CHAPTER 27

DEFORMATION UP TO BREAKING OF PERIODIC WAVES ON A BEACH

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ABSTRACT

An experimental description is presented for the transformation of periodic waves which approach breaking on a gently sloping beach. The data include the variation of wave height, phase velocity, wave surface profiles, and the maximum value of the wave height to water depth ratio $(H/h)_{max}$ around the breaking point.

The results are compared with the theories of sinusoidal and cnoidal wave shoaling, and the latter is shown in most cases to agree remarkably well when the laminar energy loss along the walls and bottom of the wave tank is included.

An empirical relation is established between wave length to water depth ratio L/h at the breaking point and the deep water wave steepness H_o/L_o. Also the maximum wave height to water depth ratio at breaking shows considerably less scattering than found previously, when plotted versus $S = h_x L/h$, h_x being bottom slope.

1. INTRODUCTION

The literature shows a considerable number of experimental investigations of the slow transformation of waves on a sloping bottom, which is denoted shoaling. In particular, data for the variation of the wave height have been reported.

Most of these results, however, do not confirm each other. Thus no definitive conclusion has been obtained so far neither about the real variation of the wave height nor as to which theory will predict the variation sufficiently accurately.

Iversen (1952)[†] presented experimental data which showed that the height of periodic waves on a sloping bottom grows much faster than predicted by the sinusoidal wave theory, and Brink-Kjær and Jonsson (1973) showed that actually the variation resemble a cnoidal wave shoaling.

Similar experiments were made by Ippen and Eagleson (1955), and Eagleson (1956) arrived at the same conclusion though the pattern was

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†The paper by Iversen (1953), 'Waves and breakers in shoaling water,' Third Conf. Coastal Engrg., Cambridge, Mass. 1952, is almost identical with Iversen (1952).

less pronounced, partly due to considerable scattering. Also the experiments by Vera-Cruz (1965) should be mentioned.

Iwagaki (1968) compared his experiments with the theory of hyperbolic waves, in which the wave profiles are approximated by parts of solitary wave profiles. As far as it has been possible to ascertain from the graphical presentation in the paper, the agreement is good for waves with small deep water steepness ($H_o/L_o < 0.005$), (though the theoretical curves have not been extended to the full region of water depths for which measurements are presented). Actually it may be shown (Svendsen, 1974) that to the first order the hyperbolic wave height varies as h^{-1}, i.e. as a shoaling solitary wave. Further, in particular waves of small deep water wave steepness (swell-type) will more and more resemble a solitary wave in shape as the water depth decreases. Hence the best agreement with hyperbolic waves should be expected for swell-type waves. For steeper waves the comparison seems inconclusive as should also be expected, as the theory does not apply to such waves.

Against this stand the solitary wave experiments by Ippen and Kulin (1954) and by Camfield and Street (1969) indicating that although a solitary wave is as far from a sinusoidal wave as well possible, the variation of its height is much better predicted by the $h^{-1/4}$ rule valid for long sinusoidal waves. This is further confirmed by the numerical calculations by Madsen and Mei (1969), which agree quite well with the results of Camfield and Street.

Finally, Wiegel (1950) claims that in general his experimental results for periodic waves on slopes 1:10.8 and 1:20 follow the linear theory.

One possible reason for these discrepancies is that all the experiments for pure solitary waves, and Wiegel's with periodic waves, have been performed on slopes which are actually too steep to allow the shoaling assumption to be valid. Another important factor is the friction losses, which can be shown to have a considerable effect on the shoaling process, in particular in a relatively narrow laboratory wave flume.

Also part of the surprisingly large scattering which appears in many experimental results for wave quantities is most likely due to the free second harmonic waves generated by the sinusoidal motion of a piston-wave-generator.

The aim of the present investigation has been to try to clear up some of these uncertainties, using the facilities for generation of waves of extremely regular and permanent form, described by Buhr Hansen, Schiolten and Svendsen (1975).

In addition to the wave height, the phase velocity and the mean water level ('set-down') have been measured, and records have been obtained for the wave surface profiles. The results are compared with theory, and since it is rather evident from previous investigations that the linear theory is doomed to fail, the major emphasis is placed on a comparison with cnoidal wave theory. Perhaps it should be added for completeness that a second or higher order Stokes theory will be out of question, too, when the Ursell parameter U (defined as HL^2/h^3)

(as in most cases) grows far beyond 30 or 40, which is about the limit for which a higher order Stokes theory is applicable.*

In each case will be discussed outcome of the comparison, and an analysis will be attempted of the possible reason for discrepancies.

