mathscr{P}\boxtimes(\mathscr{P}\oplus\underbrace{M}_{1})\to M\quad\text{and}\quad\rho:(\mathscr{P}\oplus\underbrace{M}_{1})\boxtimes\mathscr{P}\to M,$$

such that the following diagrams commute.

- Compatibility between the left action $\lambda$ and the composition product $\mu$ of $\mathscr{P}$:

$$\begin{array}{ccc}
\mathscr{P}\boxtimes\mathscr{P}\boxtimes(\mathscr{P}\oplus\underbrace{M}_{1}) & \xrightarrow{\mathscr{P}\boxtimes(\lambda+\mu)} & \mathscr{P}\boxtimes(\mathscr{P}\oplus\underbrace{M}_{1}) \\
\Big\downarrow{\scriptstyle \mu\boxtimes(\mathscr{P}\oplus M)} & & \Big\downarrow{\scriptstyle\lambda} \\
\mathscr{P}\boxtimes(\mathscr{P}\oplus\underbrace{M}_{1}) & \xrightarrow{\lambda} & M.
\end{array}$$

- Compatibility between the right action $\rho$ and the composition product $\mu$ of $\mathscr{P}$:

$$\begin{array}{ccc} (\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P} \boxtimes \mathscr{P} & \xrightarrow{(\rho+\mu)\boxtimes\mathscr{P}} & (\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P} \\ \Big\downarrow {\scriptstyle (\mathscr{P}\oplus M)\boxtimes\mu} & & \Big\downarrow {\scriptstyle \rho} \\ (\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P} & \xrightarrow{\rho} & M. \end{array}$$

- Compatibility between the left and the right action:

$$\begin{array}{ccc} \mathscr{P} \boxtimes (\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P} & \xrightarrow{(\lambda+\mu)\boxtimes\mathscr{P}} & (\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P} \\ \Big\downarrow {\scriptstyle \mathscr{P}\boxtimes(\rho+\mu)} & & \Big\downarrow {\scriptstyle \rho} \\ \mathscr{P} \boxtimes (\mathscr{P} \oplus \underbrace{M}_{1}) & \xrightarrow{\lambda} & M. \end{array}$$

The notation $\mathscr{P} \boxtimes \mathscr{P} \boxtimes (\mathscr{P} \oplus \underbrace{M}_{1})$ stands for the sub-$\mathbb{S}$-bimodule of $\mathscr{P} \boxtimes \mathscr{P} \boxtimes (\mathscr{P} \oplus M)$ with only one $M$ on the upper level. It is represented by linear combinations of 3-levelled graphs whose vertices are indexed by elements of $\mathscr{P}$ and just one of $M$ on the first level. The other $\mathbb{S}$-bimodules with just one element coming from $M$ are denoted in the same way, $\mathscr{P} \boxtimes (\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P}$ has one element of $M$ on the second level and $(\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P} \boxtimes \mathscr{P}$ has one element of $M$ on the third level.

One purpose of this notion is to define the notion of *abelian or infinitesimal* extension of a prop(erad) $\mathscr{P}$. It is defined by a prop(erad) structure on $\mathscr{P} \oplus M$, when $M$ is an infinitesimal bimodule over $\mathscr{P}$ (see [32], Section 2.4). Another important property is that, for any sub-$\mathbb{S}$-bimodule $J$ of $\mathscr{P}$, the ideal generated by $J$ in $\mathscr{P}$ is equal to the free infinitesimal $\mathscr{P}$-bimodule on $J$.

Since the prop(erad) $\mathscr{P}$ has a unit, it is equivalent to have two actions $\lambda : \mathscr{P} \boxtimes_{(1,1)} M \to M$ and $\rho : M \boxtimes_{(1,1)} \mathscr{P} \to M$ that are compatible with the composition product of prop(erad) $\mathscr{P}$. Notice that the category of infinitesimal bimodules over a prop(erad) forms an abelian category.

**Example.** Any morphism of prop(erad)s $f : \mathscr{P} \to \mathscr{Q}$ defines an infinitesimal $\mathscr{P}$-bimodule structure on $\mathscr{Q}$:

$$\mathscr{P} \boxtimes_{(1,1)} \mathscr{Q} \xrightarrow{f\boxtimes\mathscr{Q}} \mathscr{Q} \boxtimes_{(1,1)} \mathscr{Q} \xrightarrow{\mu_{\mathscr{Q}}} \mathscr{Q} \quad \text{and} \quad \mathscr{Q} \boxtimes_{(1,1)} \mathscr{P} \xrightarrow{\mathscr{Q}\boxtimes f} \mathscr{Q} \boxtimes_{(1,1)} \mathscr{Q} \xrightarrow{\mu_{\mathscr{Q}}} \mathscr{Q}.$$

**3.2. (Co)derivations.** Let $(\mathscr{P}, \mu)$ be a dg prop(erad) and $(M, \lambda, \rho)$ be an infinitesimal $\mathscr{P}$-bimodule.

**Definition** (derivation). A homogeneous morphism $\partial : \mathscr{P} \to M$ is a *homogeneous derivation* if

$$\partial \circ \mu_{(1,1)}(-,-) = \rho\big(\partial(-),-\big) + \lambda\big(-,\partial(-)\big).$$

This formula, applied to elements $p_1 \boxtimes_{(1,1)} p_2$ of $\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}$, where $p_1$ and $p_2$ are homogeneous elements of $\mathscr{P}$, gives

$$\partial \circ \mu(p_1 \boxtimes_{(1,1)} p_2) = \rho\big(\partial(p_1) \boxtimes_{(1,1)} p_2\big) + (-1)^{|\partial|\,|p_1|}\lambda\big(p_1 \boxtimes_{(1,1)} \partial(p_2)\big).$$

A *derivation* is a sum of homogeneous derivations. The set of homogenous derivations of degree $n$ is denoted by $\mathrm{Der}^n(\mathscr{P}, M)$ and the set of derivations is denoted $\mathrm{Der}^\bullet(\mathscr{P}, M)$.

**Example.** The differential of a dg prop(erad) $\mathscr{P}$ is a derivation of degree $-1$, that is an element of $\mathrm{Der}^{-1}(\mathscr{P}, \mathscr{P})$.

In this section, we only consider derivations $\mathrm{Der}(\mathscr{P}, \mathscr{Q})$, where the infinitesimal $\mathscr{P}$-bimodule structure on $\mathscr{Q}$ is given by a morphism of prop(erad)s $\mathscr{P} \to \mathscr{Q}$. In the rest of the text, we need the following lemma which makes explicit the derivations on a free prop(erad). For a prop(erad) $(\mathscr{Q}, \mu_{\mathscr{Q}})$, any graph $\mathscr{G}$ of $\mathscr{F}(\mathscr{Q})^{(n)}$ represents a class $\overline{\mathscr{G}}$ of levelled graphs of $\mathscr{Q}^{\boxtimes n}$. We recall that there is a morphism $\tilde{\mu}_{\mathscr{Q}} : \mathscr{F}(\mathscr{Q}) \to \mathscr{Q}$, the counit of adjunction, equal to $\tilde{\mu}_{\mathscr{Q}}(\mathscr{G}) := \mu_{\mathscr{Q}}^{\circ(n-1)}(\overline{\mathscr{G}})$. The morphism $\tilde{\mu}_{\mathscr{Q}}$ is the only morphism of prop(erad)s extending the map $\mathscr{Q} \xrightarrow{\mathrm{Id}} \mathscr{Q}$.

**Lemma 14.** *Let $\rho : \mathscr{F}(V) \to \mathscr{Q}$ be a morphism of prop(erad)s of degree 0. Every derivation from the free* dg *prop(erad) $\mathscr{F}(V)$ to $\mathscr{Q}$ is characterized by its restriction on $V$, that is there is a canonical one-to-one correspondence $\mathrm{Der}^n_\rho\big(\mathscr{F}(V), \mathscr{Q}\big) \cong \mathrm{Hom}^{\mathbb{S}}_n(V, \mathscr{Q})$.*

*For every morphism of* dg *$\mathbb{S}$-bimodules $\theta : V \to \mathscr{Q}$, we denote the unique derivation which extends $\theta$ by $\partial_\theta$. The image of an element $\mathscr{G}(v_1, \dots, v_n)$ of $\mathscr{F}(V)^{(n)}$ under $\partial_\theta$ is*

$$\partial_\theta\big(\mathscr{G}(v_1, \dots, v_n)\big) = \sum_{i=1}^{n} (-1)^{|\theta|.(|v_1|+\cdots+|v_{i-1}|)} \tilde{\mu}_{\mathscr{Q}}\Big(\mathscr{G}\big(\rho(v_1), \dots, \rho(v_{i-1}), \theta(v_i), \rho(v_{i+1}), \dots, \rho(v_n)\big)\Big).$$

*Proof.* Let us denote by $\theta$ the restriction of the derivation $\partial$ on $V$, that is $\theta = \partial_V : V \to \overline{\mathscr{Q}}$. From $\theta$, we can construct the whole derivation $\partial$ by induction on the weight $n$ of the free prop(erad) $\mathscr{F}(V)$ as follows.

For $n = 1$, we have $\partial^1_\theta = \theta : V \to \mathscr{Q}$. Suppose now that $\partial^n_\theta : \mathscr{F}(V)^{(n)} \to \mathscr{Q}$ is constructed and it is given by the formula of the lemma. Any simple element of $\mathscr{F}(V)^{(n+1)}$ represented by a graph with $n+1$ vertices indexed by elements of $V$ is the concatenation of a graph with $n$ vertices with an extra vertex from the top or the bottom. In the last case, $\partial^{n+1}_\theta$ is given the commutative diagram

$$\begin{array}{ccc}
\mathscr{F}(V)^{(n+1)} & \xrightarrow{\partial^{n+1}_\theta} & \mathscr{Q} \\
\uparrow{\scriptstyle \mu_{\mathscr{F}(V)}} & & \uparrow{\scriptstyle \mu_{\mathscr{Q}}} \\
V \boxtimes_{(1,1)} \mathscr{F}(V)^{(n)} & \xrightarrow{\rho \boxtimes \partial^n_\theta + \partial^n_\theta \boxtimes \rho} & \mathscr{Q} \boxtimes_{(1,1)} \mathscr{Q}.
\end{array}$$

The other case is dual. It is easy to check that the formula is still true for elements of $\mathscr{F}(V)^{(n+1)}$, that is for graphs with $n+1$ vertices. Finally, since $\rho$ is a morphism of prop(erad)s, $\partial_\theta$ is well defined and is a derivation. □

**Example.** A differential $\partial$ on a free prop(erad) $\mathscr{F}(V)$ is a derivation of $\mathrm{Der}^{-1}_{\mathrm{Id}}\big(\mathscr{F}(V), \mathscr{F}(V)\big)$ such that $\partial^2 = 0$.

**Definition** (quasi-free prop(erad)). A dg prop(erad) $\big(\mathscr{F}(V), \partial\big)$ such that the underlying prop(erad) is free is called a *quasi-free* prop(erad).

Notice that in a quasi-free prop(erad), the differential is not freely generated and is a derivation of the form given above.

Dually, let $(\mathscr{C}, \Delta^{\mathscr{C}})$ and $(\mathscr{D}, \Delta^{\mathscr{D}})$ be two coaugmented dg coprop(erad)s and let $\rho : \mathscr{C} \to \mathscr{D}$ be a morphism of coaugmented dg coprop(erad)s of degree 0. One can define the dual notion of infinitesimal comodule over a coprop(erad) and general coderivations. Since we only need coderivations between two coprop(erad)s, we do not go into such details here.

**Definition** (coderivation). A homogeneous morphism $d : \mathscr{C} \to \mathscr{D}$ is a *homogeneous coderivation* of $\rho$ if the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad d \quad} & \mathscr{D} \\ \Big\downarrow \Delta^{\mathscr{C}}_{(1,1)} & & \Big\downarrow \Delta^{\mathscr{D}}_{(1,1)} \\ \mathscr{C} \boxtimes \mathscr{C} & \xrightarrow{d \boxtimes \rho + \rho \boxtimes d} & \mathscr{D} \boxtimes \mathscr{D}. \end{array}$$

A *coderivation* is a sum of homogeneous coderivations. The space of coderivations is denoted by $\mathrm{Coder}^{\bullet}_{\rho}(\mathscr{C}, \mathscr{D})$.

**Example.** The differential of a dg coprop(erad) $\mathscr{C}$ is a coderivation of degree $-1$.

**Remark.** For a cooperad $\mathscr{D}$, we can define a more general notion of coderivation form a $\mathscr{D}$-cobimodule to $\mathscr{D}$ by a similar formula. The definition given here is a particular case. Since $\rho : \mathscr{C} \to \mathscr{D}$ is a morphism of coprop(erad)s, it provides $\mathscr{C}$ with a natural structure of $\mathscr{D}$-cobimodule.

As explained in the first section, the dual statement of Lemma 14 holds only for connected coprop(erad)s.

**Lemma 15.** *Let $\mathscr{C}$ be a connected coprop(erad) and let $\rho : \mathscr{C} \to \mathscr{F}^{\mathrm{c}}(W)$ be a morphism of augmented coprop(erad)s. Every coderivation from $\mathscr{C}$ to the cofree connected coprop(erad) $\mathscr{F}^{\mathrm{c}}(W)$ is characterized by its projection on $W$, that is there is a canonical one-to-one correspondence* $\mathrm{Coder}^{n}_{\rho}\big(\mathscr{C}, \mathscr{F}^{\mathrm{c}}(W)\big) \cong \mathrm{Hom}^{\mathbb{S}}_{n}(\overline{\mathscr{C}}, W)$.

*Proof.* The proof is similar to the one of Lemma 14 and goes by induction on $r$, where $F_r$ stands for the coradical filtration of $\mathscr{C}$. The assumption that the coprop(erad) $\mathscr{C}$ is connected ensures that the image of an element $X$ of $F_r$ under $d$ lives in $\bigoplus_{n \leqq r} \mathscr{F}^{\mathrm{c}}(W)^{(n)}$. Therefore, $d(X)$ is finite and $d$ is well defined. $\square$

We denote by $d_v$ the unique coderivation which extends a map $v : \overline{\mathscr{C}} \to W$.

**Example.** A differential $d$ on a cofree coprop(erad) $\mathscr{F}^{c}(W)$ is a coderivation of $\mathrm{Der}^{-1}_{\mathrm{Id}}\big(\mathscr{F}^{c}(W),\mathscr{F}^{c}(W)\big)$ such that $d^2=0$. By the preceding lemma, it is characterized by the composite $\mathscr{F}^{c}(W)\xrightarrow{d}\mathscr{F}^{c}(W)\twoheadrightarrow W$. Its explicit formula can be found in Lemma 22.

**Definition** (quasi-cofree coprop(erad)). A dg coprop(erad) $\big(\mathscr{F}^{c}(W),d\big)$ such that the underlying coprop(erad) is connected cofree is called a *quasi-cofree* coprop(erad).

**3.3. (De)suspension.** The homological *suspension* of a dg $\mathbb{S}$-bimodule $M$ is denoted by $sM:=\mathbb{K}s\otimes M$ with $|s|=1$, that is $(sM)_i\cong M_{i-1}$. Dually, the homological *desuspension* of $M$ is denoted by $s^{-1}M:=\mathbb{K}s^{-1}\otimes M$ with $|s^{-1}|=-1$, that is $(s^{-1}M)_i\cong M_{i+1}$.

Let $(\mathscr{P},d)$ be an augmented dg $\mathbb{S}$-bimodule, that is $\mathscr{P}=\overline{\mathscr{P}}\oplus I$. A map of augmented $\mathbb{S}$-bimodules $\mu:\mathscr{F}^{c}(\overline{\mathscr{P}})\to\mathscr{P}$ consists of a family of morphisms of dg $\mathbb{S}$-bimodules $\mu_n:\mathscr{F}^{c}(\mathscr{P})^{(n)}\to\mathscr{P}$ for each integer $n\geqq 1$. (For $n=0$, the map $\mu$ is the identity $I\to I$.) There is a one-to-one correspondence between maps $\{\mathscr{F}^{c}(\overline{\mathscr{P}})\to\mathscr{P}\}$ and maps $\{\mathscr{F}^{c}(s\overline{\mathscr{P}})\to s\mathscr{P}\}$. To each map $\mu:\mathscr{F}^{c}(\overline{\mathscr{P}})\to\mathscr{P}$, we associate the map $s\mu:\mathscr{F}^{c}(s\overline{\mathscr{P}})\to s\mathscr{P}$ defined as follows for $n\geqq 1$,

$$(s\mu)_n:\mathscr{F}^{c}(s\overline{\mathscr{P}})^{(n)}\xrightarrow{\tau_n}s^n\mathscr{F}^{c}(\overline{\mathscr{P}})^{(n)}\xrightarrow{s^{-(n-1)}}s\mathscr{F}^{c}(\overline{\mathscr{P}})^{(n)}\xrightarrow{s\otimes\mu_n}s\mathscr{P},$$

where the map $\tau_n$ moves the place of the suspension elements from the vertices outside the graph. Since it involves permutations between suspensions $s$ and elements of $\mathscr{P}$, the map $\tau_n$ yields signs by Koszul-Quillen rule. Using the fact that an element of $\mathscr{F}^{c}(\overline{\mathscr{P}})$ is an equivalent class of graphs with levels (see 1.4), one can make these signs explicit. The exact formula relating $(s\mu)$ to $\mu$ is

$$\mu\big(\mathscr{G}(p_1,\dots,p_n)\big)=(-1)^{\varepsilon(p_1,\dots,p_n)}s^{-1}(s\mu)\big(\mathscr{G}(sp_1,\dots,sp_n)\big),$$

where $\varepsilon(p_1,\dots,p_n)=(n-1)|p_1|+(n-2)|p_2|+\cdots+|p_{n-1}|$.

The degrees of $\mu$ and $s\mu$ are related by the formula $|(s\mu)_n|=|\mu_n|-(n-1)$. Therefore, the degree of $\mu_n$ is $n-2$ if and only if the degree of $(s\mu)_n$ is $-1$.

Dually, for any map of augmented $\mathbb{S}$-bimodules $\delta:\mathscr{C}\to\mathscr{F}(\overline{\mathscr{C}})$, we denote by $\delta_n$ the composite $\mathscr{C}\xrightarrow{\delta}\mathscr{F}(\overline{\mathscr{C}})\twoheadrightarrow\mathscr{F}(\overline{\mathscr{C}})^{(n)}$. There is a one-to-one correspondence between maps $\{\mathscr{C}\to\mathscr{F}(\overline{\mathscr{C}})\}$ and maps $\{s^{-1}\mathscr{C}\to\mathscr{F}(s^{-1}\overline{\mathscr{C}})\}$. To each map $\delta:\mathscr{C}\to\mathscr{F}(\overline{\mathscr{C}})$, we associate the map $s^{-1}\delta:s^{-1}\mathscr{C}\to\mathscr{F}^{c}(s^{-1}\overline{\mathscr{C}})$ defined as follows, for $n\geqq 1$,

$$(s^{-1}\delta)_n:s^{-1}\mathscr{C}\xrightarrow{s^{-(n-1)}\otimes\delta_n}s^{-n}\mathscr{F}(\overline{\mathscr{C}})^{(n)}\xrightarrow{\sigma_n}\mathscr{F}(s^{(-1)}\overline{\mathscr{C}})^{(n)}.$$

We have $|(s^{-1}\delta)_n|=|\delta_n|-(n-1)$. The degree of $\delta_n$ is $n-2$ if and only if the degree of $(s^{-1}\delta)_n$ is $-1$.

**3.4. Twisting morphism.** We generalize the notion of *twisting morphism* (or twisting cochains) of associative algebras (see E. Brown [7] and J. C. [34]) to prop(erad)s.

Let $\mathscr{C}$ be a dg coprop(erad) and $\mathscr{P}$ be a dg prop(erad). We proved in Theorem 13 that $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$ is a dg Lie-admissible algebra with the convolution product.

**Definition.** A morphism $\mathscr{C} \xrightarrow{\alpha} \mathscr{P}$, of degree $-1$, is called a *twisting morphism* if it is a solution of the *Maurer-Cartan* equation

$$D(\alpha) + \alpha \star \alpha = 0.$$

Denote by $\mathrm{Tw}(\mathscr{C}, \mathscr{P})$ the set of twisting morphisms in $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, \mathscr{P})$, that is Maurer-Cartan elements in the convolution prop(erad). Since twisting morphisms have degree $-1$, it is equivalent for them to be solution of the classical Maurer-Cartan equation in the associated dg Lie algebra, that is $D(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0$.

When $\mathscr{P}$ is augmented and $\mathscr{C}$ coaugmented, we will consider either a twisting morphism between $\mathscr{C}$ and $\mathscr{P}$, which sends $I$ to 0, or the associated morphism which sends $I$ to $I$ and $\overline{\mathscr{C}}$ to $\overline{\mathscr{P}}$.

The following constructions show that the bifunctor $\mathrm{Tw}(-,-)$ can be represented on the left and on the right.

**3.5. Bar construction.** We recall from [41], Section 4, the definition of the *bar construction* for properads and extend it to props. It is a functor

$$B : \{\text{aug. dg prop(erad)s}\} \to \{\text{coaug. dg coprop(erad)s}\}.$$

Let $(\mathscr{P}, \mu, \eta, \epsilon)$ be an augmented prop(erad). Denote by $\overline{\mathscr{P}}$ its augmentation ideal $\mathrm{Ker}(\mathscr{P} \xrightarrow{\epsilon} I)$. The prop(erad) $\mathscr{P}$ is naturally isomorphic to $\mathscr{P} = I \oplus \overline{\mathscr{P}}$. The bar construction $B(\mathscr{P})$ of $\mathscr{P}$ is a dg coprop(erad) whose underlying space is the cofree coprop(erad) $\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$ on the suspension of $\overline{\mathscr{P}}$.

The partial product of $\mathscr{P}$ induces a map of augmented $\mathbb{S}$-bimodules defined by the composite

$$\mu_2 : \overline{\mathscr{F}}^{\mathrm{c}}(\overline{\mathscr{P}}) \twoheadrightarrow \mathscr{F}^{\mathrm{c}}(\overline{\mathscr{P}})^{(2)} \cong \overline{\mathscr{P}} \boxtimes_{(1,1)} \overline{\mathscr{P}} \xrightarrow{\mu_{(1,1)}} \overline{\mathscr{P}}.$$

We have seen in the previous section that $\mu_2$ induces a map $s\mu_2$. Consider the map $\mathbb{K}s \otimes \mathbb{K}s \xrightarrow{\Pi_s} \mathbb{K}s$ of degree $-1$ defined by $\Pi_s(s \otimes s) := s$. The map $s\mu_2$ is equal to the composite

$$\begin{aligned} s\mu_2 : \overline{\mathscr{F}}^{\mathrm{c}}(s\overline{\mathscr{P}}) &\twoheadrightarrow \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})^{(2)} \cong (\mathbb{K}s \otimes \overline{\mathscr{P}}) \boxtimes_{(1,1)} (\mathbb{K}s \otimes \overline{\mathscr{P}}) \\ &\xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}} (\mathbb{K}s \otimes \mathbb{K}s) \otimes (\overline{\mathscr{P}} \boxtimes_{(1,1)} \overline{\mathscr{P}}) \xrightarrow{\Pi_s \otimes \mu_{(1,1)}} \mathbb{K}s \otimes \overline{\mathscr{P}}. \end{aligned}$$

Since $\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$ is a cofree connected coprop(erad), by Lemma 15 there exists a unique coderivation $d_2 := d_{s\mu_2} : \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}) \to \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$ which extends $s\mu_2$. When $(\mathscr{P}, d_{\mathscr{P}})$ is an augmented dg prop(erad), the differential $d_{\mathscr{P}}$ on $\mathscr{P}$ induces an internal differential $d_1$ on $\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$. The total complex of this bicomplex is the *bar construction*

$$B(\mathscr{P}, d_{\mathscr{P}}) := \left(\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}), d = d_1 + d_2\right)$$

of the augmented dg prop(erad) $(\mathscr{P}, d_{\mathscr{P}})$.

Notice that the relation $d^2 = 0$ can be understood conceptually from the Lie-admissible relations verified by the partial product of the prop(erad) $\mathscr{P}$.

**3.6. Cobar construction.** Dually, the *cobar construction* ([41], Section 4) for coprop(eard)s is a functor

$$\Omega : \{\text{coaug. dg coprop(erad)s}\} \to \{\text{aug. dg prop(erad)s}\}.$$

Let $(\mathscr{C}, \Delta, \varepsilon, u)$ be a coaugmented coprop(erad). Denote by $\overline{\mathscr{C}}$ its augmentation $\mathrm{Ker}(\mathscr{C} \xrightarrow{\varepsilon} I)$. In this case, $\mathscr{C}$ splits naturally as $\mathscr{C} = I \oplus \overline{\mathscr{C}}$. The cobar construction $\Omega(\overline{\mathscr{C}})$ of $\overline{\mathscr{C}}$ is a dg prop(erad) whose underlying space is the free prop(erad) $\mathscr{F}(s^{-1}\overline{\mathscr{C}})$ on the desuspension of $\overline{\mathscr{C}}$.

The partial coproduct of $\mathscr{C}$ induces a natural map of augmented $\mathbb{S}$-bimodules defined by

$$\Delta_2 : \overline{\mathscr{C}} \xrightarrow{\Delta_{(1,1)}} \overline{\mathscr{C}} \boxtimes_{(1,1)} \overline{\mathscr{C}} \cong \mathscr{F}(\overline{\mathscr{C}})^{(2)} \rightarrowtail \overline{\mathscr{F}}(\overline{\mathscr{C}}).$$

This map gives a map $s^{-1}\Delta_2 : s^{-1}\overline{\mathscr{C}} \to \mathscr{F}(s^{-1}\overline{\mathscr{C}})$. Consider $\mathbb{K}s^{-1}$ equipped with the diagonal map $\mathbb{K}s^{-1} \xrightarrow{\Delta_s} \mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1}$ of degree $-1$ defined by the formula $\Delta_s(s^{-1}) := s^{-1} \otimes s^{-1}$. The map $s^{-1}\Delta_2$ is equal to

$$s^{-1}\Delta_2 : \mathbb{K}s^{-1} \otimes \overline{\mathscr{C}} \xrightarrow{\Delta_s \otimes \Delta_{(1,1)}} \mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1} \otimes \overline{\mathscr{C}} \boxtimes_{(1,1)} \overline{\mathscr{C}}$$

$$\xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}} (\mathbb{K}s^{-1} \otimes \mathscr{C}) \boxtimes_{(1,1)} (\mathbb{K}s^{-1} \otimes \mathscr{C}) \cong \mathscr{F}(s^{-1}\overline{\mathscr{C}})^{(2)} \rightarrowtail \mathscr{F}(s^{-1}\overline{\mathscr{C}}).$$

Since $\mathscr{F}(s^{-1}\overline{\mathscr{C}})$ is a free prop(erad), by Lemma 14 there exists a unique derivation $\partial_2 := \partial_{s^{-1}\Delta_2} : \mathscr{F}(s^{-1}\overline{\mathscr{C}}) \to \mathscr{F}(s^{-1}\overline{\mathscr{C}})$ which extends $s^{-1}\Delta_2$. When $(\mathscr{C}, d_{\mathscr{C}})$ is an augmented dg coprop(erad), the differential $d_{\mathscr{C}}$ on $\mathscr{C}$ induces an internal differential $\partial_1$ on $\mathscr{F}(s^{-1}\overline{\mathscr{C}})$. The total complex of this bicomplex is the *cobar construction*

$$\Omega(\mathscr{C}, d_{\mathscr{C}}) := \big(\mathscr{F}(s^{-1}\overline{\mathscr{C}}), \partial = \partial_1 + \partial_2\big)$$

of the augmented dg coprop(erad) $(\mathscr{C}, d_{\mathscr{C}})$.

**3.7. Bar-cobar adjunction.** As for derivations, a morphism of prop(erad)s is characterized by the image of the indecomposable elements. We recall this fact and the dual statement in the following lemma.

**Lemma 16.** *Let $V$ be an $\mathbb{S}$-bimodule and let $\mathscr{Q}$ be a prop(erad), there is a canonical one-to-one correspondence* $\mathrm{Mor}_{\mathrm{prop(erad)s}}\big(\mathscr{F}(V), \mathscr{Q}\big) \cong \mathrm{Hom}^{\mathbb{S}}(V, \mathscr{Q})$.

*Dually, let $W$ be an $\mathbb{S}$-bimodule and let $\mathscr{C}$ be a coprop(erad), there is a canonical one-to-one correspondence* $\mathrm{Mor}_{\mathrm{coprop(erad)s}}\big(\mathscr{C}, \mathscr{F}^{\mathrm{c}}(W)\big) \cong \mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, W)$.

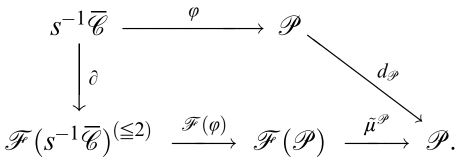
Let $(\mathscr{C}, d_{\mathscr{C}})$ be a dg coprop(erad) and $(\mathscr{P}, d_{\mathscr{P}})$ be a dg prop(erad). We will apply this result to the bar and the cobar construction of $\mathscr{P}$ and $\mathscr{C}$ respectively, that is we want to describe the space of morphisms of **dg**-prop(erad)s $\mathrm{Mor}_{\mathrm{dg\ prop(erad)s}}\big(\Omega(\mathscr{C}), \mathscr{P}\big)$ for instance. By the preceding lemma, this space is isomorphic to the space of morphisms of

$\mathbb{S}$-bimodules $\mathrm{Hom}_0^{\mathbb{S}}(s^{-1}\mathscr{C},\mathscr{P})$ of degree 0 whose unique extension commutes with the differentials. Therefore, this space of morphisms is the subspace of $\mathrm{Hom}_{-1}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$ whose elements satisfy a certain relation, which is exactly the Maurer-Cartan equation.

**Proposition 17.** *For every augmented* dg *prop(erad)* $\mathscr{P}$ *and every coaugmented* dg *coprop(erad)* $\mathscr{C}$, *there are canonical one-to-one correspondences*

$$\mathrm{Mor}_{\mathrm{dg\ prop(erad)s}}\big(\Omega(\mathscr{C}),\mathscr{P}\big) \cong \mathrm{Tw}(\mathscr{C},\mathscr{P}) \cong \mathrm{Mor}_{\mathrm{dg\ coprop(erad)s}}\big(\mathscr{C},B(\mathscr{P})\big).$$

*Proof.* Since $\Omega(\mathscr{C}) = \mathscr{F}\big(s^{-1}(\overline{\mathscr{C}})\big)$, by Lemma 16 every morphism $\varphi$ of $\mathbb{S}$-bimodules in $\mathrm{Hom}_0^{\mathbb{S}}(s^{-1}\mathscr{C},\mathscr{P})$ extends to a unique morphism of prop(erad)s between $\Omega(\mathscr{C})$ and $\mathscr{P}$. The latter one commutes with the differentials if and only if the following diagram commutes:




For an element $c \in \overline{\mathscr{C}}$, we use Sweedler's notation to denote the image of $c$ under $\Delta_2$, that is $\Delta_2(c) = \sum c' \boxtimes_{(1,1)} c''$. The diagram above corresponds to the relation

$$d_{\mathscr{P}} \circ \varphi(s^{-1}c) = \varphi \circ \partial_1(s^{-1}c) + \mu^{\mathscr{P}} \circ (\varphi \boxtimes_{(1,1)} \varphi) \circ s^{-1}\Delta_2(s^{-1}c).$$

Denote by $\alpha$ the desuspension of $\varphi$, that is $\alpha(c) = -\varphi(s^{-1}c)$. Since $\partial_1(s^{-1}c) = -s^{-1}\partial_{\mathscr{C}}(c)$, the relation becomes

$$-d_{\mathscr{P}} \circ \alpha(c) = \alpha \circ \partial_{\mathscr{C}}(c) + \mu^{\mathscr{P}} \circ (\alpha \boxtimes_{(1,1)} \alpha) \circ \Delta_2(c),$$

which is the Maurer-Cartan equation. □

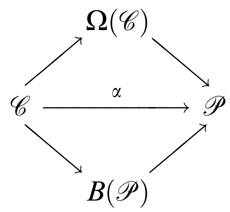
Therefore, the bar and cobar constructions form a pair of adjoint functors

$$\Omega : \{\text{coaug. dg coprop(erad)s}\} \rightleftharpoons \{\text{aug. dg prop(erad)s}\} : B.$$

If we apply the isomorphisms of Proposition 17 to $\mathscr{C} = B(\mathscr{P})$, the morphism associated to the identity on $B(\mathscr{P})$ is the *counit* of the adjunction $\epsilon : \Omega\big(B(\mathscr{P})\big) \to \mathscr{P}$. In this case, we get a universal twisting morphism $B(\mathscr{P}) \to \mathscr{P}$.

The morphism associated to the identity of $\Omega(\mathscr{C})$ when $\mathscr{P} = \Omega(\mathscr{C})$ is the *counit* of the adjunction $\mathscr{C} \to B\big(\Omega(\mathscr{C})\big)$. In this case, we get a universal twisting morphism $\mathscr{C} \to \Omega(\mathscr{C})$.

**Proposition 18.** *Any twisting morphism* $\alpha : \mathscr{C} \to \mathscr{P}$ *factors through* $B(\mathscr{P}) \to \mathscr{P}$ *and* $\mathscr{C} \to \Omega(\mathscr{C})$.



*Proof.* It is a corollary of Proposition 17. □

**3.8. Bar-cobar resolution.** In [41], Theorem 5.8, we proved that the unit of adjunction $\epsilon$ is a canonical resolution in the weight graded case. We extend this result to any dg properad here.

**Theorem 19.** *For every augmented* dg *properad* $\mathscr{P}$, *the bar-cobar construction is a resolution of* $\mathscr{P}$:

$$\epsilon : \Omega\big(B(\mathscr{P})\big) \xrightarrow{\sim} \mathscr{P}.$$

*Proof.* The bar-cobar construction of $\mathscr{P}$ is the chain complex defined on the underlying $\mathbb{S}$-bimodule $\mathscr{F}\big(s^{-1}\overline{\mathscr{F}}^{\,c}(s\overline{\mathscr{P}})\big)$. The differential $d$ is the sum of three terms $d = \partial_2 + d_2 + d_{\mathscr{P}}$, where $d_{\mathscr{P}}$ is induced by the differential on $\mathscr{P}$, $d_2$ is induced by the differential of the bar construction $B(\mathscr{P})$ and $\partial_2$ is the unique derivation which extends the partial coproduct of $\mathscr{F}^{\,c}(s\overline{\mathscr{P}})$.

Define the filtration $F_s := \bigoplus_{r \leqq s} \mathscr{F}\big(s^{-1}\overline{\mathscr{F}}^{\,c}(s\overline{\mathscr{P}})\big)_r$, where $r$ denotes the total number of elements of $\overline{\mathscr{P}}$. Let $E^{\bullet}_{st}$ be the associated spectral sequence.

This filtration is bounded below and exhaustive. Therefore, we can apply the classical convergence theorem for spectral sequences (see [46]) and prove that $E^{\bullet}$ converges to the homology of the bar-cobar construction.

We have that $E^0_{st} = \mathscr{F}_{s+t}\big(s^{-1}\mathscr{F}^{\,c}(s\overline{\mathscr{P}})\big)_s$, where $s+t$ is the total homological degree. From $d_2(F_s) \subset F_{s-1}$, $d_{\mathscr{P}}(F_s) \subset F_s$ and $\partial_2(F_s) \subset F_s$, we get that $d^0 = \partial_2 + d_{\mathscr{P}}$. The problem is now reduced to the computation of the homology of the cobar construction of the dg cofree connected coproperad $\mathscr{F}^{\,c}(s\overline{\mathscr{P}})$ on the dg $\mathbb{S}$-bimodule $s\overline{\mathscr{P}}$. This complex is equal to the bar-cobar construction of the weight graded properad $(\mathscr{P}, \mu')$, where $\mathscr{P}^{(0)} = I$ and $\mathscr{P}^{(1)} = \overline{\mathscr{P}}$, such that the composition $\mu'$ is null. We conclude using [41], Theorem 5.8. □

**Proposition 20.** *The bar-cobar resolution provides a canonical cofibrant resolution to any non-negatively graded* dg *properad.*

We refer the reader to [32], Appendix A, for the model category structure on dg prop(erads).

*Proof.* The bar-cobar resolution is quasi-free. We conclude by [32], Corollary 40. □

## 4. Homotopy (co)prop(erad)s

An associative algebra is a vector space endowed with a binary product that satisfies the strict associativity relation. J. Stasheff defined in [36] a lax version of this notion. It is the notion of an associative algebra up to homotopy or (strong) homotopy algebra. Such an algebra is a vector space equipped with a binary product that is associative only up to an infinite sequence of homotopies. In this section, we recall the generalization of this notion, that is the notion of (*strong*) *homotopy properad* due to J. Granåker [16]. We extend it to props and we also define in details the dual notion of (*strong*) *homotopy coprop*(*erad*), which will be essential to deal with minimal models in the next section. The notions of *homotopy non-symmetric* (*co*)*properad* and *homotopy non-symmetric* (*co*)*prop* are obtained in the same way.

**4.1. Definitions.** Following the same ideas as for algebras (associative or Lie, for instance), we define the notion of *homotopy* (*co*)*prop*(*erad*) via (co)derivations and (co)free (co)prop(erad)s.

**Definition** (homotopy prop(erad)). A structure of *homotopy prop*(*erad*) on an augmented dg $\mathbb{S}$-bimodule $(\mathscr{P}, d_{\mathscr{P}})$ is a coderivation $d$ of degree $-1$ on $\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$ such that $d^2 = 0$.

A structure of homotopy prop(erad) is equivalent to a structure of quasi-cofree coprop(erad) on $s\overline{\mathscr{P}}$. We call the latter the (*generalized*) *bar construction of* $\mathscr{P}$ and we still denote it by $B(\mathscr{P})$. Since $\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$ is a cofree connected coprop(erad), by Lemma 15 the coderivation $d$ is characterized by the composite

$$s\mu : \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}) \xrightarrow{d} \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}) \twoheadrightarrow s\mathscr{P},$$

that is $d = d_{s\mu}$. The map $s\mu$ of degree $-1$ is equivalent to a unique map $\mu : \mathscr{F}^{\mathrm{c}}(\overline{\mathscr{P}}) \to \mathscr{P}$, such that $\mu_n : \mathscr{F}^{\mathrm{c}}(\overline{\mathscr{P}})^{(n)} \to \mathscr{P}$ has degree $n-2$. The condition $d^2 = 0$ written with the $\{\mu_n\}_n$ is made explicit in Proposition 23.

**Example.** A dg prop(erad) is a homotopy prop(erad) such that every map $\mu_n = 0$ for $n \geqq 3$. In this case, $\big(\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}), d\big)$ is the bar construction of $\mathscr{P}$.

We define the notion of *homotopy coprop*(*erad*) by a direct dualization of the previous arguments.

**Definition** (homotopy coprop(erad)). A structure of *homotopy coprop*(*erad*) on an augmented dg $\mathbb{S}$-bimodule $(\mathscr{C}, d_{\mathscr{C}})$ is a derivation $\partial$ of degree $-1$ on $\mathscr{F}(s^{-1}\overline{\mathscr{C}})$ such that $\partial^2 = 0$.

A structure of homotopy coprop(erad) is equivalent to a structure of quasi-free prop(erad) on $s^{-1}\overline{\mathscr{C}}$. We call the latter the (*generalized*) *cobar construction of* $\mathscr{C}$ and we still denote it by $\Omega(\mathscr{C})$. By Lemma 14, the derivation $\partial$ is characterized by its restriction on $s^{-1}\overline{\mathscr{C}}$:

$$s^{-1}\Delta : s^{-1}\overline{\mathscr{C}} \rightarrowtail \overline{\mathscr{F}}(s^{-1}\overline{\mathscr{C}}) \xrightarrow{\partial} \overline{\mathscr{F}}(s^{-1}\overline{\mathscr{C}}),$$

that is $\partial = \partial_{s^{-1}\Delta}$. The map $s^{-1}\Delta$ of degree $-1$ is equivalent to a map $\Delta : \mathscr{C} \to \mathscr{F}(\overline{\mathscr{C}})$, such that the component $\Delta_n : \mathscr{C} \to \mathscr{F}(\overline{\mathscr{C}})^{(n)}$ has degree $n-2$. The condition $\partial^2 = 0$ is equivalent to relations for the $\{\Delta_n\}_n$ that we make explicit in Proposition 24.