2. DESCRIPTION OF EXPERIMENTAL FACILITIES AND PROCEDURE

The waves are generated by a flap-type wave generator in a flume 33 m long, 60 cm wide, with a plane beach sloping 1:35 (see Fig. 1). The motion of the wave generator is controlled by a PDP 8 mini-computer, which generates a command signal of the form

$$\xi = e_1 \sin\omega t + e_2 \sin(2\omega t+\beta_2) \qquad (1)$$

(Buhr Hansen and Svendsen, 1974). Fig. 2 shows a comparison between a resulting measured wave profile (with almost no free second harmonic components) and a second order Stokes wave. The parameter $U = HL^2/h^3$ is about 2. It appears that even for U as large as 40 - 50 (which is far up in the cnoidal region), the waves generated by (1) remain of clean and constant form.

This is important because the waves in the constant depth part of the wave flume represent the initial conditions for the shoaling process. (It may be noticed that it has no meaning to consider whether the wave produced by (1) is a 'Stokes' or a 'cnoidal' wave, as long as it is of constant form.)

The water surface variation is recorded by a resistance wave height transducer (two silver wires, 0.17 mm diameter, 5 mm apart), the signal of which is scanned by the computer 400 times per second. The transducer is mounted on a carriage which is moving slowly along the flume. Vertical irregularities of the rails for the carriage are eliminated by storing in the computer a zero level correction, which is obtained from the wave transducer during a carriage-run without waves.

In the experiments, the computer determines on line the height H of each wave, the mean water level $\overline{\eta}$, and by means of an additional wave height transducer, the phase velocity c. At the same time selected wave profiles are stored. After each experiment the results may be plotted out on an ordinary pen-recorder.

In all experiments reported, the still water depth was 36.0 cm and the plane slope was 1:35. The wave frequency varied between 0.3 Hz and 1.2 Hz, the wave height in the constant depth part of the flume between 3.5 cm and 10 cm.

The calibration factor for the wave transducers was determined by linear regression on 10 - 12 data points corresponding to 1 cm increments in the submergence of the transducer. In all experiments the minimum submergence was more than 1.5 cm, corresponding to a regression coefficient larger than 0.999.

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*The limit referred to corresponds to the largest value of H for which there is no secondary wave crest in the trough. In the second order this means that $U < (L/h)^3 \pi^{-1} (3 \coth^3 kh - \coth kh)^{-1}$, the maximum of which is 26.3 for $kh \to 0$ (see Svendsen and Jonsson, 1976). For higher order theories U may be slightly larger.

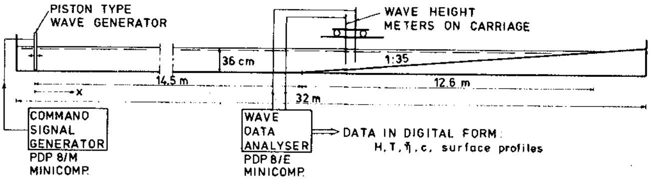

Fig. 1 Experimental facility

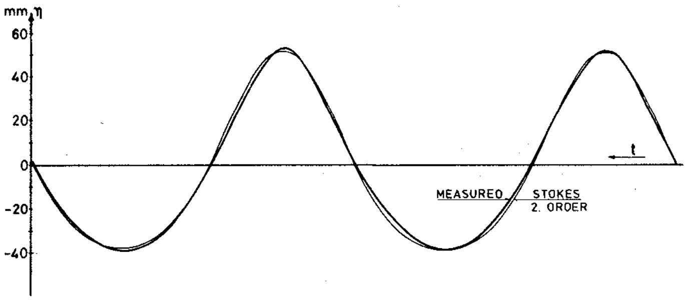

Fig. 2 Comparison between profile measured in front of the wave generator and Stokes second order approximation with the same wave height and period