**Example.** A dg coprop(erad) is a homotopy coprop(erad) such that every map $\Delta_n = 0$ for $n \geqq 3$. In this case, $\big(\mathscr{F}(s^{-1}\overline{\mathscr{C}}), \partial\big)$ is the cobar construction of $\mathscr{C}$.

When $\mathscr{P}$ is concentrated in arity $(1,1)$, the definition of a homotopy properad on $\mathscr{P}$ is exactly the same as the definition of a strong homotopy algebra given by J. Stasheff in [36]. Dually, when $\mathscr{C}$ is concentrated in arity $(1,1)$, we get the notion of strong homotopy coassociative algebra.

When $\mathscr{P}$ is concentrated in arity $(1,n)$ for $n \geqq 1$, we have the notion of *strong homotopy operad* (see [43]). The dual notion gives the definition of a *strong homotopy cooperad.*

**Remark.** By abstract nonsense, the notion of homotopy prop(erad) should also come from Koszul duality for colored operads (see [44]). There exists a colored operad whose "algebras" are (partial) prop(erad)s. Such a colored operad is quadratic (the associativity relation of the partial product of a prop(erad) is an equation between compositions of two elements). It should be a Koszul colored operad. An "algebra" over the Koszul resolution of this colored operad is exactly a homotopy prop(erad).

**4.2. Admissible subgraph.** Let $\mathscr{G}$ be a connected graph directed by a flow and denote by $\mathscr{V}$ its set of vertices. We define a partial order on $\mathscr{V}$ by the following covering relation: $i \prec j$ if $i$ is below $j$ according to the flow and if there is no vertex between them. In this case, we say that $i$ and $j$ are *adjacent* (see also [41], p. 34). Examples of adjacent and non-adjacent vertices can be found in Figure 2.

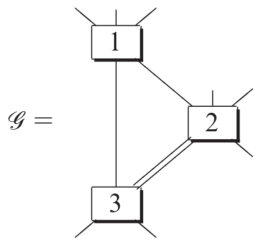



Figure 2. The vertices 1, 2 and 2, 3 are adjacent. The vertices 1 and 3 are not adjacent.

Denote this poset by $\Pi_{\mathscr{G}}$ and consider its Hasse diagram $\mathscr{H}(\mathscr{G})$, that is the diagram composed by the elements of the poset with one edge between two of them, when they are related by a covering relation. See Figure 3 for an example.

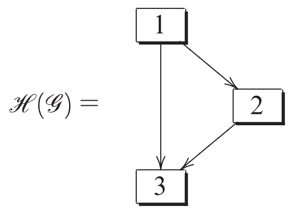



Figure 3. The Hasse diagramm associated to the graph of Figure 2.

Actually, $\mathscr{H}(\mathscr{G})$ is obtained from $\mathscr{G}$ be removing the external edges and by replacing several edges between two vertices by only one edge. Since $\mathscr{G}$ is connected and directed by a flow, the Hasse diagram $\mathscr{H}(\mathscr{G})$ has the same properties. A *convex subset* $\mathscr{V}'$ of $\mathscr{V}$ is a set of vertices of $\mathscr{G}$ such that for every pair $i \leqq j$ in $\mathscr{V}'$ the interval $[i, j]$ of $\Pi_{\mathscr{G}}$ is included in $\mathscr{V}'$. If $\mathscr{G}$ is a connected graph of genus 0, the set of vertices of any connected subgraph of $\mathscr{G}$ is convex. This property does not hold any more for connected graphs of higher genus.

**Lemma 21.** *Let $\mathscr{G}$ be a connected directed graph without oriented loops and let $\mathscr{G}'$ be a connected subgraph of $\mathscr{G}$. The set of vertices of $\mathscr{G}'$ is convex if and only if the contraction of $\mathscr{G}'$ inside of $\mathscr{G}$ gives a graph without oriented loops.*

A connected subgraph $\mathscr{G}'$ with this property is called *admissible* in [16]. We denote by $\mathscr{G}/\mathscr{G}'$ the graph obtained by the contraction of $\mathscr{G}'$ inside $\mathscr{G}$. See Figure 4 for an example of an admissible subgraph and an example of a non-admissible subgraph of $\mathscr{G}$. By extension, an admissible subgraph of a non-necessarily connected graph is a union of admissible subgraphs (eventually empty) of each connected component.

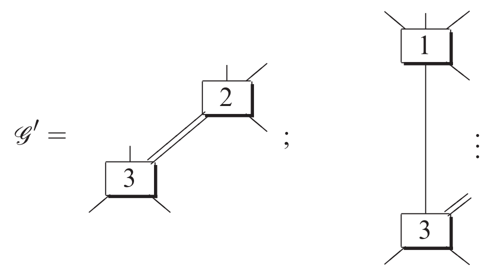



Figure 4. Example of an admissible subgraph $\mathscr{G}'$ of $\mathscr{G}$ and an example of a non-admissible subgraph of $\mathscr{G}$.

**4.3. Interpretation in terms of graphs.** Let $\mu : \mathscr{F}^{c}(\bar{\mathscr{P}}) \to \mathscr{P}$ be a morphism of augmented dg $\mathbb{S}$-bimodules. We denote by $\mu(\mathscr{G}(p_1, \dots, p_n))$ the image of an element $\mathscr{G}(p_1, \dots, p_n)$ of $\mathscr{F}^{c}(\bar{\mathscr{P}})^{(n)}$ under $\mu$. Let $\mathscr{G}'$ be an admissible subgraph of $\mathscr{G}$ with $k$ vertices. Denote by $\mathscr{G}/\mu\mathscr{G}'(p_1, \dots, p_n)$ the element of $\mathscr{F}^{c}(\bar{\mathscr{P}})^{(n-k+1)}$ obtained by composing $\mathscr{G}'(p_{i_1}, \dots, p_{i_k})$ in $\mathscr{G}(p_1, \dots, p_n)$ under $\mu$. When the $p_i$ and $\mu$ are not of degree zero, this composition induces natural signs that we make explicit in the sequel. Let us start with a representative element of a class of graphs $\mathscr{G}(p_1, \dots, p_n)$ whose vertices are indexed by elements $p_i$, that is to say we have chosen an order between the $p_i$ (see Section 1.4). The vertices of $\mathscr{G}'$ are indexed by elements $p_{i_1}, \dots, p_{i_k}$. We denote by $J = (i_1, \dots, i_k)$ the associated ordered subset of $[n] = \{1, \dots, n\}$ and $p_J = p_{i_1}, \dots, p_{i_k}$. Since $\mathscr{G}'$ is an admissible subgraph, its set of vertices forms a convex subset of the set of vertices of $\mathscr{G}$ (or a disjoint union of convex subsets if $\mathscr{G}$ is not connected). Therefore, it is possible to change the order of the vertices of $\mathscr{G}$ such that the vertices of $\mathscr{G}'$ are next to each others. That is there exist two ordered subsets $I_1$ and $I_2$ of $[n]$ such that the underlying subsets $I_1$, $I_2$ and $J$ without order form a partition of $[n]$ and such that $\mathscr{G}(p_1, \dots, p_n) = (-1)^{\varepsilon_1}\mathscr{G}(P_{I_1}, P_J, P_{I_2})$. The sign $(-1)^{\varepsilon_1}$ is given by the Koszul-Quillen sign rule from the permutation of the $p_i$. Now we can apply $\mu$ to get

$$\mathscr{G}/\mu\mathscr{G}'(p_1, \dots, p_n) = (-1)^{\varepsilon_1+\varepsilon_2}\mathscr{G}/\mathscr{G}'\big(P_{I_1}, \mu(\mathscr{G}'(P_J)), P_{I_2}\big),$$

where $\varepsilon_2 = |P_{I_1}|.|\mu|$. It is an easy exercise to prove that this definition of the signs does not depend on the different choices.

**Lemma 22.** *Let $v$ be a map $\mathscr{F}^c(W) \to W$ of degree $-1$. The unique coderivation $d_v \in \mathrm{CoDer}^{-1}_{\mathrm{Id}}\big(\mathscr{F}^c(W), \mathscr{F}^c(W)\big)$ which extends $v$ is given by*

$$d_v\big(\mathscr{G}(w_1, \dots, w_n)\big) = \sum_{\mathscr{G}' \subset \mathscr{G}} \mathscr{G}/v\mathscr{G}'(w_1, \dots, w_n),$$

*where the sum runs over admissible subgraphs $\mathscr{G}'$ of $\mathscr{G}$.*

*Proof.* This formula defines a coderivation. Since the composite of $d_v$ with the projection on $W$ is equal to $v$, we conclude by the uniqueness property of coderivations of Lemma 15. □

**Proposition 23.** *A map $\mu : \mathscr{F}^c(\overline{\mathscr{P}}) \to \mathscr{P}$ defines a structure of homotopy prop(erad) on the augmented* dg $\mathbb{S}$*-bimodule $\mathscr{P}$ if and only if, for every $\mathscr{G}(p_1, \dots, p_n)$ in $\mathscr{F}^c(\overline{\mathscr{P}})$, we have*

$$\sum_{\mathscr{G}' \subset \mathscr{G}} (-1)^{\varepsilon(\mathscr{G}', p_1, \dots, p_n)} \mu\big(\mathscr{G}/\mu\mathscr{G}'(p_1, \dots, p_n)\big) = 0,$$

*where the sum runs over admissible subgraphs $\mathscr{G}'$ of $\mathscr{G}$.*

*Proof.* By definition, $\mu$ induces a structure of homotopy prop(erad) if and only if $d_{s\mu}^2 = 0$. This last condition holds if and only if the composite $\mathrm{proj}_{s\mathscr{P}} \circ d_{s\mu}^2 = (s\mu) \circ d_{s\mu}$ is zero, where $\mathrm{proj}_{s\mathscr{P}}$ is the projection on $s\mathscr{P}$. From Lemma 22, this is equivalent to

$$\sum_{\mathscr{G}' \subset \mathscr{G}} (s\mu)\big(\mathscr{G}/(s\mu)\mathscr{G}'(sp_1, \dots, sp_n)\big) = 0,$$

where the sum runs over admissible subgraphs $\mathscr{G}'$ of $\mathscr{G}$. Recall from Section 3.3 that the signs between $(s\mu)$ and $\mu$ are

$$\mu\big(\mathscr{G}(p_1, \dots, p_n)\big) = (-1)^{\varepsilon(p_1, \dots, p_n)} s^{-1}(s\mu)\big(\mathscr{G}(sp_1, \dots, sp_n)\big),$$

where $\varepsilon(p_1, \dots, p_n) = (n-1)|p_1| + (n-2)|p_2| + \cdots + |p_{n-1}|$. Therefore, $\mu$ induces a structure of homotopy prop(erad) if and only if

$$\sum_{\mathscr{G}' \subset \mathscr{G}} (-1)^{\varepsilon(\mathscr{G}', p_1, \dots, p_n)} \mu\big(\mathscr{G}/\mu\mathscr{G}'(p_1, \dots, p_n)\big) = 0,$$

where $(-1)^{\varepsilon(\mathscr{G}', p_1, \dots, p_n)}$ is product of the sign coming from the composition with $s\mu$ and the sign coming from the formula between $\mu$ and $s\mu$. □

**Remark.** In the case of associative algebras, the graphs involved are ladders (branches, directed graphs just one incoming edge and one outgoing edge for each vertex) and we recover exactly the original definition of J. Stasheff [36].

Dually, we have the following characterization of homotopy coprop(erad)s. Let $\mathscr{G}$ be a graph whose $i^{\text{th}}$ vertex has $n$ inputs and $m$ outputs. For every graph $\mathscr{G}'$ with $n$ inputs and $m$ outputs, denote by $\mathscr{G} \circ_i \mathscr{G}'$ the graph obtained by inserting $\mathscr{G}'$ in $\mathscr{G}$ at the place of the $i^{\text{th}}$ vertex.

**Proposition 24.** *A map* $\Delta : \mathscr{C} \to \mathscr{F}(\overline{\mathscr{C}})$ *defines a structure of homotopy coprop(erad) on the augmented* dg $\mathbb{S}$*-bimodule* $\mathscr{C}$ *if and only if, for every* $c \in \overline{\mathscr{C}}$, *we have*

$$\sum (-1)^{\rho(\mathscr{G}_i^2, c_1, \ldots, c_l)} \mathscr{G}^1 \circ_i \mathscr{G}_i^2(c_1, \ldots, c_{i-1}, c_1', \ldots, c_k', c_{i+1}, \ldots, c_l) = 0,$$

*where the sum runs over elements* $\mathscr{G}^1(c_1, \ldots, c_l)$ *and* $\mathscr{G}_i^2(c_1', \ldots, c_k')$ *such that* $\Delta(c) = \sum \mathscr{G}^1(c_1, \ldots, c_l)$ *and* $\Delta(c_i) = \sum \mathscr{G}_i^2(c_1', \ldots, c_k')$.

*Proof.* By definition, $\Delta$ induces a structure of homotopy coprop(erad) if and only if $\partial_{s^{-1}\Delta}^2 = 0$. Since $\partial_{s^{-1}\Delta}$ is a derivation, $\partial_{s^{-1}\Delta}^2 = 0$ is equivalent to $\partial_{s^{-1}\Delta} \circ (s^{-1}\Delta)(s^{-1}c) = 0$, for every $c \in \overline{\mathscr{C}}$. Denote

$$(s^{-1}\Delta)(s^{-1}c) = \sum \mathscr{G}^1(s^{-1}c_1, \ldots, s^{-1}c_l) \quad \text{and} \quad (s^{-1}\Delta)(s^{-1}c_i) = \sum \mathscr{G}_i^2(s^{-1}c_1', \ldots, s^{-1}c_k').$$

By the explicit formula for $\partial_{s^{-1}\Delta}$ given in Lemma 14 applied to $\rho = \mathrm{Id}_{\mathscr{F}(s^{-1}\overline{\mathscr{C}})}$, we have

$$\begin{aligned} \partial_{s^{-1}\Delta} \circ (s^{-1}\Delta)(s^{-1}c) &= \partial_{s^{-1}\Delta}\big(\sum \mathscr{G}^1(s^{-1}c_1, \ldots, s^{-1}c_l)\big) \\ &= \sum \mathscr{G}^1 \circ_i \mathscr{G}_i^2(s^{-1}c_1, \ldots, s^{-1}c_{i-1}, s^{-1}c_1', \ldots, s^{-1}c_k', s^{-1}c_{i+1}, \ldots, s^{-1}c_l) \\ &= 0. \end{aligned}$$

We get back to the map $\Delta$ with the formula

$$\Delta(c) = (-1)^{\varepsilon(c_1, \ldots, c_l)} \sum \mathscr{G}^1(c_1, \ldots, c_l),$$

where $\varepsilon(c_1, \ldots, c_l) = (l-1)|c_1| + (l-2)|c_2| + \cdots + |c_{l-1}|$. We conclude as in the proof of Proposition 23. □

**4.4. Homotopy non-symmetric (co)prop(erad).** It is straightforward to generalize the two previous subsections to non-symmetric (co)prop(erad)s. One has just to consider non-labelled graphs instead of graphs with leaves, inputs and outputs labelled by integers. Therefore, there is a bar and a cobar construction between non-symmetric dg prop(erad)s and non-symmetric dg coprop(erad)s. The notion that will be used in the sequel is the notion of *homotopy non-symmetric prop*(*erad*). It is defined by a coderivation on the non-symmetric cofree (connected) coprop(erad). Equivalently, we can describe it in terms of non-labelled graphs like in Proposition 23. The chain complex defining the cohomology of a gebra over a non-symmetric prop(erad) has always such a structure (see [32], Section 2).

**4.5. Homotopy properads and associated homotopy Lie algebras.** It was proven in [18] that for any operad, $\mathscr{P} = \{\mathscr{P}(n)\}$, the vector space $\bigoplus_n \mathscr{P}(n)$ has a natural structure of Lie algebra which descends to the space of coinvariants $\bigoplus_n \mathscr{P}(n)_{\mathbb{S}_n}$, which is isomorphic to the space of invariants $\bigoplus_n \mathscr{P}(n)^{\mathbb{S}_n}$. In [43] this result was generalized to homotopy operads and the associated $L_\infty$-algebras. In this section, we further extend the results of [18], [43] from homotopy operads to arbitrary homotopy prop(erad)s: $\mathscr{P} = \{\mathscr{P}(m,n)\}$.

Recall that a structure of $L_\infty$-algebra on $\mathfrak{g}$ is given by a square-zero coderivation on $\mathscr{S}^c(s\mathfrak{g})$, where $\mathscr{S}^c(s\mathfrak{g})$ stands for the cofree cocommutative coalgebra on the suspension of $\mathfrak{g}$. Hence, such a structure is completely characterized by the image of the coderivation on $s\mathfrak{g}$, $\mathscr{S}^c(s\mathfrak{g}) \to s\mathfrak{g}$. Equivalently, an $L_\infty$-algebra is an algebra over the minimal (Koszul) resolution of the operad $\mathscr{L}ie$. We refer the reader to [32], Section 1, for more details on $L_\infty$-algebras.

Let $\mathscr{P}$ be an $\mathbb{S}$-bimodule. We denote by $\bigoplus \mathscr{P}$ the direct sum of all the components of $\mathscr{P}$, that is $\bigoplus_{m,n} \mathscr{P}(m,n)$. We consider the map $\Theta : \mathscr{S}^c(\bigoplus \mathscr{P}) \to \mathscr{F}^c(\mathscr{P})$ defined by $\Theta(p_1 \odot \cdots \odot p_n) := \sum \mathscr{G}(p_1, \ldots, p_n)$, where the sum runs over the classes of graphs under the action of the automorphism group of the graph. This sum is finite and since a graph is a quotient of a levelled graph (see Section 1.4), the signs are well defined.

**Theorem 25.** *Let $\mathscr{P}$ be a homotopy properad, the direct sum $\bigoplus \mathscr{P}$ of its components has an induced $L_\infty$-structure.*

*Proof.* We define the partial cotriple coproduct of a cofree coprop(erad) by the composite:

$$\Delta' : \mathscr{F}^c(V) \xrightarrow{\tilde{\Delta}} \mathscr{F}^c\big(\mathscr{F}^c(V)\big) \twoheadrightarrow \mathscr{F}^c\big(V, \underbrace{\mathscr{F}^c(V)}_{1}\big),$$

where $\mathscr{F}^c\big(V, \underbrace{\mathscr{F}^c(V)}_{1}\big)$ represents graphs indexed by elements of $V$ and one element of $\mathscr{F}^c(V)$. Similarly, we define the partial cotriple coproduct of the cofree cocommutative coalgebra by

$$\delta' : \mathscr{S}^c(V) \xrightarrow{\tilde{\delta}} \mathscr{S}^c\big(\mathscr{S}^c(V)\big) \twoheadrightarrow \mathscr{S}^c\big(V, \underbrace{\mathscr{S}^c(V)}_{1}\big).$$

Let $s\mu : \mathscr{F}^c(s\overline{\mathscr{P}}) \to s\overline{\mathscr{P}}$ be a map of degree $-1$ defining a homotopy properad structure on $\mathscr{P}$, that is the composite

$$\mathscr{F}^c(s\overline{\mathscr{P}}) \xrightarrow{\ \Delta'\ } \mathscr{F}^c\big(s\overline{\mathscr{P}}, \underbrace{\mathscr{F}^c(s\overline{\mathscr{P}})}_{1}\big) \xrightarrow{\mathscr{F}^c(s\overline{\mathscr{P}}, s\mu)} \mathscr{F}^c(s\overline{\mathscr{P}}) \xrightarrow{\ s\mu\ } s\overline{\mathscr{P}}$$

is zero. A map $l : \mathscr{S}^c(s\mathfrak{g}) \to s\mathfrak{g}$ induces a square-zero coderivation on $\mathscr{S}^c(s\mathfrak{g})$ means that the following composite is equal to zero:

$$\mathscr{S}^c(s\mathfrak{g}) \xrightarrow{\ \delta'\ } \mathscr{S}^c\big(s\mathfrak{g}, \underbrace{\mathscr{S}^c(s\mathfrak{g})}_{1}\big) \xrightarrow{\mathscr{S}^c(s\mathfrak{g}, l)} \mathscr{S}^c(s\mathfrak{g}) \xrightarrow{\ l\ } s\mathfrak{g}.$$

We define the induced $L_\infty$-structure by

$$l : \mathscr{S}^c\big(s(\bigoplus \overline{\mathscr{P}})\big) \xrightarrow{\Theta} \mathscr{F}^c(s\overline{\mathscr{P}}) \xrightarrow{s\mu} s\mathscr{P}.$$

The relation of the $L_\infty$-structure for $l$ lifts to the relation of the homotopy prop(erad) by the following commutative diagram:

$$\begin{array}{ccccccc}
\mathscr{S}^c\big(s(\bigoplus\bar{\mathscr{P}})\big) & \xrightarrow{\delta'} & \mathscr{S}^c\big(s(\bigoplus\bar{\mathscr{P}}), \underbrace{\mathscr{S}^c\big(s(\bigoplus\bar{\mathscr{P}})\big)}_{1}\big) & \xrightarrow{\mathscr{S}^c(s\bar{\mathscr{P}}, l)} & \mathscr{S}^c\big(s(\bigoplus\bar{\mathscr{P}})\big) & \xrightarrow{l} & s(\bigoplus\bar{\mathscr{P}}) \\
\downarrow{\scriptstyle\Theta} & & & & & \nearrow{\scriptstyle s\mu} & \\
\mathscr{F}^c(s\bar{\mathscr{P}}) & \xrightarrow{\Delta'} & \mathscr{F}^c\big(s\bar{\mathscr{P}}, \underbrace{\mathscr{F}^c(s\bar{\mathscr{P}})}_{1}\big) & \xrightarrow{\mathscr{F}^c(s\bar{\mathscr{P}}, s\mu)} & \mathscr{F}^c(s\bar{\mathscr{P}}), & &
\end{array}$$

which concludes the proof. □

When $\mathscr{P}$ is a (strict) prop(erad), the induced structure is the (strict) Lie algebra coming from the anti-symmetrization of the Lie-admissible algebra of Proposition 4. Theorem 25 generalizes the well-known fact that a homotopy (associative) algebra is a homotopy Lie algebra by anti-symmetrization of the structure maps.