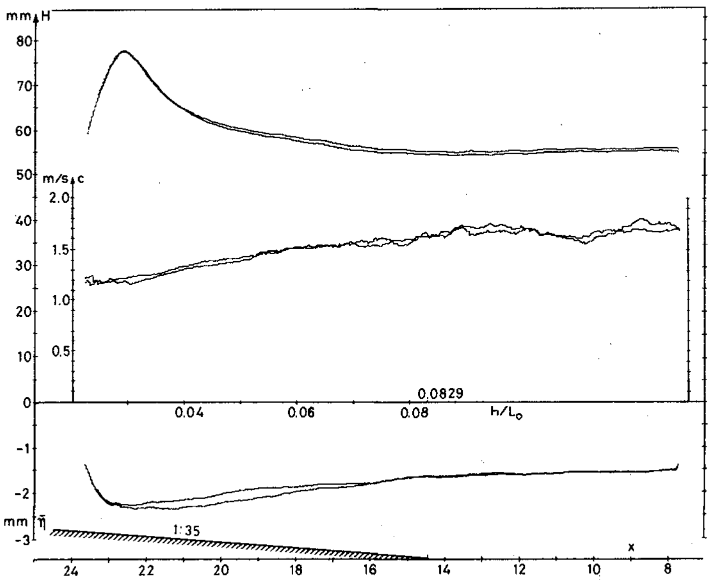

Fig. 3 Illustration of reproducibility of experiments

The reproducibility of the experiments is illustrated in Fig. 3 showing two records of the same experiment. The figure also shows that even most of the small and apparently inexplicable irregularities in the records are obviously repeated exactly the same way. Some may be due to irregularities in the wave flume (though the accuracy of alignment of sides and bottom is well below 1 mm) but most of them seemed to be generated by either capillary waves, secondary waves generated by the breaking process, and perhaps the rest of the free second harmonics.

3. THE SHOALING ASSUMPTION

The notion of wave shoaling or wave transformation on a beach was introduced on an intuitive basis by Rayleigh (1911). In his approach there are three more or less independent assumptions involved:

(a) The wave will — to the first approximation in bottom slope — continuously adjust its form so that surface profile, phase and particle velocities, pressure variation, etc. can be determined from the horizontal bottom theory, applying the local values of water depth and wave height.

(b) The wave energy flux through a vertical section is constant, which implies that the reflection is negligible.

(c) The number of waves remain constant during the shoaling process so that the wave period T is conserved.

Essentially each of these assumptions requires a 'sufficiently gently varying water depth', but how gently will actually depend on the wave theory considered. This question can be analysed theoretically by rigorous perturbation expansions including the effect of the bottom slope h_x.

Rayleigh, of course, presented the ideas in terms of the linear wave theory, and for that case it may be shown that the shoaling assumptions will be satisfied provided the relative change in water depth over a wave length is of the same order of magnitude as the wave steepness (or smaller), i.e.

$$S \equiv h_x L/h = O(H/L) \tag{2}$$

For higher order Stokes waves only smaller values of h_x are allowed (depending on the order considered), and for (first order) cnoidal waves, Svendsen (1974) showed that a consistent shoaling theory requires $S = O(h/L)^3$.

In conclusion we notice that in all cases the parameter S occurs and that shoaling conditions imply that S is too small to be of importance. This will be discussed further later on.

4. THE WAVE HEIGHT VARIATION

Experimental Results

Since the wave period is assumed to be constant, one of the principal problems in wave shoaling is to determine the wave height H as a function of the water depth h.

Figs. 4 and 5 show the recorded variation of the wave height for deep water wave steepnesses ranging from 0.0039 to 0.064. Both dimensionless wave period $T\sqrt{g/h}$, wave height to water depth ratio H/h in the constant depth part of the flume, and the theoretically determined deep water wave steepness H_o/L_o are given in each figure.

There are two experimental curves in each figure. One represent each individually measured wave height, the other a moving average over the length of the reflection pattern.

Though the curves for the individual wave heights seem to show a continuous variation they are actually step-curves. This is because the carriage with the wave transducer moves 2 - 4 cm (depending on the wave period) along the wave flume during one wave period, i.e. between each new result for the wave height.

In the following the origin of the theoretical curves is described and discussed, but first we consider the effect of energy loss due to friction.