The same statement holds for the space of coinvariant elements and the space of invariant elements.

**Theorem 26.** *Let $\mathscr{P}$ be a homotopy properad, the total space of coinvariant elements $\bigoplus \mathscr{P}_{\mathbb{S}}$ and the total space of invariant elements $\bigoplus \mathscr{P}^{\mathbb{S}}$ have an induced $L_\infty$-structure.*

*Proof.* We apply the same arguments as in the proof of Proposition 6. □

We prove below that the maps $\mathscr{P} \to \bigoplus \mathscr{P}$ and $\mathscr{P} \to \bigoplus \mathscr{P}^{\mathbb{S}}$ are functors for the category of the homotopy prop(erad)s to one of homotopy Lie algebras (see Proposition 34). The same result holds for non-symmetric homotopy properads as well.

**4.6. Homotopy convolution prop(erad).** In this section, we extend the definition of the convolution prop(erad) to the homotopy case.

**Theorem 27.** *When $(\mathscr{C}, \Delta)$ is a (non-symmetric) homotopy coprop(erad) and $(\mathscr{P}, \mu)$ is a (non-symmetric) prop(erad), the convolution prop(erad) $\mathscr{P}^{\mathscr{C}} = \mathrm{Hom}(\mathscr{C}, \mathscr{P})$ is a homotopy (non-symmetric) prop(erad).*

*The same result holds when $\mathscr{C}$ is a (non-symmetric) coprop(erad) and $\mathscr{P}$ a homotopy (non-symmetric) prop(erad).*

*Proof.* To an element $\mathscr{G}(f_1, \dots, f_n)$ of $\mathscr{F}^c(\bar{\mathscr{P}}^{\bar{\mathscr{C}}})^{(n)}$, we consider the map

$$\tilde{\mathscr{G}}(f_1, \dots, f_n) : \mathscr{F}(\bar{\mathscr{C}})^{(n)} \to \mathscr{F}^c(\bar{\mathscr{P}})^{(n)}$$

defined by $\mathscr{G}'(c_1, \dots, c_n) \mapsto (-1)^{\xi}\mathscr{G}\big(f_1(c_1), \dots, f_n(c_n)\big)$ if $\mathscr{G}' \cong \mathscr{G}$ and 0 otherwise, where $\xi = \sum_{i=2}^{n} |f_i|(|c_1| + \cdots + |c_{i-1}|)$. We define maps $\mu_n : \mathscr{F}^c(\bar{\mathscr{P}}^{\bar{\mathscr{C}}})^{(n)} \to \mathscr{P}^{\mathscr{C}}$ by the formula

$$\mu_n\big(\mathscr{G}(f_1, \dots, f_n)\big) := \tilde{\mu}_{\mathscr{P}} \circ \tilde{\mathscr{G}}(f_1, \dots, f_n) \circ \Delta_n.$$

The degree of $\Delta_n$ is $n-2$ and the degree of $\tilde{\mu}_{\mathscr{P}}$ is zero. Therefore, the degree of $\mu_n$ is $n-2$.

The map $\mu$ verifies the relation of Proposition 23:

$$\sum_{\mathscr{G}'\subset\mathscr{G}} \pm\mu\big(\mathscr{G}/\mu\mathscr{G}'(f_1,\dots,f_n)\big)$$

$$= \sum \pm\tilde{\mu}_{\mathscr{P}} \circ \widetilde{\mathscr{G}/\mathscr{G}'}\big(f_1,\dots,\mu_k\big(\mathscr{G}'(f_{i_1},\dots,f_{i_k})\big),\dots,f_n\big)\circ\Delta_l$$

$$= \sum \pm\tilde{\mu}_{\mathscr{P}} \circ \widetilde{\mathscr{G}/\mathscr{G}'}\big(f_1,\dots,\tilde{\mu}_{\mathscr{P}}\circ\widetilde{\mathscr{G}'}(f_{i_1},\dots,f_{i_k})\circ\delta_k,\dots,f_n\big)\circ\Delta_l,$$

where the sum runs over admissible subgraphs $\mathscr{G}'$ of $\mathscr{G}$. We denote by $k$ the number of vertices of $\mathscr{G}'$ and $l = n-k+1$. We use the generic notation $i$ for the new vertex of $\mathscr{G}/\mathscr{G}'$ obtained after contracting $\mathscr{G}'$. For every element $c\in\overline{\mathscr{C}}$, we denote by $\Delta(c) = \sum \mathscr{G}^1(c_1,\dots,c_l)$ and $\Delta(c_i) = \sum \mathscr{G}_i^2(c_1',\dots,c_k')$. The associativity of the product of $\mathscr{P}$ gives

$$\sum_{\mathscr{G}'\subset\mathscr{G}} (-1)^{\varepsilon(\mathscr{G}',f_1,\dots,f_n)}\,\mu\big(\mathscr{G}/\mu\mathscr{G}'(f_1,\dots,f_n)\big)(c)$$

$$= \tilde{\mu}_{\mathscr{P}}\circ\tilde{\mathscr{G}}(f_1,\dots,f_n)\circ\Big(\sum(-1)^{\rho(\mathscr{G}_i^2,c_1,\dots,c_l)}\,\mathscr{G}^1\circ_i\mathscr{G}_i^2(c_1,\dots,c_1',\dots,c_k',\dots,c_l)\Big).$$

Since $(\mathscr{C},\Delta)$ is a homotopy coprop(erad), the last term vanishes by Proposition 24.

The same statement in the non-symmetric case is proven in the same way and the dual statement also. □

**Remark.** In the particular case when $\mathscr{C}$ is a homotopy coalgebra and $\mathscr{P}$ an associative algebra, $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ is a homotopy algebra. In the same way, when $\mathscr{C}$ is a homotopy operad and $\mathscr{P}$ an operad, $\mathrm{Hom}(\mathscr{C},\mathscr{P})$ is a homotopy operad (see [43], Lemma 5.10).

**Theorem 28.** *When $(\mathscr{C},\Delta)$ is a homotopy coprop(erad) and $(\mathscr{P},\mu)$ is a prop(erad) (or when $(\mathscr{C},\Delta)$ is a coprop(erad) and $(\mathscr{P},\mu)$ is a homotopy prop(erad)), the total space of the convolution prop(erad) $\mathscr{P}^{\mathscr{C}} = \mathrm{Hom}(\mathscr{C},\mathscr{P})$ is a homotopy Lie algebra.*

*The total subspace $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$ of invariant elements is a sub-$L_\infty$-algebra.*

*Proof.* The first part is a direct corollary of Theorem 27 and Theorem 25. Since the structure maps of this $L_\infty$-algebra are composite of equivariant maps ($\Delta_n$, $\tilde{\mu}_{\mathscr{P}}$), they induce an $L_\infty$-algebra structure on the total space of $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$. (This is similar to the one used in the proof of Proposition 11.) □

In the latter case, the $L_\infty$-'operations' or homotopies are explicitly given by the following formula. The image of $f_1,\dots,f_n\in\mathrm{Hom}^{\mathbb{S}}(\mathscr{C},\mathscr{P})$ under $l_n$, for $n>1$, is given by

$$l_n(f_1,\dots,f_n) = \sum_{\sigma\in\mathbb{S}_n}(-1)^{\mathrm{sgn}(\sigma,f_1,\dots,f_n)}\tilde{\mu}_{\mathscr{P}}\circ(f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(n)})\circ\Delta_n,$$

where $(-1)^{\mathrm{sgn}(\sigma,f_1,\dots,f_n)}$ is the Koszul-Quillen sign appearing after permutating the $f_i$ with $\sigma$. The first 'operation' $l_1$ is the differential, that is $l_1(f) := D(f) = d_{\mathscr{P}}\circ f - (-1)^{|f|}f\circ d_{\mathscr{C}}$.

In this homotopy Lie algebra, the generalized Maurer-Cartan equation is well defined since the formal infinite sum $Q(\alpha) := \sum_{n \geqq 1} \frac{1}{n!} l_n(\alpha, \dots, \alpha)$ is in fact equal to the composite $D + \tilde{\mu}_{\mathscr{P}} \circ \mathscr{F}(\alpha) \circ \Delta$ in $\mathrm{Hom}(\mathscr{C}, \mathscr{P})$, when $\mathscr{C}$ is a homotopy coprop(erad) and to $D + \mu \circ \mathscr{F}(\alpha) \circ \tilde{\Delta}_{\mathscr{C}}$ when $\mathscr{P}$ is a homotopy prop(erad). (See [32], Section 1.3, for the general definition of filtered $L_\infty$-algebras.)

**Definition.** Let $(\mathscr{C}, \Delta)$ be a homotopy coprop(erad) and $(\mathscr{P}, \mu)$ be a prop(erad) (or $(\mathscr{C}, \Delta)$ a coprop(erad) and $(\mathscr{P}, \mu)$ a homotopy properad). A morphism $\mathscr{C} \xrightarrow{\alpha} \mathscr{P}$, of degree $-1$, is called a *twisting morphism* if it is a solution of the (generalized) Maurer-Cartan equation

$$Q(\alpha) := \sum_{n \geqq 1} \frac{1}{n!} l_n(\alpha, \dots, \alpha) = 0,$$

in the homotopy Lie algebra $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, \mathscr{P})$. We denote this set by $\mathrm{Tw}(\mathscr{C}, \mathscr{P})$.

We can represent the bifunctor $\mathrm{Tw}(-,-)$ in the same as in the strict case (see Proposition 17).

**Proposition 29.** *Let $(\mathscr{C}, \Delta)$ be a homotopy coprop(erad) and $(\mathscr{P}, \mu)$ be a prop(erad). There is a natural bijection*

$$\mathrm{Mor}_{\mathrm{dg\ prop(erad)s}}\big(\Omega(\mathscr{C}), \mathscr{P}\big) \cong \mathrm{Tw}(\mathscr{C}, \mathscr{P}).$$

*Let $(\mathscr{C}, \Delta)$ be a coprop(erad) and $(\mathscr{P}, \mu)$ be a homotopy prop(erad). There is a natural bijection*

$$\mathrm{Tw}(\mathscr{C}, \mathscr{P}) \cong \mathrm{Mor}_{\mathrm{dg\ coprop(erad)s}}\big(\mathscr{C}, \mathscr{B}(\mathscr{P})\big).$$

*Proof.* The proof is a direct generalization of the proof of Proposition 17. □

**4.7. Morphisms of homotopy (co)prop(erad)s.** In this section, we recall the notion of morphism between two homotopy properads due to [16]. We extend it to homotopy (co)props and make them explicit in terms of Maurer-Cartan elements in some convolution $L_\infty$-algebra.

Since a homotopy properad is equivalent to its associated (generalized) bar construction, the notion of *morphism of homotopy properads* (or *weak morphism*) is defined as follows.

**Definition** ([16]). Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two homotopy prop(erad)s. A morphism between $\mathscr{P}_1$ and $\mathscr{P}_2$ is a morphism of dg coprop(erad)s between their bar constructions: $B(\mathscr{P}_1) \to B(\mathscr{P}_2)$.

A morphism of dg coprop(erad)s $\Phi : B(\mathscr{P}_1) = \mathscr{F}^c(s\overline{\mathscr{P}}_1) \to B(\mathscr{P}_2) = \mathscr{F}^c(s\overline{\mathscr{P}}_2)$ is characterized by its image on $s\overline{\mathscr{P}}_2$. We denote by $s^{-1}\varphi : B(\mathscr{P}_1) \to \overline{\mathscr{P}}_2$ the composite of $\Phi$ with the projection on $s\overline{\mathscr{P}}_2$ followed by the desuspension. Notice that the degree of $s^{-1}\varphi$ is $-1$.

By Proposition 29, $\Phi$ is a morphism of dg coprop(erad)s if and only if $s^{-1}\varphi$ is a Maurer-Cartan element in $\mathrm{Hom}^{\mathbb{S}}\big(B(\mathscr{P}_1), \mathscr{P}_2\big)$, that is

$$Q(s^{-1}\varphi) = \sum_{n \geqq 1} \frac{1}{n!} l_n(s^{-1}\varphi, \dots, s^{-1}\varphi) = D(s^{-1}\varphi) + \mu_{\mathscr{P}_2} \circ \mathscr{F}^{\mathrm{c}}(s^{-1}\varphi) \circ \tilde{\Delta} = 0,$$

where $\tilde{\Delta}$ is the coproduct map $B(\mathscr{P}_1) = \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}_1) \to \mathscr{F}^{\mathrm{c}}\big(\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}_1)\big)$.

**Proposition 30.** *A morphism of* $\mathbb{S}$*-bimodules* $\varphi : B(\mathscr{P}_1) \to s\overline{\mathscr{P}}_2$ *induces a morphism of homotopy properads between* $\mathscr{P}_1$ *and* $\mathscr{P}_2$ *if and only if* $s^{-1}\varphi$ *is a Maurer-Cartan element in the* $L_\infty$*-algebra* $\mathrm{Hom}^{\mathbb{S}}\big(B(\mathscr{P}_1), \mathscr{P}_2\big)$, *that is* $Q(s^{-1}\varphi) = 0$.

Like in Section 4.3, we make explicit the above definition in terms of graphs.

**Proposition 31.** *A map* $s^{-1}\varphi : B(\mathscr{P}_1) \to \overline{\mathscr{P}}_2$ *is a morphism of homotopy prop(erad)s if and only if, for every class of graphs* $\mathscr{G}$ *under the action of the automorphism group, the following relation holds*:

$$\sum s\mu_k^{\mathscr{P}_2}(\mathscr{G}/\varphi\mathscr{G}_1 \sqcup \cdots \sqcup \varphi\mathscr{G}_k) = \sum \varphi\big(\mathscr{G}/(s\mu^{\mathscr{P}_1})\mathscr{G}'\big),$$

*where the first sum runs over all partitions of the graph* $\mathscr{G}$ *into admissible subgraphs* $\mathscr{G}_1 \sqcup \cdots \sqcup \mathscr{G}_k$ *and where the second sum runs over all admissible subgraphs* $\mathscr{G}'$ *of* $\mathscr{G}$. *Once again, the signs are induced by Koszul-Quillen rule, when applied to elements* $sp_1, \dots, sp_n$, *such that* $n$ *is the number of vertices of* $\mathscr{G}$.

*Proof.* The map $s^{-1}\varphi : B(\mathscr{P}_1) \to \overline{\mathscr{P}}_2$ induces a unique morphism of coprop(erad)s $\Phi : B(\mathscr{P}_1) \to B(\mathscr{P}_2)$ which commutes with the differentials if and only if the above relation is verified. (The left-hand term is the projection on $\overline{\mathscr{P}}_2$ of the composite $d_{B(\mathscr{P}_2)} \circ \Phi$ and the right-hand term is the projection on the same space of the composite $\Phi \circ d_{B(\mathscr{P}_1)}$, that is $\varphi \circ d_{B(\mathscr{P}_1)}$.) □

When applied to $A_\infty$-algebras, the underlying graphs are ladders and this proposition gives the classical notion of weak morphisms, that is morphisms between $A_\infty$-algebras.

Dually, we define the notion of morphisms between homotopy coprop(erad)s.

**Definition.** Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two homotopy prop(erad)s. A morphism between $\mathscr{C}_1$ and $\mathscr{C}_2$ is a morphism of dg prop(erad)s between their cobar constructions: $\Omega(\mathscr{C}_1) \to \Omega(\mathscr{C}_2)$.

A morphism of dg prop(erad)s $\Psi : \Omega(\mathscr{C}_1) = \mathscr{F}(s^{-1}\overline{\mathscr{C}}_1) \to \Omega(\mathscr{C}_2) = \mathscr{F}(s^{-1}\overline{\mathscr{C}}_2)$ is characterized by the image of $s^{-1}\overline{\mathscr{C}}_1$. We denote by $s^{-1}\psi : \overline{\mathscr{C}}_1 \to \Omega(\mathscr{C}_2)$ the desuspension of the restriction of $\Psi$ on $s^{-1}\overline{\mathscr{C}}_1$. By Proposition 29, $\Psi$ is a morphism of dg prop(erad)s if and only if $s^{-1}\psi$ is a twisting morphism in $\mathrm{Hom}^{\mathbb{S}}\big(\overline{\mathscr{C}}_1, \Omega(\mathscr{C}_2)\big)$, that is

$$Q(s^{-1}\psi) = \sum_{n \geqq 1} \frac{1}{n!} l_n(s^{-1}\psi, \dots, s^{-1}\psi) = D(s^{-1}\psi) + \tilde{\mu} \circ \mathscr{F}(s^{-1}\psi) \circ \Delta_{\mathscr{C}_1} = 0,$$

where $\tilde{\mu}$ is the composition map $\mathscr{F}\big(\Omega(\mathscr{C}_2)\big) = \mathscr{F}\big(\mathscr{F}(s^{-1}\overline{\mathscr{C}}_2)\big) \to \mathscr{F}(s^{-1}\overline{\mathscr{C}}_2) = \Omega(\mathscr{C}_2)$.

**Proposition 32.** *A morphism of* $\mathbb{S}$*-bimodules* $\psi : s^{-1}\overline{\mathscr{C}}_1 \to \Omega(\mathscr{C}_2)$ *induces a morphism of homotopy coproperads between* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *if and only if* $s^{-1}\psi$ *is a Maurer-Cartan element in the* $L_\infty$*-algebra* $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}_1, \Omega(\mathscr{C}_2))$, *that is* $Q(s^{-1}\psi) = 0$.

We now prove that the convolution prop(erad) is a construction functorial with respect to the first argument.

**Theorem 33.** *Let* $\Psi$ *be a morphism of homotopy coprop(erad)s between* $\mathscr{C}_1$ *and* $\mathscr{C}_2$. *Let* $\mathscr{P}$ *be a prop(erad). There exists a natural morphism of homotopy prop(erad)s between* $\mathrm{Hom}(\mathscr{C}_2, \mathscr{P})$ *and* $\mathrm{Hom}(\mathscr{C}_1, \mathscr{P})$ *induced by* $\Psi$.

The same statement holds in the non-symmetric case.

*Proof.* Let $\Psi$ denote the morphism of dg prop(erads) $\Omega(\mathscr{C}_1) \to \Omega(\mathscr{C}_1)$ and $s^{-1}\psi$ the induced twisting morphism $\overline{\mathscr{C}}_1 \to \Omega(\mathscr{C}_2)$, that is $Q(s^{-1}\psi) = 0$. We define the morphism of coprop(erad)s $\Phi : B(\mathrm{Hom}(\mathscr{C}_2, \mathscr{P})) \to B(\mathrm{Hom}(\mathscr{C}_1, \mathscr{P}))$ by its image $\varphi$ on $s\,\overline{\mathrm{Hom}}(\mathscr{C}_1, \mathscr{P}) = \mathrm{Hom}(s^{-1}\overline{\mathscr{C}}_1, \overline{\mathscr{P}})$ as follows. Let

$$\mathscr{G}(f_1, \dots, f_n) \in B(\mathrm{Hom}(\mathscr{C}_2, \mathscr{P})) = \mathscr{F}^{\mathrm{c}}(s\,\overline{\mathrm{Hom}}(\mathscr{C}_1, \mathscr{P})) = \mathscr{F}^{\mathrm{c}}(\mathrm{Hom}(s^{-1}\overline{\mathscr{C}}_1, \overline{\mathscr{P}})).$$

The image of $\mathscr{G}(f_1, \dots, f_n)$ under $\varphi$ is equal to the composite

$$\varphi(\mathscr{G}(f_1, \dots, f_n)) : s^{-1}\overline{\mathscr{C}}_1 \xrightarrow{\ \psi\ } \mathscr{F}(s^{-1}\overline{\mathscr{C}}_2) \xrightarrow{\tilde{\mathscr{G}}(f_1,\dots,f_n)} \mathscr{F}(\overline{\mathscr{P}}) \xrightarrow{\ \tilde{\mu}_{\mathscr{P}}\ } \mathscr{P}.$$

It remains to prove that $s^{-1}\varphi$ is a twisting element in $\mathrm{Hom}(B(\mathrm{Hom}(\mathscr{C}_2, \mathscr{P})), \mathrm{Hom}(\mathscr{C}_1, \mathscr{P}))$, that is $Q(s^{-1}\varphi) = 0$. By the definition of $Q$ in this homotopy prop(erad) and by the 'associativity' of $\tilde{\mu}_{\mathscr{P}}$, $Q(s^{-1}\varphi)(\mathscr{G}(f_1, \dots, f_n))$ is equal to the composite

$$\overline{\mathscr{C}}_1 \xrightarrow{\ \Delta_{\mathscr{C}_1}\ } \mathscr{F}(\overline{\mathscr{C}}_1) \xrightarrow{\mathscr{F}(s^{-1}\psi)} \mathscr{F}(\mathscr{F}(s^{-1}\overline{\mathscr{C}}_2)) \xrightarrow{\ \tilde{\mu}\ } \mathscr{F}(s^{-1}\overline{\mathscr{C}}_2) \xrightarrow{\tilde{\mathscr{G}}(f_1,\dots,f_n)} \mathscr{F}(\overline{\mathscr{P}}) \xrightarrow{\ \tilde{\mu}_{\mathscr{P}}\ } \mathscr{P},$$

where $\tilde{\mu}$ is the 'triple' map associated to the free prop(erad) $\mathscr{F}(s^{-1}\overline{\mathscr{C}}_2)$. Therefore $Q(s^{-1}\varphi)(\mathscr{G}(f_1, \dots, f_n)) = \tilde{\mu}_{\mathscr{P}} \circ \tilde{\mathscr{G}}(f_1, \dots, f_n) \circ Q(s^{-1}\psi)$ which vanishes since $Q(s^{-1}\psi) = 0$. □

The dual statement is also true and can be proved in the same way. It will appear in a future work of the second author in relation with the transfer of algebraic structures up to homotopy through a deformation-retract (homological perturbation lemma).

**Proposition 34.** *The constructions given in Theorem* 25 *and Theorem* 26 *provide us with three functors,*

Category of homotopy properads $\to$ Category of homotopy Lie algebras.

*Proof.* Let $\Phi : B(\mathscr{P}_1) \to B(\mathscr{P}_2)$ be a morphism of coprop(erad)s defining a morphism of homotopy prop(erad)s between $\mathscr{P}_1$ and $\mathscr{P}_2$. The associated projection $\varphi$ verifies $Q(s^{-1}\varphi) = 0$, that is

$$\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}_1) \xrightarrow{\ \tilde{\Delta}\ } \mathscr{F}^{\mathrm{c}}(\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}_1)) \xrightarrow{\mathscr{F}^{\mathrm{c}}(s^{-1}\varphi)} \mathscr{F}^{\mathrm{c}}(\overline{\mathscr{P}}_2) \xrightarrow{\ \mu_{\mathscr{P}_2}\ } \mathscr{P}_2$$

equals 0. We define the map

$$f : \mathscr{S}^c\big(s(\bigoplus \bar{\mathscr{P}}_1)\big) \xrightarrow{\Theta} \mathscr{F}^c(s\bar{\mathscr{P}}_1) \xrightarrow{\varphi} s(\bigoplus \bar{\mathscr{P}}_2).$$

The map $f$ is a morphism of $L_\infty$-algebras. Its desuspension $s^{-1}f$ verifies the Maurer-Cartan equation in the $L_\infty$-algebra $\mathrm{Hom}\big(\mathscr{S}^c\big(s(\bigoplus \bar{\mathscr{P}}_1)\big),\ \oplus \bar{\mathscr{P}}_2\big)$ (see [8]). The Maurer-Cartan equation for $s^{-1}f$ lifts to the Maurer-Cartan equation for $s^{-1}\varphi$ via $\Theta$, that is the following diagram is commutative:

$$\begin{array}{ccccccc}
\mathscr{S}^c\big(s(\bigoplus \bar{\mathscr{P}}_1)\big) & \xrightarrow{\tilde{\delta}} & \mathscr{S}^c\big(\mathscr{S}^c\big(s(\bigoplus \bar{\mathscr{P}}_1)\big)\big) & \xrightarrow{\mathscr{S}^c(s^{-1}f)} & \mathscr{S}^c(\bigoplus \bar{\mathscr{P}}_2) & \xrightarrow{l_{\oplus\mathscr{P}_2}} & \bigoplus \bar{\mathscr{P}}_2 \\
\downarrow{\scriptstyle\Theta} & & & & & \nearrow{\scriptstyle\mu_{\mathscr{P}_2}} & \\
\mathscr{F}^c(s\bar{\mathscr{P}}_1) & \xrightarrow{\tilde{\Delta}} & \mathscr{F}^c\big(\mathscr{F}^c(s\bar{\mathscr{P}}_1)\big) & \xrightarrow{\mathscr{F}^c(s^{-1}\varphi)} & \mathscr{F}^c(\bar{\mathscr{P}}_2), & &
\end{array}$$

which concludes the proof. □

**Corollary 35.** *Let $\Psi$ be a morphism of homotopy coprop(erad)s between $\mathscr{C}_1$ and $\mathscr{C}_2$. Let $\mathscr{P}$ be a prop(erad). There exists a natural morphism of $L_\infty$-algebras between* $\mathrm{Hom}(\mathscr{C}_2, \mathscr{P})$ *and* $\mathrm{Hom}(\mathscr{C}_1, \mathscr{P})$ *induced by $\Psi$. Its restriction to* $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}_2, \mathscr{P})$ *gives a natural morphism of $L_\infty$-algebras between* $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}_2, \mathscr{P})$ *and* $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}_1, \mathscr{P})$.

*Proof.* The first part is a direct corollary of Theorem 33 and Proposition 34. Since these constructions are composite of equivariant maps, they are stable on the space of invariant elements $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}_2, \mathscr{P})$ and $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}_1, \mathscr{P})$. □

## 5. Models

In this section, we recall the definitions of *minimal* and *quadratic model* for properads and we formally extend them to props. Recall that a model is a quasi-free resolution. Our viewpoint here is to classify properads according to the form of their minimal model, when it exists. For instance, a properad is *Koszul* if and only if it admits a quadratic model. To clarify the genus of some resolutions, we introduce the notion of *contractible prop(erad)s*. Such properads have genus 0 quadratic models.

**5.1. Minimal models.** Recall that a *quasi-free* prop(erad) is a (dg) prop(erad) whose underlying $\mathbb{S}$-bimodule, that is forgetting the differential map, is a free prop(erad) $\mathscr{F}(M)$. It is not necessarily a free dg prop(erad) since the differential $\partial$ may not be freely generated by the differential of $M$.

**Definition** (model). Let $\mathscr{P}$ be a prop(erad). A *model* of $\mathscr{P}$ is a quasi-free prop(erad) $\big(\mathscr{F}(M), \partial\big)$ equipped with a quasi-isomorphism $\mathscr{F}(M) \xrightarrow{\sim} \mathscr{P}$.

Theorem 19 proves that every augmented prop(erad) has a canonical model given by the bar-cobar construction. Some prop(erad)s admit more simple models. The differential $\partial$ of a quasi-free prop(erad) $\mathscr{F}(M)$ is by definition a derivation. Lemma 14 shows that it is characterized by its restriction $\partial_M : M \to \mathscr{F}(M)$ on $M$.

**Definition** (decomposable differential). The differential $\partial$ of a quasi-free prop(erad) is called *decomposable* if the image of its restriction to $M$, $\partial_M : M \to \mathscr{F}(M)$, is composed by decomposable elements, that is $\mathrm{Im}(\partial_M) \subset \bigoplus_{n \geqq 2} \mathscr{F}(M)^{(n)}$.

**Definition** (minimal model). A model $(\mathscr{F}(M), \partial)$ is called *minimal* if its differential $\partial$ is decomposable.

**5.2. Form of minimal models.** From Theorem 19, we know that every augmented (dg) properad admits a resolution of the form $\Omega(B(P))$. A natural way to get a minimal model from this would be to consider the homology of the bar construction, try to endow it with a structure of homotopy coproperad and then take the generalized cobar construction of it. In this section, we prove that when minimal models exist, they are of this form.

**Proposition 36.** *Let $(\mathscr{F}(M), \partial)$ be a quasi-free properad with a decomposable differential generated by a non-negatively graded $\mathbb{S}$-module $M$. Then the homology of the bar construction $B(\mathscr{F}(M))$ of $(\mathscr{F}(M), \partial)$ is equal to the suspension of $M$.*

*Proof.* The bar construction of the dg-properad $\mathscr{P} := \mathscr{F}(M)$ is defined by the underlying $\mathbb{S}$-bimodule $B(\mathscr{P}) := \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}) = \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{F}}(M))$. The differential $d$ is the sum of two terms $d_0 + \tilde{\partial}$. The component $\tilde{\partial}$ comes from $\partial$ and $d_0$ is the unique coderivation which extends the partial product of $\mathscr{F}(M)$.

Consider the filtration $F_s := \bigoplus_{r \leqq s} \mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{F}}(M))_r$, where $r$ is the sum of the degrees of the elements of $M$. Let's denote by $E^{\bullet}_{st}$ the associated spectral sequence.

Since the chain complex $M$ is bounded below, this filtration is bounded below $F_{-1} = 0$. It is obviously exhaustive, therefore the classical theorem of convergence of spectral sequences shows that $E^{\bullet}$ converges to the homology of $B(\mathscr{F}(M))$.

We have $\tilde{\partial}(F_s) \subset F_{s-1}$ and $d_0(F_s) \subset F_s$. Hence, the first term of the spectral sequence is $E^0_{st} = \mathscr{F}^{\mathrm{c}}_{s+t}(s\overline{\mathscr{F}}(M))_s$, where $s + t$ is the total homological degree, and $d^0 = d_0$. We have reduced the problem to computing the homology of the bar construction of the free properad on $M$, which is equal to $\Sigma M$ by [41], Corollary 5.10 (where we choose to put each element of $M$ in weight 1). □

The next proposition shows that, when a minimal model of a properad $\mathscr{P}$ exists, it is necessarily given by a quasi-free properad on the homology of the bar construction of $\mathscr{P}$.

**Theorem 37.** *Let $\mathscr{P}$ be an augmented* dg *properad and let $(\mathscr{F}(M), \partial)$ be a minimal model of $\mathscr{P}$. The $\mathbb{S}$-bimodule $sM$ is isomorphic to the homology of the bar construction of $\mathscr{P}$.*

*Proof.* In [41], we proved in Proposition 4.9 that the bar construction preserves quasi-isomorphisms. Therefore, the bar construction of $\mathscr{F}(M)$ is quasi-isomorphic to the bar construction of $\mathscr{P}$. We conclude by Proposition 36. □

We denote by $\mathscr{P}^{\mathrm{i}} := H_{\bullet}(B(\mathscr{P}))$ the homology of the bar construction of $\mathscr{P}$. When $(\mathscr{F}(s^{-1}\mathscr{P}^{\mathrm{i}}), \partial)$ is a minimal model of $\mathscr{P}$, the derivation $\partial$ is equivalent to a structure of homotopy coproperad on $\mathscr{P}^{\mathrm{i}}$ such that $\delta_1 = 0$. That is $(\mathscr{F}(s^{-1}\mathscr{P}^{\mathrm{i}}), \partial)$ is the generalized cobar construction $\Omega(\mathscr{P}^{¡})$ of the homotopy coproperad $\mathscr{P}^{¡}$. As a conclusion, we have the following corollary which gives the form of minimal models.

**Corollary 38.** *A minimal model of an augmented* dg *properad $\mathscr{P}$ is always the cobar construction $\Omega(\mathscr{P}^{¡})$ on the homology of $B(\mathscr{P})$ endowed with a structure of homotopy coproperad.*

In the sequel, we will only consider props freely generated by a properad, in the sense of the horizontal (concatenation) product. The minimal model of such props is given by the generalized cobar construction of the associated homotopy coproperad, viewed as a homotopy coprop. And the result of the preceding lemma still holds.

**5.3. Quadratic models and Koszul duality theory.** In general, it is a difficult problem to find the minimal model of a prop(erad). One can first compute the homology of the bar construction and then provide a structure of homotopy coproperad on it, that is with higher homotopy cooperations. For some weight graded properads, there exist simple minimal models which are given by the Koszul duality theory. These properads are called Koszul.

**Definition** (quadratic differential). The differential $\partial$ of a quasi-free prop(erad) is called *quadratic* if the image of $\partial_M : M \to \mathscr{F}(M)$ is in $\mathscr{F}(M)^{(2)}$.

**Definition** (quadratic model). A model $(\mathscr{F}(M), \partial)$ is called *quadratic* if its differential $\partial$ is quadratic.

When $\mathscr{P}$ is a weight graded properad, its bar construction splits as a direct sum of finite chain complexes indexed by the weight (cf. [41], Section 7.1.1). In this case, we can talk about top dimensional homology groups.

**Theorem 39.** *Let $\mathscr{P}$ be a weight graded properad concentrated in homological degree* 0. *The following assertions are equivalent.*

(1) *The homology of $B(\mathscr{P})$ is concentrated in top dimension.*

(2) *The $\mathbb{S}$-bimodule $\mathscr{P}^{¡}$ is a strict coproperad.*

(3) *The properad $\mathscr{P}$ admits a quadratic model*: $\Omega(\mathscr{P}^{¡}) \xrightarrow{\sim} \mathscr{P}$.

*Proof.* (1) $\Rightarrow$ (2) is given by [41], Proposition 7.2.

(2) $\Rightarrow$ (3) is given by [41], Theorem 5.9. When $\mathscr{P}^{¡}$ has a structure of strict coproperad, its cobar construction is a resolution of $\mathscr{P}$ and the differential of it is quadratic.

(3) $\Rightarrow$ (1) Since $\mathscr{P}$ is isomorphic to $\mathscr{F}(M_0)/(\partial(M_1))$, whith $\partial$ quadratic, this presentation is quadratic. Define an extra weight on $M$ by the formula $\omega(M_n) := n+1$. With this weight, the quasi-isomorphism $\mathscr{F}(M) \xrightarrow{\rho} \mathscr{P}$ is a morphism of weight graded dg properads. The induced morphism $B(\rho)$ on the bar construction preserves this grading. Therefore we have $H_n(B(\mathscr{P})^{(n)}) = H_n(B(\mathscr{F}(M))^{(n)}) = (sM)_n$ and the homology of the bar construction of $\mathscr{P}$ is concentrated in top dimension. □

In this case, the properad $\mathscr{P}$ is called a *Koszul* properad. The coproperad $\mathscr{P}^{\text{i}}$ is its *Koszul dual* and $\mathscr{P}$ has a quadratic model which is the cobar construction on $\mathscr{P}^{\text{i}}$. In other words, a properad is Koszul when its bar construction is *formal*, that is when $B(\mathscr{P})$ is quasi-isomorphic to its homology $\mathscr{P}^{\text{i}}$ as a dg coproperad. This case is simple and particularly efficient. When $\mathscr{P} = F(V)/(R)$ has a quadratic presentation with a finite dimensional space of generators $V$, then the linear dual (twisted by the signature representation) of the coproperad $\mathscr{P}^{\text{i}}$ is a properad equal, up to suspension, to $\mathscr{P}^{!} = F(V^{\vee})/(\mathscr{R}^{\perp})$ where $V^{\vee}$ is the linear dual of $V$ twisted by the signature representation. This relation provides a concrete method to compute the minimal model of Koszul properads. The next step is to be able to prove that it is Koszul. Koszul duality theory provides a smaller chain complex $\mathscr{P}^{\text{i}} \boxtimes \mathscr{P}$ which is acyclic if and only if the properad $\mathscr{P}$ is Koszul. Therefore, there are simple methods to show that a properad is Koszul. When a properad is defined by two Koszul properads with a distributive law, [41], Proposition 8.4 shows that it is Koszul. In the operadic case, there are basically two other efficient methods. First if the homology of the free $\mathscr{P}$-algebra is acyclic then the operad $\mathscr{P}$ is Koszul (see [10], Proposition 5.3.5). Finally, when the operad is set theoretic, we can use the associated poset to prove that it is Koszul (see [39]).

**5.4. Homotopy Koszul properads.** If a properad is Koszul, then we have clearly cut means to construct its minimal model. However, the ordinary notion of Koszulness does not cover many important examples. For example, the properad of associative bialgebras is not Koszul since it is not quadratic and any Koszul properad has a quadratic presentation by [41], Corollary 7.5. So we are left in such cases with no concrete methods of proving that a particular properad $\mathscr{P}$ admits a minimal model, and, if so, constructing it explicitly. It is already a highly non-trivial problem in general to find the set of generators for a minimal model, not speaking about the differential. In this section we extend the notion of Koszulness in such a way that some of the above problems become effectively solvable.

**Definition.** Let $\mathscr{P} = \mathscr{F}(V)/(\mathscr{R})$ be a properad generated by an $\mathbb{S}$-bimodule $V = \{V(m,n)\}$ concentrated in degree zero, and with an ideal generated by $\mathscr{R} \subset \mathscr{F}(V)^{(\geqq 2)}$. Let $\pi_k : \mathscr{F}(V) \to \mathscr{F}(V)^{(k)}$ be the natural projection, and let us set,

$$\mathscr{R}_k := \pi_k(\mathscr{R}), \quad \text{for } k = 2, 3, \dots .$$

Let us also denote by $\mathscr{P}^{(\geqq k)}$ the image of $\mathscr{F}(V)^{(\geqq k)}$ under the natural epimorphism $\mathscr{F}(V) \twoheadrightarrow \mathscr{P}$.

The properad $\mathscr{P}$ is called *homotopy Koszul* if

(i) the quadratic properad $\mathscr{P}_2 := \mathscr{F}(V)/(\mathscr{R}_2)$ is Koszul,

(ii) $\mathscr{P}$ and $\mathscr{P}_2$ are isomorphic as $\mathbb{S}$-bimodules,

(iii) there is an extra grading on the properad $\mathscr{P} = \bigoplus_{\lambda} \mathscr{P}(\lambda)$, with $\mathscr{P}(\lambda)$ being a collection of finite-dimensional $\mathbb{S}$-bimodules.

In practice the conditions (i)–(iii) above are often not hard to check (see examples below). As an extra grading one can use, for example, the path grading of a free properad introduced by Kontsevich and studied in [29]. The main motivation behind the definition is the following.