Energy Loss due to Friction

This effect was taken into account in the theoretical curves by reducing the energy flux at each station with the energy lost since the previous station in the calculation, due to friction along the bottom and along the side walls.

In these calculations laminar boundary layers were assumed in all cases although the longest and highest waves according to Jonsson (1966) should have turbulent boundary layers, at least close to the breaking point. The effect, however, of introducing the turbulent value of the wave energy loss on the last part of the slope appears to be insignificant.

In the calculation of friction losses were used wave particle velocities determined by the linear theory. This also applies to the region where the wave height variation was calculated from the cnoidal theory. In fact it is a reasonable simplification since the friction only 'eats' a minor part of the energy flux anyway.

Linear Wave Shoaling

It is not surprising that the present results confirm the conclusion quoted in the introduction from other investigations, namely that shoreward of the point of minimum wave height, i.e. roughly $h/L_o = 0.10$, the linear theory predicts divergingly smaller wave heights than measured. This is evident from Fig. 4 b where the linear curve is shown through to breaking.

On the other hand we notice that as long as the deep water steepness is less than 3 - 4% (Figs. 4 a and 5 a), the linear wave theory seems to work quite well in deeper water. This is of particular interest because the cnoidal theory cannot be applied for $h/L_o \gtrsim 0.10$.

In fact, for the small wave steepnesses the agreement is better than in the interpretation of Iversen's measurements given by Brink-Kjær and Jonsson (1973). They found that even for smaller wave steepnesses an appreciable discrepancy seemed to develop between the linear theory and