**Theorem 40.** *If a properad $\mathscr{P}$ is homotopy Koszul, then it admits a minimal model of the form $\big(\mathscr{F}(s\bar{\mathscr{P}}_2^{\text{¡}}),\delta\big)$, where $\mathscr{P}_2^{\text{¡}}$ is the coproperad Koszul dual to $\mathscr{P}_2$.*

*Proof.* Consider the bounded above increasing filtration $F_{-p}\mathscr{P} := \mathscr{P}^{(\geqq p)}$ of the properad $\mathscr{P}$. As $F_{-p}\mathscr{P} \cap P(\lambda)$ are finite-dimensional vector spaces, the spectral sequences associated with this filtration (see below) have good convergence properties. Since $\mathscr{P}$ is isomorphic to $\mathscr{P}_2$ as an $\mathbb{S}$-bimodule, the associated graded properad

$$\bigoplus_{p\geqq 0} \frac{\mathscr{P}^{(\geqq p)}}{\mathscr{P}^{(\geqq p+1)}}$$

is isomorphic to $\mathscr{P}_2$ as a properad. Then we have

**Claim 1.** *The homologies of the bar constructions, $B(\mathscr{P})$ and $B(\mathscr{P}_2)$, are isomorphic as $\mathbb{S}$-bimodules, i.e. $H_\bullet\big(B(\mathscr{P})\big) \simeq \mathscr{P}_2^{\text{¡}}$ as $\mathbb{S}$-bimodules.*

Indeed, the filtration $F_{-p}\mathscr{P} := \mathscr{P}^{(\geqq p)}$ induces an associated filtration of the complex $B(\mathscr{P})$ (as differential in $B(\mathscr{P})$ is built from compositions in $\mathscr{P}$ which respect the filtration $F_{-p}\mathscr{P}$). By the above observation, the $0^{\text{th}}$ term, $E^0$, of the associated spectral sequence, $\{E^r, d^r\}$, is exactly the complex $B(\mathscr{P}_2)$, $E^0_{pq} = B(\mathscr{P}_2)^{(-p)}_{p+q}$ and $d^0 = d_{B(\mathscr{P}_2)}$. As $\mathscr{P}_2$ is Koszul, $E^1 = H_\bullet\big(B(\mathscr{P}_2)\big)$ is exactly the Koszul dual coproperad $\mathscr{P}_2^{\text{¡}}$, that is $E^1_{pq} = 0$ for $q \neq -2p$ and $E^1_{pq} = H_{-p}\big(B(\mathscr{P}_2)^{(-p)}\big) = (\mathscr{P}_2^{\text{¡}})^{(-p)}$ when $q = -2p$. The induced differentials, $d^r$ for $r \geqq 1$, are zero because of the homological degree 0 assumption on $\mathscr{P}$. Thus the spectral sequence $\{E^r, d^r\}$ degenerates at the first term. The extra grading on the properad $\mathscr{P}$ induces an extra grading $\lambda$ on $B(\mathscr{P})$ which makes $F_{-p}\big(B(\mathscr{P})\big) \cap B(\mathscr{P})(\lambda)$ into a bounded filtration of $B(\mathscr{P})(\lambda)$. Hence it converges to $H_\bullet\big(B(\mathscr{P})\big)(\lambda)$ by the Classical Convergence Theorem 5.5.1 of [46], thereby proving Claim 1.

Choosing a homological splitting of the complex $B(\mathscr{P})$,

$$H_\bullet\big(B(\mathscr{P})\big) \underset{p}{\overset{i}{\rightleftarrows}} B(\mathscr{P}) \circlearrowleft h,$$

one can use dual transfer formulae of [16] for homotopy coproperads to induce on the $\mathbb{S}$-bimodule $H_\bullet\big(B(\mathscr{P})\big) \simeq \mathscr{P}_2^{\text{¡}}$ the associated strongly homotopy coproperad structure, that is a differential, $\delta$, in the free properad[2] $\mathscr{F}\big(s^{-1}H_\bullet\big(\bar{B}(\mathscr{P})\big)\big) = \Omega\big(H_\bullet\big(B(\mathscr{P})\big)\big)$ generated by

---

[2] In fact the Granåker formulae provide us in general with a differential $\delta$ in a *completed* (with respect to the number of vertices) free properad: there is no guarantee that such $\delta$ applied to a generator is a *finite* sum of terms but we can only be sure that $\delta$ is continuous with respect to the topology induced by the number of vertices filtration. However, our assumption on existence of an extra gradation in $\mathscr{P}$ implies that $\delta$ is well-defined in the ordinary category of properads: it is *finite* on every generator so that $\big(\mathscr{F}\big(s^{-1}H_\bullet\big(\bar{B}(\mathscr{P})\big)\big),\delta\big)$ makes sense without completion.

It is important to notice that had we chosen to work with topological properads (with topology induced by the number of vertices or genus filtrations), the condition (iii) in the definition of homotopy Koszulness can be safely omitted—Theorem 40 stays true in the category of (completed) topological properads because all the spectral sequences we used in the proof stay convergent by classical Complete Convergence Theorem 5.5.10 (see [46], p. 139). As an example of the deformation quantization prop [30] shows, working with topological prop(erad)s is unavoidable in application of the theory of prop(erad)s to geometry and mathematical physics.

$H_\bullet\big(B(\mathscr{P})\big)$. In general, this differential is *not* quadratic, i.e. the induced homotopy coproperad structure on $H_\bullet\big(B(\mathscr{P})\big)$ is *not* equal to the coproperad structure on $\mathscr{P}_2^{\text{¡}}$. Moreover, the chosen homological splitting provides us canonically with a morphism of homotopy coproperads which extends $i$,

$$H_\bullet\big(B(\mathscr{P})\big) \to B(\mathscr{P}),$$

i.e. with a morphism of dg properads,

$$\phi : \big(\mathscr{F}\big(s^{-1}H_\bullet\big(\bar{B}(\mathscr{P})\big)\big),\delta\big) \to \Omega\big(B(\mathscr{P})\big).$$

As $\Omega\big(B(\mathscr{P})\big) \xrightarrow{\sim} \mathscr{P}$ is a resolution of $\mathscr{P}$ by Theorem 19, the required Theorem 40 follows immediately from the following

**Claim 2.** *Under the assumption on the properad $\mathscr{P}$ the morphism $\phi$ is a quasi-isomorphism.*

Indeed, the introduced above filtration of the bar construction, $B(\mathscr{P})$, induces a filtration $F_{-p}H_\bullet\big(B(\mathscr{P})\big)$ of its homology with the associated graded coproperad being exactly $\mathscr{P}_2^{\text{¡}}$. This filtration of $H_\bullet\big(B(\mathscr{P})\big)$ induces in turn a filtration of the complex $\big(\mathscr{F}\big(s^{-1}H_\bullet\big(\bar{B}(\mathscr{P})\big)\big),\delta\big)$. The $0^{\text{th}}$ term of the associated spectral sequence is precisely the minimal model, $\big(\mathscr{F}(s^{-1}\bar{\mathscr{P}}_2^{\text{¡}}),\delta\big)$, of the properad $\mathscr{P}_2$. As the latter is Koszul by assumption, its homology is equal to $\mathscr{P}_2$. By homological degree assumption on $\mathscr{P}$, the induced differential on the next term of the spectral sequence vanishes so that it degenerates. The extra grading assumption on $\mathscr{P}$ implies that this spectral sequence converges to the homology $\big(\mathscr{F}\big(s^{-1}H_\bullet\big(\bar{B}(\mathscr{P})\big)\big),\delta\big)$ which is equal, therefore, as an $\mathbb{S}$-bimodule to $\mathscr{P}_2 \simeq \mathscr{P}$. This fact completes the proof of Claim 2 and hence of the theorem. □

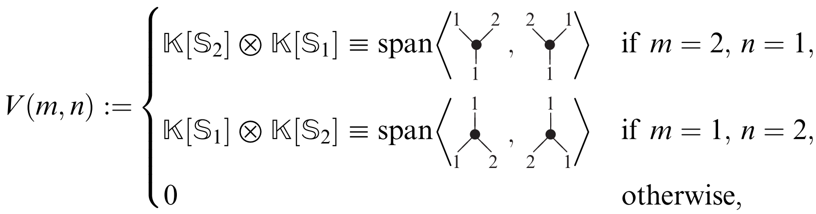
The operad $\mathscr{P}_2$ is Koszul means that the differential of the minimal model $\big(\Omega(\mathscr{P}_2^{\text{¡}}),\delta_2\big)$ is quadratic, that is $\delta_2 : s^{-1}\bar{\mathscr{P}}_2^{\text{¡}} \to \mathscr{F}(s^{-1}\bar{\mathscr{P}}_2^{\text{¡}})^{(2)}$. Since the transfer of homotopy coproperad structures does not change the map $\Delta_2$ defining the homotopy coproperad structure on $H_\bullet\big(B(\mathscr{P})\big)$ but just add extra terms $\Delta_n$, for $n \geqq 3$, the final differential $\delta$ defining a minimal model of $\mathscr{P}$ is equal to $\delta_2$ plus extra terms $\delta_n$ for $n \geqq 3$ such that $\delta_n : s^{-1}\bar{\mathscr{P}}_2^{\text{¡}} \to \mathscr{F}(s^{-1}\bar{\mathscr{P}}_2^{\text{¡}})^{(n)}$, that is to say, $\delta$ is a perturbation of $\delta_2$.

The coproperad $\mathscr{P}_2^{\text{¡}}$ is computable by Koszul duality theory. Therefore the above theorem gives an immediate estimate of the set of generators for a minimal model of a homotopy Koszul properad. Moreover, the differential in this quasi-free model can in principle be computed via ordinary homotopy transfer formulae.

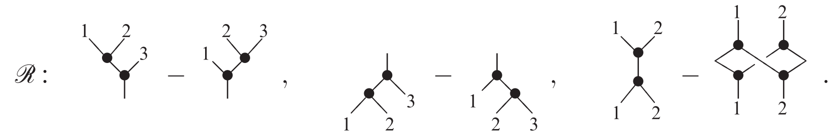
The class of properads which are homotopy Koszul but not Koszul is non-empty and contains an important example of the properad, $\mathscr{A}ss\mathscr{B}i$, of (co)associative bialgebras which can be defined as a quotient,

$$\mathscr{A}ss\mathscr{B}i := \mathscr{F}(V)/(\mathscr{R})$$

of the free properad, $\mathscr{F}(V)$, generated by the $\mathbb{S}$-bimodule $V = \{V(m,n)\}$,



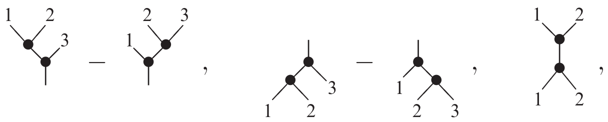

representing a binary product and a binary coproduct without symmetries, modulo the ideal generated by relations




These relations stand respectively for the associativity of the product, the coassociativity of the coproduct and the relation between them, that is the coproduct is a morphism of algebras or equivalently the product is a morphism of coalgebras. As the ideal contains 4-vertex graphs, the properad $\mathscr{A}ss\mathscr{B}i$ is not quadratic. Hence $\mathscr{A}ss\mathscr{B}i$ can not be Koszul in the ordinary sense. However, we have the following

**Proposition 41.** *The properad $\mathscr{A}ss\mathscr{B}i$ is homotopy Koszul.*

*Proof.* (i) The properad $\mathscr{A}ss\mathscr{B}i_2$ is Koszul as it is generated by the bimodule $V$ with the relations,

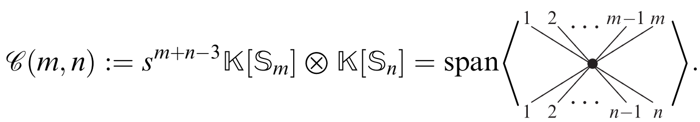


which verify the Distributive Law (see [41], Section 5.6 and Proposition 8.5).

(ii) The $\mathbb{S}$-bimodule isomorphism $\mathscr{A}ss\mathscr{B}i \simeq \mathscr{A}ss\mathscr{B}i_2$ was established in [9].

(iii) The ideal generated by $\mathscr{R}$ preserves the path grading (see [29] for its definition and main properties) of the free properad $\mathscr{F}(V)$ and hence induces an associated filtration on $\mathscr{A}ss\mathscr{B}i$ which satisfies the last condition in the definition of a homotopy Koszulness properad. $\square$

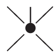
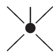
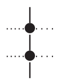
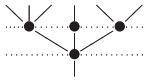
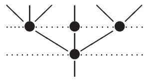
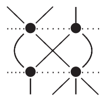
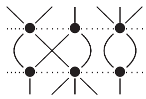
**Corollary 42** (cf. [28]). *The properad $\mathscr{A}ss\mathscr{B}i$ admits a minimal resolution, $\mathscr{F}(\mathscr{C})$, generated by the $\mathbb{S}$-bimodule $\mathscr{C} = \{\mathscr{C}(m,n)\}_{m,n \geqq 1, m+n \geqq 3}$, with*



*Proof.* The Koszul dual properad of $\mathscr{A}ss\mathscr{B}i_2$ is the properad generated by a binary product and a binary coproduct which are associative and coassociative. All the composites with the product and the coproduct vanish except . The only non-vanishing elements of this properad are obtained by composing first some products and then coproducts. We conclude that $\mathscr{A}ss\mathscr{B}i_2^{\text{i}}(m,n) = s^{m-2}\mathbb{K}[\mathbb{S}_m] \otimes s^{n-2}\mathbb{K}[\mathbb{S}_n]$ for $m, n \geqq 1$, $m+n \geqq 3$ and zero otherwise. Then Theorem 40 implies the claim. $\square$

We refer the reader to Section 6.2 for another application of the notion of homotopy Koszulness.

**5.5. Models for associative algebras, non-symmetric operads, operads, properads, props.** There are several different notions of algebraic objects in the literature that are used to model the operations acting on some algebraic category. We briefly recall them in the following table.

| | | | | | |
|---|---|---|---|---|---|
| operations |  |  no symmetry |  |  |  |
| composition |  |  planar |  non-planar |  |  |
| monoidal category | (Vect, $\otimes$) | (gVect, $\circ$) | ($\mathbb{S}$-Mod, $\circ$) | ($\mathbb{S}$-biMod, $\boxtimes_c$) | ($\mathbb{S}$-biMod, $\boxtimes$) |
| monoid | associative algebras | non-symmetric operads | operads | properads | props |
| modules | modules | non-symmetric algebras | algebras | (bial)gebras | (bial)gebras |
| free monoid | ladders (tensor module) | planar trees | trees | connected graphs | graphs |

To each pair of such objects, there is a forgetful functor and a left adjoint:

$$\text{associative algebras} \rightleftarrows \text{non-symmetric operads} \rightleftarrows \text{operads} \rightleftarrows \text{properads} \rightleftarrows \text{props}.$$

Let us make them explicit.

• To any prop $\mathscr{P}$, the associated properad $\mathscr{U}^{\text{props}}_{\text{properads}}(\mathscr{P})$ is given by the same underlying $\mathbb{S}$-bimodule where we only consider vertical compositions of operations based on connected graphs. That is we forget the horizontal composition. Its left adjoint $\mathscr{F}^{\text{props}}_{\text{properads}}(\mathscr{P})$ is given by the free symmetric tensor on $\mathscr{P}$ for the horizontal tensor product. (This functor was introduced in [41], Section 1, where it is denoted by $\mathscr{S}$.) In other words, we freely generate horizontal compositions from a properad to get a prop.

• The operad obtained from a properad $\mathscr{P}$ is the $\mathbb{S}$-module

$$\mathscr{U}^{\text{properads}}_{\text{operads}}(\mathscr{P})(n) := \mathscr{P}(1,n)$$

equipped with the restriction to one rooted trees composition. Its left adjoint functor is $\mathscr{F}^{\text{properads}}_{\text{operads}}(\mathscr{P})(m,n) := \mathscr{P}(n)$ for $m = 1$ and 0 for $m > 1$.

• For any operad $\mathscr{P}$, we consider the non-symmetric operad $\mathscr{U}^{\text{operads}}_{\text{non-symm. operads}}(\mathscr{P}) = \mathscr{P}$ where we forget the action of the symmetric group. The left adjoint is given by

$$\mathscr{F}^{\text{operads}}_{\text{non-symm. operads}}(\mathscr{P})(n) = \mathscr{P}(n) \otimes \mathbb{K}[\mathbb{S}_n]$$

(see M. Aguiar and M. Livernet [3]).

• The pair of adjoint functors between associative algebras and non-symmetric operads is defined in the same pair of functors between operads and properads. In one way, we just consider the unital operation (arity (1)) of a non-symmetric operad. In this other way, for an associative algebra we define a non-symmetric operad concentrated in arity (1).

**Proposition 43.** *All these functors are exact, that is the image of a quasi-isomorphism is a quasi-isomorphism.*

*Proof.* It is trivial for the forgetful functors and for the functors $F^{\text{operads}}_{\text{operads}}$ and $F^{\text{non-symm. operads}}_{\text{ass. algebras}}$ because the underlying dg-module does not change. Since the functor $F^{\text{operads}}_{\text{non-symm. operads}}$ is given by tensoring $\mathbb{S}_n$-modules with the flat $\mathbb{K}$ module $\mathbb{K}[\mathbb{S}_n]$ (the characteristic of $\mathbb{K}$ is 0), it is exact. Over a field of characteristic 0, the functor $F^{\text{props}}_{\text{properads}}$ is also exact. □

This proposition justifies the following philosophy. To study the deformation theory of elements of an algebraic category, that is a class of gebras (modules, algebras, bialgebras), one should first model this category using the simplest possible object of the previous table. For instance, associative, diassociative, dendriform algebras [22] are encoded each time by a non-symmetric operad. Commutative, Lie, preLie, Gerstenhaber, Poisson algebras are modelled by operads. Lie bialgebras, infinitesimal Hopf algebras [2], (associative) bialgebras (see [32], Section 3.3) are representations of properads. Non-unital infinitesimal Hopf algebras, semi Hopf algebras, Lie bialgebras [23] can only be represented by a prop.

Then to study the deformation theory of this algebraic category, that is to define the stable notion up to homotopy (see 6.1) or the deformation complex (see [32], Section 2), one has to find a cofibrant resolution (bar-cobar, minimal model for instance) of the related operad, properad or prop $\mathscr{P}$. This resolution contains all the necessary data since a resolution for the induced prop is "freely" obtained by the free exact functor.

**5.6. Models generated by genus 0 differentials.** Let $\mathscr{A}$ be a category of gebras defined by some products and some coproducts with relations that can be written as linear combinations of connected graphs of genus 0, for example Lie bialgebras, Frobenius bialgebras, infinitesimal bialgebras (see [11], [41]). In this case, the class of gebras can be faithfully modelled with a smaller algebraic object called a dioperad [11].

A dioperad is a properad with only compositions of operations based on genus 0 connected graphs. Hence, there is a natural forgetful functor from properads to dioperads. To any properad $\mathscr{P}$, the associated dioperad $\mathscr{U}^{\text{properads}}_{\text{dioperads}}(\mathscr{P})$ has the same underlying $\mathbb{S}$-bimodule and we only consider vertical compositions of operations based on connected graphs of genus 0. Let us denote by $\square$ the restriction of $\boxtimes$ to genus 0 graphs. With this notation, a dioperad is a monoid $(\mathscr{D}, \mu_{\mathscr{D}})$ in the monoidal category $(\mathbb{S}\text{-biMod}, \square)$. From now on, let us denote the genus in exponent. For instance, $\mathscr{F}^0$ will denote the free dioperad functor $\mathscr{F}^{\text{dioperads}}_{\mathbb{S}\text{-biMod}}$ and $\mathscr{F}$ will simply denote the free properad functor $\mathscr{F}^{\text{properads}}_{\mathbb{S}\text{-biMod}}$.

**Proposition 44.** *The left adjoint of the forgetful functor*

$$\mathscr{U}^{\text{properads}}_{\text{dioperads}}(\mathscr{P}) : \text{Properads} \to \text{Dioperads}$$

*is given by*

$$\mathscr{F}(\mathscr{D})/I,$$

*where $I$ is the (properadic) ideal generated by the image under $\mu_{\mathscr{D}} - \mathrm{Id}$ of $\mathscr{F}^0(\mathscr{D})^{(2)}$, that is the connected graphs of genus 0 with two vertices.*

*In other words, this construction is the quotient of the free properad on $\mathscr{D}$, considered as an $\mathbb{S}$-bimodule, by the (dioperadic) composition of any pair of adjacent vertices with only one edge in between.*

Notice that this construction is the same as the universal enveloping algebra of a Lie algebra. Therefore, we will often call it the *universal enveloping properad of a dioperad* and $\mathscr{F}^{\text{properads}}_{\text{dioperads}}$ the *universal enveloping functor*.

*Proof.* The proof is the same as the proof of the universal property of the universal enveloping algebra of a Lie algebra. Hence it is left to the reader. $\square$

A direct corollary gives that the universal enveloping properad of a dioperad defined by generators and relations is a properad given by the same generators and relations.