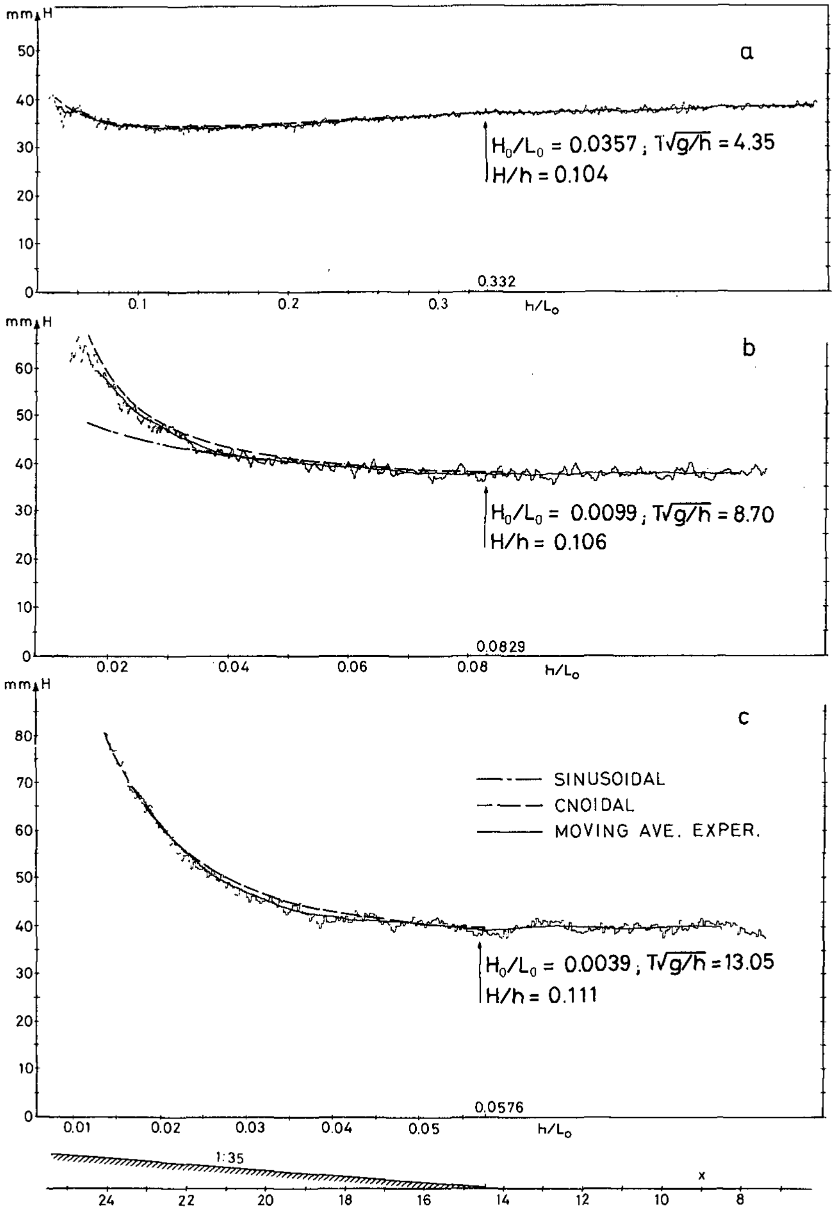

Fig. 4 Variation of wave height for three different wave steepnesses

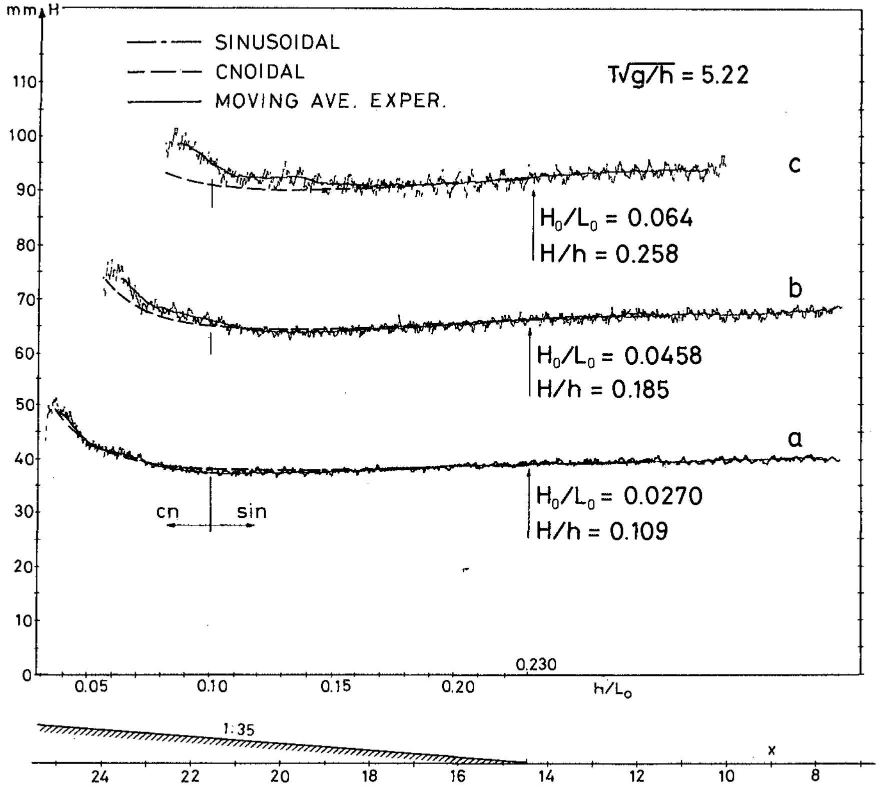

Fig. 5 Variation of wave height, rather steep waves

the measurement as h/L_o decreased towards the value 0.10. Fig. 6 shows an example of this where H_o/L_o is 3.58%, and the linear theory yields results up to 8% higher than the measurements (i.e. a minimum value of H/H_o = 0.913 against 0.85 measured).

It has turned out that the major reason for this discrepancy is that friction has been neglected in Brink-Kjær and Jonsson's calculation of the theoretical curves. In particular in Iversen's case, with a wave flume only 30 cm wide and a horizontal bottom depth of 77.8 cm in the case considered, the friction along the side walls has a considerable effect. Taking this into account brings the theoretical minimum value of H/H_o down to 0.869 in the case shown in Fig. 6, and this must be considered in fair agreement with the measured value. The theoretical variation with friction included is shown in the figure. It may also be noticed that the wave heights measured by Iversen are most likely influenced by the fact that the first 4.6' (= 1.40 m) of Iversen's slope are steeper (1:5.75 = 0.174) than the value 1:13.8 = 0.072 referred to as the slope for the experiment.