**Corollary 45.** *Let $\mathscr{D}$ be a dioperad defined by generators and relations: $\mathscr{D} = \mathscr{F}^0(V)/(R)$, where $(R)$ is the (dioperadic) ideal generated by $R$. The universal enveloping properad is equal to*

$$\mathscr{F}^{\text{properads}}_{\text{dioperad}}(\mathscr{D}) = \mathscr{F}(V)/(R),$$

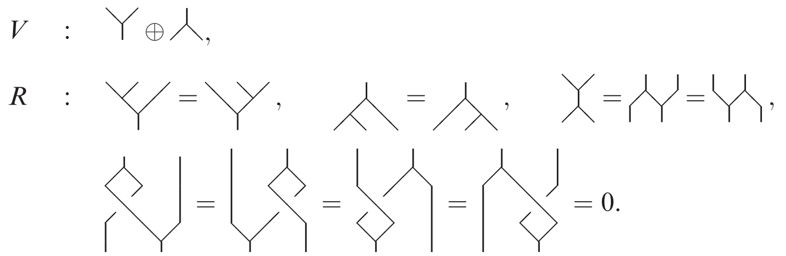
*where $(R)$ is the (properadic) ideal generated by $R$.*

Even if an algebraic category $\mathscr{A}$ can be modelled by a dioperad, the induced cofibrant resolution of this dioperad does not contain all the data necessary for the study of deformation theory of $\mathscr{A}$ because the universal enveloping functor $\mathscr{F}^{\text{properads}}_{\text{dioperads}}$ is not exact as the following counter-example shows.

Let $\varepsilon\mathscr{B}i$ be the properad which models infinitesimal bialgebras (see [41], Section 2.9). We consider its Koszul dual properad without the relation $\diamondsuit = 0$. Let us denote it by $\mathcal{NC}$-$\mathcal{F}rob$ because it models some kind of non-commutative Frobenius bialgebras. An $\mathcal{NC}$-$\mathcal{F}rob$-bialgebra is a vector space $X$ equipped with a binary associative product $\mu : X \otimes X \to X$ and a binary coassociative coproduct $\Delta : X \to X \otimes X$ such that $\Delta$ is a morphism of bimodules. This means

$$\Delta \circ \mu = (\mathrm{Id} \otimes \mu) \circ (\Delta \otimes \mathrm{Id}) = (\mu \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta).$$

The graphical picture of all the relations is the following:




Since the relations are linear combinations of connected graphs of genus 0, this category is faithfully modelled by the dioperad $\mathcal{NC}$-$\mathcal{F}rob^0 = \mathcal{F}^0(V)/(R)$. The exponent 0 stands for the restriction to graphs of genus 0. It was proved in [11] that $\mathcal{NC}$-$\mathcal{F}rob^0$ is a Koszul dioperad, since its Koszul dual dioperad $\varepsilon\mathcal{B}i^0$ is Koszul by means of distributive laws. That is the dioperad $\mathcal{NC}$-$\mathcal{F}rob^0$ admits a quadratic dioperadic (genus 0) model $(\mathcal{F}^0(\mathcal{C}), \partial^0) \xrightarrow{\sim} \mathcal{NC}$-$\mathcal{F}rob^0$, where $\mathcal{C}$ is the codioperad $s^{-1}\varepsilon\mathcal{B}i^{0\vee}$. (Notice that there is no direct proof of this fact.) The differential $\partial^0$ splits each element of $\mathcal{C}$ into two vertices with only one edge in between.

Consider now the properad $\mathcal{NC}$-$\mathcal{F}rob = \mathcal{F}(V)/(R)$, which is the image under the universal enveloping functor $\mathcal{F}_{\text{dioperads}}^{\text{properads}}$ of $\mathcal{F}rob^0$ by Corollary 45. The image of the chain complex $(\mathcal{F}^0(\mathcal{C}), \partial^0)$ under the functor $\mathcal{F}_{\text{dioperads}}^{\text{properads}}$ is the quasi-free properad on $\mathcal{C}$ with the differential $\partial^0$, that is the cobar construction of $\mathcal{C}$, where this later is considered as a coproperad. The homology of this chain complex is not concentrated in degree 0.

We build a cycle based on graphs of genus 2 from the following picture:

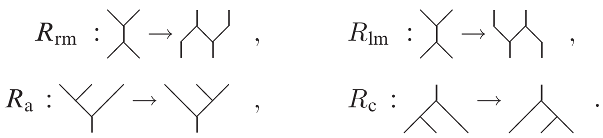
$$\begin{array}{ccc}
\mu \circ \Delta \circ \mu \circ \Delta & \xrightarrow{-\mu \circ R_{\mathrm{lm}} \circ \Delta} & \mu \circ (\mu \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta) \circ \Delta \\
\Big\downarrow {\scriptstyle \mu \circ R_{\mathrm{rm}} \circ \Delta} & & \Big\downarrow {\scriptstyle \mu \circ (\mu \otimes \mathrm{Id}) \circ R_{\mathrm{c}}} \\
\mu \circ (\mathrm{Id} \otimes \mu) \circ (\Delta \otimes \mathrm{Id}) \circ \Delta & \xrightarrow{-R_{\mathrm{a}} \circ (\Delta \otimes \mathrm{Id}) \circ \Delta} & \mu \circ (\mu \otimes \mathrm{Id}) \circ (\Delta \otimes \mathrm{Id}) \circ \Delta,
\end{array}$$

where $R_{\mathrm{rm}}$ stands for the "right module" relation $\mu \circ \Delta \to (\mathrm{Id} \otimes \mu) \circ (\Delta \otimes \mathrm{Id})$, $R_{\mathrm{lm}}$ for the "left module" relation $\mu \circ \Delta \to (\mu \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta)$, $R_{\mathrm{a}}$ the associativity relation

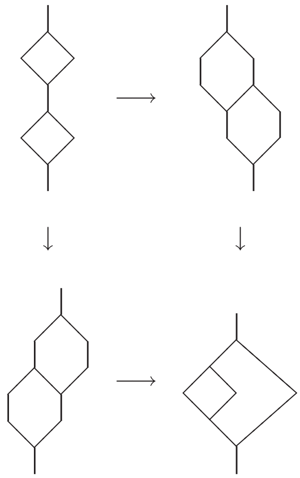
$$\mu \circ (\mu \otimes \mathrm{Id}) \to \mu \circ (\mathrm{Id} \otimes \mu)$$

and $R_{\mathrm{c}}$ the coassociativity relation $(\Delta \otimes \mathrm{Id}) \circ \Delta \to (\mathrm{Id} \otimes \Delta) \circ \Delta$.

The graphical picture is as follows:




Then, the cycle is based upon the following picture:



We denote with the same notation the corresponding homotopies, that is elements of $\mathscr{C}$:

$$\partial_0(R_{\mathrm{rm}}) = \mu \circ \Delta - (\mathrm{Id} \otimes \mu) \circ (\Delta \otimes \mathrm{Id}), \quad \partial_0(R_{\mathrm{lm}}) = \mu \circ \Delta - (\mu \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta),$$

$$\partial_0(R_{\mathrm{a}}) = \mu \circ (\mu \otimes \mathrm{Id}) - \mu \circ (\mathrm{Id} \otimes \mu), \quad \partial_0(R_{\mathrm{c}}) = (\Delta \otimes \mathrm{Id}) \circ \Delta - (\mathrm{Id} \otimes \Delta) \circ \Delta.$$

The previous picture proves that

$$\xi := \mu \circ R_{\mathrm{rm}} \circ \Delta - \mu \circ R_{\mathrm{lm}} \circ \Delta - R_{\mathrm{a}} \circ (\Delta \otimes \mathrm{Id}) \circ \Delta + \mu \circ (\mu \otimes \mathrm{Id}) \circ R_{\mathrm{c}}$$

is a cycle in $\big(\mathscr{F}(\mathscr{C}), \partial^0\big)$, that is $\partial^0(\xi) = 0$.

**Lemma 46.** *The cycle $\xi$ is not a boundary under $\partial^0$.*

*Proof.* The degree of $\xi$ is 1. Suppose that there exists an element $\zeta$ of degree 2 such that $\partial^0(\zeta) = \xi$. This element belongs to

$$\zeta \in \mathscr{F}(\mathscr{C}_0 \oplus \underbrace{\mathscr{C}_1}_{(2)}) \oplus \mathscr{F}(\mathscr{C}_0 \oplus \underbrace{\mathscr{C}_2}_{(1)}).$$

Let us denote by $\zeta = \zeta_1 + \zeta_2$ each component. The image under the quadratic differential $\partial^0$ of any element of $\mathscr{F}(\underbrace{\mathscr{C}_0}_{(k)} \oplus \underbrace{\mathscr{C}_2}_{(1)})$ is an element of $\mathscr{F}(\underbrace{\mathscr{C}_0}_{(k+1)} \oplus \underbrace{\mathscr{C}_1}_{(1)})$. And since the genus of the differential $\partial^0$ is 0, $\zeta_2$ is in $\mathscr{F}^2(\underbrace{\mathscr{C}_0}_{(1)} \oplus \underbrace{\mathscr{C}_2}_{(1)})$, that is the part of genus 2 of $\mathscr{F}(\mathscr{C}_0 \oplus \mathscr{C}_2)$. The $\mathbb{S}$-bimodule $\mathscr{C}_0$ is equal to $V = \curlyvee \oplus \curlywedge$, that is binary. Hence $\mathscr{F}(\underbrace{\mathscr{C}_0}_{(1)} \oplus \underbrace{\mathscr{C}_2}_{(1)})$ is concentrated in genus 0 and 1, which proves $\zeta_2 = 0$.

Since the image of $\mathscr{F}(\underbrace{\mathscr{C}_0}_{(k)} \oplus \underbrace{\mathscr{C}_1}_{(2)})$ under $\partial^0$ is in $\mathscr{F}(\underbrace{\mathscr{C}_0}_{(k+2)} \oplus \underbrace{\mathscr{C}_1}_{(1)})$, $\zeta_1$ must belong to $\mathscr{F}(\mathscr{C}_1)^{(2)}$. More precisely, $\zeta_1$ is an element of $\mathscr{F}^2(\mathscr{C}_1)^{(2)}$ because the differential $\partial^0$ preserves the genus. The $\mathbb{S}$-bimodule $\mathscr{C}_1$ is generated by the four elements $R_{\mathrm{rm}} \in \mathscr{C}(2,2)$, $R_{\mathrm{lm}} \in \mathscr{C}(2,2)$, $R_{\mathrm{a}} \in \mathscr{C}(1,3)$ and $R_{\mathrm{c}} \in \mathscr{C}(3,1)$. The only way to get an element of genus 2 is to graft one element from $\mathscr{C}(1,3)$ to an element from $\mathscr{C}(3,1)$. Finally $\zeta$ is linear combination of $R_{\mathrm{c}} \circ \sigma \circ R_{\mathrm{a}}$, with $\sigma \in \mathbb{S}_3$. And in this case, $\partial^0(\zeta)$ cannot contain elements like $\mu \circ R_{\mathrm{rm}} \circ \Delta - \mu \circ R_{\mathrm{lm}} \circ \Delta$ whence the contradiction. $\square$

This counter-example answers a question raised by [29], that is the functor $\mathscr{F}_{\text{dioperads}}^{\text{props}}$ is not exact.

**Theorem 47.** *The universal enveloping functor* $\mathscr{F}_{\text{dioperads}}^{\text{properads}}$ *is not exact.*

For this reason, we are reluctant to include dioperads in the preceding table. It is not enough in general to find a resolution of the genus 0 part of a properad to generate a complete resolution of it. Nevertheless, it is sometimes the case. We have emphasized the class of properads that admits a quadratic model, that is Koszul properad. We do the same thing with properads for which there exists a model with a genus 0 differential.

**Definition** (contractible properad). We call *contractible properad* any properad $\mathscr{P}$ that admits a model $\big(\mathscr{F}(\mathscr{C}), \partial^0\big) \xrightarrow{\sim} \mathscr{P}$ with $\partial^0|_{\mathscr{C}} : \mathscr{C} \to \mathscr{F}^0(\mathscr{C})$, that is the part of genus 0 of the free properad on $\mathscr{C}$.

It is equivalent to ask that $\mathscr{C}$ is a homotopy coproperad with structure maps $\delta_n : \mathscr{C} \to \mathscr{F}^0(\mathscr{C})^{(n)}$ with image of genus 0. In other words, $\mathscr{C}$ is a *homotopy codioperad.*

**Proposition 48.** *Let* $\mathscr{P} = \mathscr{F}(V)/(R)$ *be a properad defined by genus* 0 *relations,* $R \subset \mathscr{F}^0(V)$. *The properad* $\mathscr{P}$ *is a contractible properad if and only if the associated dioperad* $\mathscr{D} := \mathscr{F}^0(V)/(R)$ *admits a quasi-free (dioperadic) resolution* $\big(\mathscr{F}^0(\mathscr{C}), \partial^0\big) \xrightarrow{\sim} \mathscr{D}$, *which is a quasi-isomorphism preserved by the universal enveloping functor* $\mathscr{F}_{\text{dioperads}}^{\text{properads}}$.

*Proof.* If $\mathscr{P}$ is contractible, we denote by $\big(\mathscr{F}(\mathscr{C}), \partial^0\big) \xrightarrow{\sim} \mathscr{P}$ its genus 0 differential model. Since $\partial^0$ preserves the genus, the chain complex $\big(\mathscr{F}(\mathscr{C}), \partial^0\big)$ is equal to the direct sum of sub-complexes $\bigoplus_{g \geqq 0} \big(\mathscr{F}^g(\mathscr{C}), \partial^0\big)$. Hence, the genus 0 chain complex is a resolution of $\mathscr{D}$. And by Corollary 45 the image under the universal enveloping functor $\mathscr{F}_{\text{dioperads}}^{\text{properads}}$ of the quasi-isomorphism $\big(\mathscr{F}^0(\mathscr{C}), \partial^0\big) \xrightarrow{\sim} \mathscr{D}$ is the resolution $\big(\mathscr{F}(\mathscr{C}), \partial^0\big) \xrightarrow{\sim} \mathscr{P}$. The other way is trivial. $\square$

A Koszul contractible properad $\mathscr{P}$ is a properad with a minimal model $\big(\mathscr{F}(\mathscr{C}), \partial^0\big) \xrightarrow{\sim} \mathscr{P}$ whose differential $\partial^0$ is quadratic and genus 0. It is equivalent to say that $\mathscr{C}$ is a codioperad. If a properad $\mathscr{P} = \mathscr{F}(V)/(R)$ with genus 0 relations is contractible

Koszul, then the associated dioperad $\mathscr{D} = \mathscr{F}^0(V)/(R)$ is Koszul in the sense of [11]. But it is not true that any Koszul dioperad is a Koszul contractible properad as the example of $\mathscr{NC}$-$\mathscr{F}rob$ shows. Lemma 46 shows that it is not contractible. Moreover we shall see below that it is not Koszul as a properad either.

**Proposition 49.** *Let $\mathscr{P} = \mathscr{F}(V)/(R)$ be a Koszul properad defined by a finite dimensional $\mathbb{S}$-bimodule $V$ and by genus 0 relations, $R \subset \mathscr{F}^0(V)$. If the Koszul dual properad of $\mathscr{P}$ is equal, as an $\mathbb{S}$-bimodule, to the Koszul dual dioperad of the associated dioperad $\mathscr{D} := \mathscr{F}^0(V)/(R)$ then the properad $\mathscr{P}$ is contractible.*

*Proof.* In this case, the Koszul dual coproperad $\mathscr{P}^{¡} = \mathscr{P}^{!\vee}$ is equal to the Koszul dual dioperad $\mathscr{D}_{¡} = \mathscr{D}^{!\vee}$. Hence the image of the partial coproduct $\Delta_{(1,1)} : \mathscr{P}^{¡} \to \mathscr{P}^{¡} \boxtimes \mathscr{P}^{¡}$ is actually in $\mathscr{P}^{¡} \square \mathscr{P}^{¡}$ which is the part of genus 0 of $\mathscr{P}^{¡} \boxtimes \mathscr{P}^{¡}$. $\square$

The Koszul dual properad is equal to the Koszul dual dioperad if and only if the part of genus $> 0$ of $\mathscr{P}^!$ vanished, that is $\mathscr{F}^g(V^\vee)/(R^\perp) = 0$ for $g > 0$. Proposition 49 allows us to give examples of Koszul contractible properads. One way to prove that a properad is Koszul is by means of *distributive laws* (see [41], Proposition 8.4). Let $\mathscr{P}$ be a quadratic properad of the form $\mathscr{P} = \mathscr{F}(V, W)/(R \oplus D \oplus S)$, where $R \subset \mathscr{F}^{(2)}(V)$, $S \subset \mathscr{F}^{(2)}(W)$ and where

$$D \subset (I \oplus \underbrace{W}_{1}) \boxtimes_c (I \oplus \underbrace{V}_{1}) \oplus (I \oplus \underbrace{V}_{1}) \boxtimes_c (I \oplus \underbrace{W}_{1}).$$

The two pairs of $\mathbb{S}$-bimodules $(V, R)$ and $(W, S)$ generate two properads denoted $\mathscr{A} := \mathscr{F}(V)/(R)$ and $\mathscr{B} := \mathscr{F}(W)/(S)$.

**Definition** (distributive law). Let $\lambda$ be a morphism of $\mathbb{S}$-bimodules

$$\lambda : (I \oplus \underbrace{W}_{1}) \boxtimes_c (I \oplus \underbrace{V}_{1}) \to (I \oplus \underbrace{V}_{1}) \boxtimes_c (I \oplus \underbrace{W}_{1})$$

such that the $\mathbb{S}$-bimodule $D$ is defined by the image of

$$(\mathrm{id}, -\lambda) : (I \oplus \underbrace{W}_{1}) \boxtimes_c (I \oplus \underbrace{V}_{1})$$
$$\to (I \oplus \underbrace{W}_{1}) \boxtimes_c (I \oplus \underbrace{V}_{1}) \oplus (I \oplus \underbrace{V}_{1}) \boxtimes_c (I \oplus \underbrace{W}_{1}).$$

We call $\lambda$ a *distributive law* and denote $D$ by $D_\lambda$ if the two following morphisms are injective:

$$\begin{cases} \underbrace{\mathscr{A}}_{1} \boxtimes_c \underbrace{\mathscr{B}}_{2} \to \mathscr{P}, \\ \underbrace{\mathscr{A}}_{2} \boxtimes_c \underbrace{\mathscr{B}}_{1} \to \mathscr{P}. \end{cases}$$

The last condition must be seen as a coherence axiom, which ensures that the natural morphism $\mathscr{A} \boxtimes \mathscr{B} \to \mathscr{P}$ is injective. In this case, [42], Proposition 8.4 states that $\mathscr{P}$ is Koszul if $\mathscr{A}$ and $\mathscr{B}$ are Koszul. A properad is called *binary* if it is generated by binary products and coproducts.

**Proposition 50.** *Let $\mathscr{D} = \mathscr{F}^0(V)/(R)$ be a binary Koszul dioperad defined by a distributive law such that $V$ is finite dimensional. Then the associated properad $\mathscr{P} := \mathscr{F}(V)/(R)$ is Koszul and contractible.*

*Proof.* If a binary dioperad $\mathscr{D}$ defined by a distributive law verifies the hypotheses of [11], Proposition 5.9, then the associated properad $\mathscr{P}$ is also defined by distributive law and verifies the hypotheses of [41], Proposition 8.4. In this case, the Koszul dual coproperad, given by [41], Proposition 8.2, has a genus 0 coproduct. □

**Corollary 51.** *The properads $\mathscr{B}i\mathscr{L}ie$ of Lie bialgebras and $\varepsilon\mathscr{B}i$ of infinitesimal Hopf algebras are Koszul contractible.*

In this case, the Koszul dual (co)dioperad provides the good space of "homotopies" for the resolution of the properad. Therefore, it gives the proper notion of homotopy $\mathscr{P}$-gebra (see 6.1). An example of this fact, for $\mathscr{B}i\mathscr{L}ie$, can be found in [32], Section 3.2, see also [12], [31].

**Remark.** Dually, in this case, the products of operations based on strictly positive genus graphs of the Koszul dual properad always vanish. If $g$ denotes the genus of the underlying graph, it means that any such product is equivalent to products based on graphs with $g$ simple loops $\diamond$, using the relations of the products and the relations of coproducts. Therefore, it is zero because of the relation $\diamond = 0$ in the Koszul dual properad. This statement is a non-trivial result about the coherence of the relations of a properad.