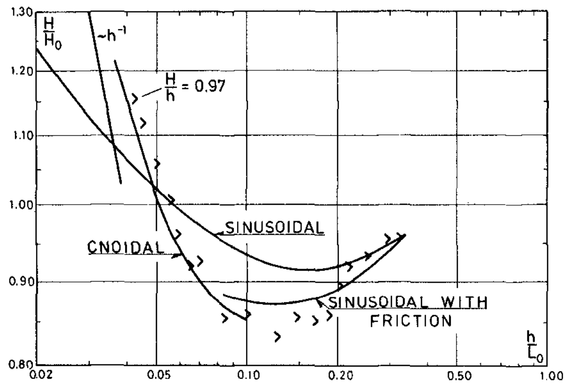

Fig. 6
The effect of friction losses on Iversen's results

For the experiment shown in Fig. 4 a, H_o/L_o is almost the same (3.57%) and here the measured minimum value of H/H_o is 0.875 against the calculated value (including friction) of 0.889.

From Fig. 5 a-c, however, we see that if the deep water steepness increases, the wave height to water depth ratio will grow to large values already outside the cnoidal region. In the case of H_o/L_o = 6.4% (Fig. 5 c) the wave actually breaks at $h/L_o \simeq 0.10$, so that the entire shoaling process has been determined by the linear theory. And quite obviously, linear theory cannot handle the large values of H/h.

Since we here at the breaking point have $U_{max} \sim 45$ ($H/h \sim 0.71$ and $L/h \sim 8$) it seems likely that the problem could be overcome by using a second or third order Stokes shoaling theory.

Cnoidal Wave Shoaling

The theory of cnoidal wave shoaling used here was developed by Svendsen and Brink-Kjær (1972) who on the same intuitive basis as Rayleigh solved and tabulated the variation of the wave height. A more direct presentation of the results can be found in Skovgaard et al. (1974).

Perhaps it should be mentioned that this theory is based on (8) (see Sect. 6). A slightly different version will appear if (10) (in which $\sqrt{1 + A\,H/h}$ is substituted by $1 + \frac{1}{2} A\,H/h$) is used, and other differences of similar nature are possible too. Formally all these versions (as e.g. Shuto (1974) and Ostrovskiy and Pelinovskiy (1970) are equal in that they only differ in the way the small terms are handled. For practical applications, however, where H/h is not really as small as envisaged in the theory they result in considerable differences in the numerical results for e.g. the wave height variation, in particular as we approach the breaking point. In our numerical calculations we have found that the best fit to the measurements is obtained by using the version developed by Svendsen and Brink-Kjær.

As appears from Fig. 4 b and c, the combined linear-cnoidal shoaling model fits the experimental data surprisingly well in those cases where the H/h-ratio remains small for $h/L_o > 0.10$. The predictions even follow the development all the way to the breaking point, although the theory should not be applicable there.

It should be emphasized, however, that essentially this only indicates that the relationship between cnoidal energy flux and wave height shows a realistic variation with water depth. The absolute value of the energy flux is determined from the wave height in the constant depth part of flume and may not be correct (and other cnoidal wave properties as e.g. the position of the mean water level may be even rather inaccurately predicted by the same theory). This must be recalled in those experiments were $h/L_o \gtrsim 0.10$ in the constant depth part of the flume. Then linear wave theory is applied for $h/L_o > 0.10$, and at $h/L_o = 0.10$ the theoretical result must be matched with the cnoidal shoaling, which is used shoreward of that point.* Svendsen and Brink-Kjær (1972) matched the two theories by assuming continuity in energy flux. However logical this approach seems it results in a discontinuity in wave height at the matching point.

Since, however, neither of the two theories yields the exact energy flux for a given wave height it may be argued that it is equally correct to match the wave heights, which we know are continuous, and accept a discontinuity in the theoretically determined energy flux (which is approximate anyway) at the matching point. This is actually the method chosen here. Finally is mentioned that in the numerical evaluations the still water depth has been corrected for wave set-down.

Discussion

As mentioned the figures show a reasonable agreement, though discrepancies up to 6 - 8% in wave height develop close to the breaking point. This cannot surprise, however, since the energy flux used for the calculations was based on the assumptions that $H \ll h$, that the horizontal velocity u is constant over the water depth, and that the excess pressure p^+ due to the wave is constant too, and proportional to the local

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*In principle any point shoreward of $h/L_o \sim 0.10$ could be chosen as the matching point between the two theories.

value of the surface elevation. As is evident from e.g. velocity measurements in waves near breaking (see e.g. Iwagaki and Sakai, 1976), the velocity is far from constant over the water depth, and the constancy of p^+ actually represents neglection of the vertical accelerations in this context.