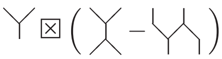
To any binary properad $\mathscr{P}$, we associate a properad $\mathscr{P}_\diamond$ which encodes $\mathscr{P}$-gebras satisfying the extra loop relation $\diamond = 0$. Since the properad $\mathscr{B}i\mathscr{L}ie$ is Koszul, its Koszul dual properad $\mathscr{F}rob_\diamond$ is also Koszul by Koszul duality theory. This means that $\mathscr{F}rob_\diamond$ has a quadratic model. Since the properad $\mathscr{B}i\mathscr{L}ie$ has non trivial higher genus compositions, this model is not contractible, that is the boundary map creates higher genus graphs. The example $\mathscr{F}rob_\diamond$ provides an example of a Koszul non-contractible properad. (We do not know how to prove this result without the help of Koszul duality for properads.)

Let $\mathscr{C}$ denote the Koszul dual coproperad of $\mathscr{N}\mathscr{C}$-$\mathscr{F}rob$, that is $\mathscr{C} = s^{-1}\varepsilon\mathscr{B}i_\diamond^\vee$. Recall that a properad $\mathscr{P}$ is Koszul if and only if the cobar construction of the Koszul dual coproperad $\Omega(\mathscr{P}^{\text{i}}) = \big(\mathscr{F}(\mathscr{P}^{\text{i}}), \partial\big)$ is a resolution of $\mathscr{P}$. This statement is not true for $\mathscr{N}\mathscr{C}$-$\mathscr{F}rob$. The cycle $\xi$ given above induces a non-trivial element in homology.

**Lemma 52.** *The cycle $\xi$ is not a boundary under $\partial$.*

*Proof.* We use the same notations as in Lemma 46 but applied to $\partial$ instead of $\partial^0$. The space $\mathscr{C}_1$ is generated by the elements $R_{\text{lm}}$, $R_{\text{rm}}$, $R_{\text{a}}$, $R_{\text{c}}$ and some $R_i$ for $i = 1, \dots, 4$. For the same reason, $\zeta_1$ must be an element of $\mathscr{F}(\mathscr{C}_1)^{(2)}$. Since the image under $\partial$ of any element of $\mathscr{C}_1$ is a graph with two adjacent vertices indexed by $\curlyvee$ or $\curlywedge$, the element $\mu \circ R_{\text{lm}} \circ \Delta$ cannot belong to $\partial(\zeta_1)$. Hence $\mu \circ R_{\text{lm}} \circ \Delta$ must be an element of $\partial(\zeta_2)$. Since $\partial$ is quadratic, there exists an element $S$ in $\mathscr{C}_2$ such that $\partial(S) = \mu \circ R_{\text{lm}} + \cdots$ or $\partial(S) = R_{\text{lm}} \circ \Delta + \cdots$. Such an $S$ has to be an element of either $\varepsilon\mathscr{B}i_\diamond^\vee(1,2)^{(3)}$ or $\varepsilon\mathscr{B}i_\diamond^\vee(2,1)^{(3)}$.

Consider the first case, the second one being symmetrical. The only element in $\varepsilon\mathscr{B}i_{\diamond}^{\vee}(1,2)^{(3)}$ whose partial coproduct includes $\mu \circ R_{\mathrm{lm}}$ is the dual of the composite of



in $\varepsilon\mathscr{B}i_{\diamond}(1,2)^{(3)}$. The associativity relation and the loop relation in $\varepsilon\mathscr{B}i_{\diamond}$ show that this composite is equal to zero, which concludes the proof. $\square$

**Theorem 53.** *The properad $\mathscr{N}\mathscr{C}$-$\mathscr{F}rob$ of non-commutative Frobenius bialgebras and the properad $\varepsilon\mathscr{B}i_{\diamond}$ of involutive infinitesimal bialgebras are not Koszul.*

We hope that this helps to clarify the general picture of models for prop(erad)s.

## 6. Homotopy $\mathscr{P}$-gebra

In this section, we define the notion of $\mathscr{P}$-gebra up to homotopy or *homotopy $\mathscr{P}$-gebra*. We make explicit structures of homotopy $\mathscr{P}$-gebras in terms of Maurer-Cartan elements. We also define and make explicit morphisms of homotopy $\mathscr{P}$-algebras, when $\mathscr{P}$ is an operad, in terms of Maurer-Cartan elements in an $L_\infty$-algebra. This last part uses the notion of homotopy Koszul (colored) operads defined in the previous section.

**6.1. $\mathscr{P}$-gebra, $\mathscr{P}_{(n)}$-gebra and homotopy $\mathscr{P}$-gebra.** Let $\mathscr{P}$ be a dg prop(erad) and $\Omega(\mathscr{C})$ be a model of $\mathscr{P}$.

**Definition** (homotopy $\mathscr{P}$-gebra). A structure of *homotopy $\mathscr{P}$-gebra* on a dg module $X$ is a morphism of dg prop(erad)s: $\Omega(\mathscr{C}) \to \mathrm{End}_X$.

Any $\mathscr{P}$-gebra is a homotopy $\mathscr{P}$-gebra of particular type. In this case, the morphism of dg-properads factors through $\mathscr{P}$, that is $\Omega(\mathscr{C}) \xrightarrow{\sim} \mathscr{P} \to \mathrm{End}_X$. For the Koszul operads $\mathscr{A}ss$, $\mathscr{C}om$, $\mathscr{L}ie$, this notion coincides with homotopy associative, commutative, Lie algebras. For the properads $\mathscr{B}i\mathscr{L}ie$ and $\mathscr{A}ss\mathscr{B}i$, we get the notions of homotopy Lie bialgebras and homotopy bialgebras. Since $\mathscr{B}i\mathscr{L}ie$ is contractible, the explicit definition given in [12], [31] coincides with this one.

[32], Theorem 5 shows that a structure of homotopy $\mathscr{P}$-gebra on $X$ is equivalent to a morphism of $\mathbb{S}$-bimodules in $s^{-1}\,\mathrm{Hom}_0^{\mathbb{S}}(\mathscr{C}, \mathrm{End}_X)$ which is a Maurer-Cartan element in the $L_\infty$-convolution algebra $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, \mathrm{End}_X)$.

**Theorem 54.** *A $\mathscr{P}$-gebra structure on $X$ is equivalent to a Maurer-Cartan element in* $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, \mathrm{End}_X)$.

This notion is well defined and independent of the choice of a model. By [32], Theorem 13, if $\Omega(\mathscr{C}_1)$ and $\Omega(\mathscr{C}_2)$ are two models of $\mathscr{P}$, then the convolution $L_\infty$-algebras are quasi-isomorphic, which induces a bijection between the set of Maurer-Cartan elements.

We can discuss the form of the solutions of the Maurer-Cartan equation. It gives the following definition.

**Definition** ($\mathscr{P}_{(n)}$-gebra). A dg module $X$ endowed with a Maurer-Cartan element $\gamma$ in $\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, \mathrm{End}_X)$ such that $\gamma(c) = 0$ for every $c \in \mathscr{C}_{k>n}$ is called a *$\mathscr{P}_{(n)}$-gebra.*

This notion is the direct generalization of the notion of $A_{(n)}$-algebra of Stasheff [36] or $L_{(n)}$-algebras. A $\mathscr{P}_{(n)}$-gebra is a homotopy $\mathscr{P}$-gebra with strict relations from degree $n$.

**6.2. Morphisms of homotopy $\mathscr{P}$-algebras as Maurer-Cartan elements.** Another application of the notion of homotopy Koszul can be found in the study of morphisms between homotopy $\mathscr{P}$-algebras. A *colored properad* is an operad such that the inputs and outputs are labelled by an extra labelling and such that the composition is coherent with respect to this extra labelling. That is if the 'colors' (labelling) do not match, the composition of operations vanishes. It is proven in [44] how to extend Koszul duality of operads to colored operads. It is straightforward to generalize Theorem 40 to this case.

Let $\mathscr{P} = \mathscr{F}(V)/(R)$ be a Koszul operad. One can define the 2-colored operad $\mathscr{P}_{\bullet\to\bullet}$ by $\mathscr{P} = \mathscr{F}(V_1 \oplus V_2 \oplus f)/(R_1 \oplus R_2 \oplus R_{\bullet\to\bullet})$, where $V_1$ and $R_1$ (resp. $V_2$ and $R_2$) are copies of $V$ and $R$ with inputs and outputs labelled by the color 1 (resp. 2), $f$ is a generator of arity $(1,1)$ which goes from 1 to 2 and $R_{\bullet\to\bullet}$ is generated by $v \circ f^{\otimes n} - f \circ v$ for any element $v \in V(n)$ (see [26] for more details). The purpose of this definition lies in the following result. A structure of $\mathscr{P}_{\bullet\to\bullet}$-algebra is the data of two $\mathscr{P}$-algebras with a morphism of $\mathscr{P}$-algebras between them.

**Lemma 55.** *When $\mathscr{P}$ is Koszul generated by a finite dimensional $\mathbb{S}$-module $V$ such that $V(1) = 0$, the 2-colored operad $\mathscr{P}_{\bullet\to\bullet}$ is homotopy Koszul.*

*Proof.* (i) The operad $(\mathscr{P}_{\bullet\to\bullet})_2$ is equal to $\mathscr{F}(V_1 \oplus V_2 \oplus f)/(R_1 \oplus R_2 \oplus R_\bullet)$, where $R_\bullet = f \circ V_1$. Hence, it is equal to $(\mathscr{P}_{\bullet\to\bullet})_2 \cong \mathscr{P}_1 \oplus \mathscr{P}_2 \oplus \mathscr{P} \circ (I \oplus \underbrace{f}_{\geqq 1})$. Its Koszul dual is equal to $(\mathscr{P}_{\bullet\to\bullet})_2^{\mathrm{i}} = \mathscr{P}_1^{\mathrm{i}} \oplus \mathscr{P}_2^{\mathrm{i}} \oplus s(f \circ \mathscr{P}^{\mathrm{i}})$. Therefore, $\big(\mathscr{F}(s^{-1}\overline{\mathscr{P}}_1^{\mathrm{i}} \oplus s^{-1}\overline{\mathscr{P}}_2^{\mathrm{i}} \oplus f \circ \overline{\mathscr{P}}^{\mathrm{i}}), \delta_2\big)$ is a quadratic model of $(\mathscr{P}_{\bullet\to\bullet})_2$, because $\delta_2$ is equal to 3 copies of the Koszul resolution of $\mathscr{P}$.

(ii) Since $\mathscr{P}_{\bullet\to\bullet} \cong \mathscr{P}_1 \oplus \mathscr{P}_2 \oplus \mathscr{P} \circ (I \oplus \underbrace{f}_{\geqq 1})$, it is equal to $(\mathscr{P}_{\bullet\to\bullet})_2$.

(iii) Since $V$ is finite dimensional and $V(1) = 0$, the filtration with the number of leaves gives a suitable filtration. □

In this case, the minimal model of $\mathscr{P}_{\bullet\to\bullet}$ is given by $\big(\mathscr{F}(s^{-1}\overline{\mathscr{P}}_1^{\mathrm{i}} \oplus s^{-1}\overline{\mathscr{P}}_2^{\mathrm{i}} \oplus f \circ \overline{\mathscr{P}}^{\mathrm{i}}), \delta\big)$ by Theorem 40.

**Proposition 56.** *An algebra over the model of $\mathscr{P}_{\bullet\to\bullet}$ is the data of two homotopy $\mathscr{P}$-algebras with a homotopy (or weak) morphism between them.*

*Proof.* A morphism of 2-colored operads $\big(\mathscr{F}(s^{-1}\overline{\mathscr{P}}^{\mathrm{i}}_{\bullet\to\bullet 2}), \delta\big) \to \mathrm{End}_{X,Y}$ defines a homotopy $\mathscr{P}$-algebra structure on $X$ and $Y$. The component on $\{\mathrm{Hom}(X^{\otimes n}, Y)\}_{n\geqq 1}$ is equivalent to a morphism of dg $\mathscr{P}^{\mathrm{i}}$-coalgebras $\mathscr{P}^{\mathrm{i}}(X) \to \mathscr{P}^{\mathrm{i}}(Y)$, that is between the bar constructions of $X$ and $Y$. □

**Theorem 57.** *Morphisms of homotopy $\mathscr{P}$-algebras between $X$ and $Y$ are in one-to-one correspondence with Maurer-Cartan elements in the $L_\infty$-algebra*

$$\big(\mathrm{Hom}^{\mathbb{S}}(\mathscr{P}^{\text{¡}}_{\bullet\to\bullet 2}, \mathrm{End}_{X,Y}), \delta\big).$$

Notice that this result was already proved by hands in [8] in the case of homotopy Lie algebras.

Finally, a structure of homotopy $\mathscr{P}$-algebra on $X$ is a Maurer-Cartan element in the strict Lie algebra $\mathrm{Hom}^{\mathbb{S}}(\mathscr{P}^{\text{¡}}, \mathrm{End}_X)$, whereas a morphism of homotopy $\mathscr{P}$-algebras between $X$ and $Y$ is a (generalized) Maurer-Cartan element in the homotopy Lie algebra $\mathrm{Hom}^{\mathbb{S}}(\mathscr{P}^{\text{¡}}, \mathrm{End}_{X,Y})$. The conceptual explanation of this phenomenon is the following. In the first case, we have a quadratic model of the Koszul operad $\mathscr{P}$ and the second case, we use a non-quadratic model of the homotopy Koszul 2-colored operad $\mathscr{P}_{\bullet\to\bullet}$.

## References


[1] *John Frank Adams*, Infinite loop spaces, Ann. Math. Stud. **90**, Princeton University Press, Princeton, N.J., 1978.

[2] *M. Aguiar*, Infinitesimal Hopf algebras, New trends in Hopf algebra theory (La Falda 1999), Contemp. Math. **267** (2000), 1–29.

[3] *Marcelo Aguiar* and *Muriel Livernet*, The associative operad and the weak order on the symmetric groups, J. Homot. Relat. Struct. **2** (2007), no. 1, 57–84.

[4] *Clemens Berger* and *Benoit Fresse*, Combinatorial operad actions on cochains, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 1, 135–174.

[5] *C. Berger* and *I. Moerdijk*, Axiomatic homotopy theory for operads, Comment. Math. Helv. **78** (2003), no. 4, 805–831.

[6] *D. Borisov* and *Yu I. Manin*, Generalized operads and their inner cohomomorphisms, Progr. Math. **265** (2008), 247–308.

[7] *E. H. Brown*, *Jr*., Twisted tensor products. I, Ann. Math. (2) **69** (1959), 223–246.

[8] *V. A. Dolgushev*, Erratum to: A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold, arXiv:math.QA/0703113 (2007).

[9] *Benjamin Enriquez* and *Pavel Etingof*, On the invertibility of quantization functors, J. Algebra **289** (2005), no. 2, 321–345.

[10] *B. Fresse*, Koszul duality of operads and homology of partition posets, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic $K$-theory, Contemp. Math. **346** (2004), 115–215.

[11] *W. L. Gan*, Koszul duality for dioperads, Math. Res. Lett. **10** (2003), no. 1, 109–124.

[12] *W. L. Gan*, On a theorem of Conant-Vogtmann, Math. Res. Lett. **12** (2005), no. 5-6, 749–757.

[13] *M. Gerstenhaber*, The cohomology structure of an associative ring, Ann. Math. (2) **78** (1963), 267–288.

[14] *M. Gerstenhaber* and *A. Voronov*, Homotopy $G$-algebras and moduli space operad, Internat. Math. Res. Notices (1995), no. 3, 141–153.

[15] *E. Getzler* and *M. M. Kapranov*, Modular operads, Compos. Math. **110** (1998), no. 1, 65–126.

[16] *Johan Granåker*, Strong homotopy properads, Int. Math. Res. Not. IMRN **14** (2007).

[17] *D. Guin* and *J.-M. Oudom*, On the Lie envelopping algebra of a pre-Lie algebra, arXiv.org:math/0404457 (2004).

[18] *M. Kapranov* and *Yu. Manin*, Modules and Morita theorem for operads, Amer. J. Math. **123** (2001), no. 5, 811–838.

[19] *R. M. Kaufmann*, On spineless cacti, Deligne's conjecture and Connes-Kreimer's Hopf algebra, Topology **46** (2007), no. 1, 39.

[20] *M. Kontsevich* and *Y. Soibelman*, Deformations of algebras over operads and the Deligne conjecture, Conférence Moshé Flato 1999, Vol. I (Dijon), Math. Phys. Stud. **21** (2000), 255–307.

[21] *T. Lada* and *M. Markl*, Symmetric brace algebras, Appl. Categ. Struct. **13** (2005), no. 4, 351–370.

[22] *J.-L. Loday*, *A. Frabetti*, *F. Chapoton*, and *F. Goichot*, Dialgebras and related operads, Lect. Notes Math. **1763**, Springer-Verlag, Berlin 2001.

[23] *J.-L. Loday*, Generalized bialgebras and triples of operads, arXiv:math.QA/0611885 (2006).
[24] *J.-L. Loday* and *M. Ronco*, On the structure of cofree Hopf algebras, J. reine angew. Math. **592** (2006), 123–155.
[25] *Saunders Mac Lane*, Categorical algebra, Bull. Amer. Math. Soc. **71** (1965), 40–106.
[26] *M. Markl*, Homotopy algebras are homotopy algebras, Forum Math. **16** (2004), no. 1, 129–160.
[27] *M. Markl*, Intrinsic brackets and the $l_\infty$-deformation theory of bialgebras, arXiv.org:math/0411456 (2004).
[28] *M. Markl*, A resolution (minimal model) of the PROP for bialgebras, J. Pure Appl. Alg. **205** (2006), no. 2, 341–374.
[29] *M. Markl* and *A. Voronov*, PROPped up graph cohomology, arXiv:math.QA/0307081 (2003).
[30] *S. A. Merkulov*, Prop profile of deformation quantization and graph complexes with loops and wheels, 2004.
[31] *S. A. Merkulov*, Prop profile of poisson geometry, Commun. Math. Phys. **262** (2006), 117–135.
[32] *S. A. Merkulov* and *B. Vallette*, Deformation theory of representations of prop(erad)s II, J. reine angew. Math., to appear.
[33] *J. E. McClure* and *J. H. Smith*, A solution of Deligne's Hochschild cohomology conjecture, Recent progress in homotopy theory (Baltimore, MD, 2000), Contemp. Math. **293** (2002), 153–193.
[34] *John C. Moore*, Differential homological algebra, Actes du Congrès International des Mathématiciens (Nice 1970), Tome 1, Gauthier-Villars, Paris (1971), 335–339.
[35] *D. Quillen*, Rational homotopy theory, Ann. Math. (2) **90** (1969), 205–295.
[36] *J. D. Stasheff*, Homotopy associativity of $H$-spaces, I, II, Trans. Amer. Math. Soc. **108** (1963), 275–292; ibid. **108** (1963), 293–312.
[37] *D. E. Tamarkin*, Another proof of M. Kontsevich formality theorem, arXiv:math/9803025 (1998).
[38] *B. Vallette*, Free monoid in monoidal abelian categories, Appl. Categ. Struct. **17**, no. 1 (2009), 43–61.
[39] *B. Vallette*, Homology of generalized partition posets, J. Pure Appl. Algebra **208** (2006), no. 2, 699–725.
[40] *B. Vallette*, Manin products, Koszul duality, Loday algebras and Deligne conjecture, J. reine angew. Math. **620** (2008), 105–164.
[41] *B. Vallette*, A Koszul duality for props, Trans. Amer. Math. Soc. **359** (2007), 4865–4993.
[42] *B. Vallette*, Transfer of algebraic homotopy structures, in preparation (2009).
[43] *P. Van der Laan*, Operads up to Homotopy and Deformations of Operad Maps, arXiv:math.QA/0208041 (2002).
[44] *P. Van der Laan*, Coloured Koszul duality and strongly homotopy operads, arXiv:math.QA/0312147 (2003).
[45] *Alexander A. Voronov*, Homotopy Gerstenhaber algebras, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud. **22** (2000), 307–331.
[46] *C. A. Weibel*, An introduction to homological algebra, Cambridge Stud. Adv. Math. **38**, Cambridge University Press, Cambridge 1994.

---

Department of Mathematics, Stockholm University, 10691 Stockholm, Sweden
e-mail: sm@math.su.se

Laboratoire J.A. Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
e-mail: bruno.vallette@math.unice.fr

Eingegangen 28. September 2007, in revidierter Fassung 29. April 2008