A few comments will also be appropriate about the matching procedure between linear and cnoidal wave theory.

The wave tables prepared by Skovgaard et al. (1974) are based on continuity in energy flux. The tables may also be used, however, for calculations with continuity in wave height at the matching point, if it is noticed that the corresponding shift in energy flux is represented by a formal shift in the deep water wave steepness. The procedure is illustrated in the appendix.

As mentioned in Sect. 3, a proper measure of the steepness of the sloping bottom is the parameter $S = h_x L/h$.

Since L is approximately proportional to $\sqrt{gh}$, a plane slope will correspond to $S \sim h_x h^{-1/2}$ so that for fixed h_x the value of S grows with decreasing water depth, indicating that the slope appears steeper and steeper to the waves as they propagate shoreward. Hence the shoaling condition (which requires S small) will sooner or later be invalidated. With reference to the assumptions in Sect. 3 this would cause appreciable reflection and disintegration of the wave form (T not constant). No such phenomena were observed in the experiments recorded here, which suggests that S in all cases have been small enough.

5. WAVE SURFACE PROFILES

Let us assume that a rigorous perturbation expansion is carried out for waves on a sufficiently gently sloping bottom. Then the shoaling of the wave will represent the first approximation, but even though the slope of the bottom does not directly influence this first order solution, a second order solution exists which will represent the first approximation to the effect of the bottom slope, which has the effect of making the wave skew.

Svendsen (1974) carried out the calculations for this second order solution in the case of cnoidal waves, and the result for the skew wave profiles can be written

$$\eta = \eta^{(0)} + \eta^{(1)} \tag{3}$$

where $\eta^{(0)}$ is the constant depth cnoidal wave profile

$$\eta^{(0)} = H\left(\beta(m) + cn^2(2K\theta, m)\right), \quad \theta = \frac{t}{T} - \frac{x}{L} \tag{4}$$

K being the complete elliptic integral of the first kind, m its parameter, and β a function of m. $\eta^{(1)}$ is the above mentioned first approximation to the effect of the sloping bottom. $\eta^{(1)}$ is given by

$$\frac{\eta^{(1)}}{h} \approx 3\, S \frac{L}{h} T \sqrt{g/h}\; f\left(\theta; U, \frac{H}{h}\right) \tag{5}$$

where

$$f\left(\theta; U, \frac{H}{h}\right) = \frac{\eta_\theta^{(0)}}{\sqrt{gh}} \int \eta_\theta^{(0)-2} \int \eta_\theta^{(0)} \int \left[2\left(c\eta^{(0)}\right)_h - c_h\, \eta^{(0)}\right] d\xi\, d\zeta\, d\theta \tag{6}$$

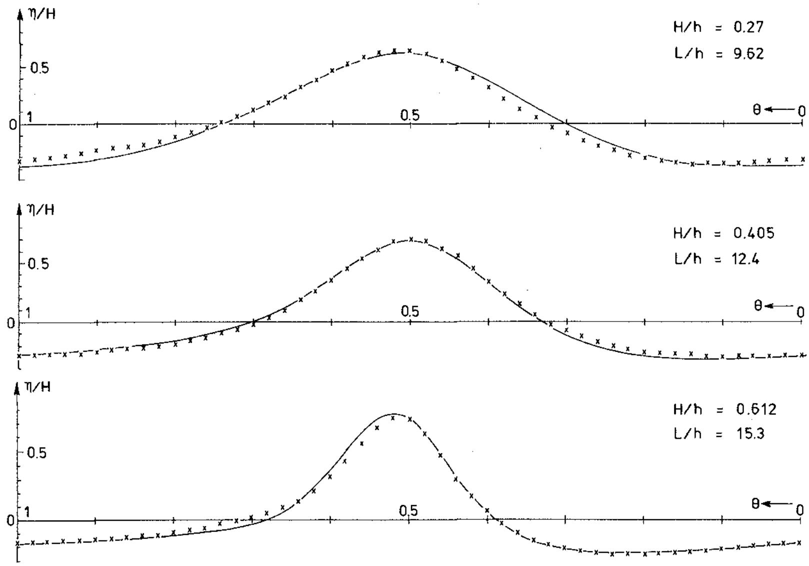

Fig. 7 Wave profiles for wave with H_o/L_o = 0.0165 on slope h_x = 1:35

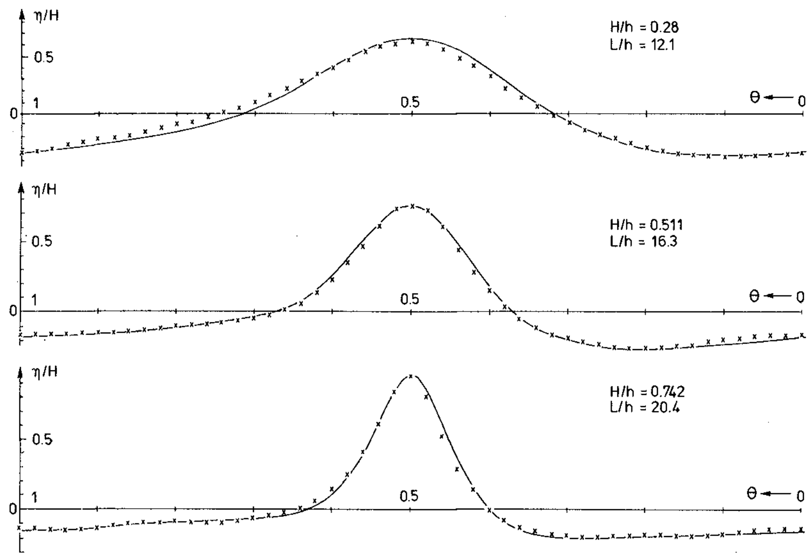

Fig. 8 Wave profiles for wave with H_o/L_o = 0.0114 on slope h_x = 1:35

with indexes $_\theta$ and $_h$ denoting partial differentiation with respect to θ and h, respectively, under the restriction that the energy flux E_f is constant. This solution has been evaluated for a number of cases corresponding to wave profiles measured in the experiments. As input for the evaluation of the theoretical profiles has been used the wave period, the local measured wave height, the water depth (including set-down), and the bottom slope.

Figs. 7 and 8 show a comparison between measured and calculated profiles, in each case for three different phases of the deformation of a wave towards the breaking point.

It is immediately evident from the figures that the agreement is good even for wave height to water depth ratios as large as 0.75. It should, however, also be noticed that all the cases in the figures correspond to situations where the deviation from the symmetrical (ordinary) cnoidal wave profile is small. In other words, situations where the shoaling assumption about a local equilibrium is still valid. This is required also in the theory for $\eta^{(1)}$ because $\eta^{(1)}$ has to be a small perturbation on $\eta^{(0)}$. Consequently the large deformations which rapidly develop just before breaking cannot be predicted by this theory.

An interesting feature is that the skewness of the surface slopes is not so pronounced in the wave crest. The major effect of sloping bottom is concentrated in the wave trough, which has its deepest point right in front of the next wave crest.

6. PHASE VELOCITY

The measurements of the phase velocity c were obtained by measuring the time (in milliseconds) it took the wave crest (identified digitally by the computer) to travel the distance between two wave gauges placed 20.0 cm apart in the direction of wave travel.

The results obtained in this way are rather sensitive to small changes in the shape of the wave crest between the two wave gauges. As a consequence, the individual measurements show a considerable scattering (± 10 - 25%). This is particularly pronounced for the very small wave steepnesses. Consequently the scattering is much reduced when the waves steepen on the slope.

The results for c presented in Fig. 9 a - d represent a moving average over a number of waves. The results have further been confirmed by a different method based on measuring electrically the time it takes the wave to travel between two pointed metal-rods placed 20 cm apart. The mean value of the measurements obtained in this way confirmed that the results obtained from the computer when the pointed ends of the metal-rods were placed at a level close to the wave crest.

It can be mentioned that one of the reasons for the large scattering in the experimental results is that small free second harmonic waves still exist in the flume. Such disturbances result in phase velocities which are constant in time, but vary from point to point. Hence the tendency mentioned in Sect. 2, that even the irregularities are reproduced when an experiment is repeated.

The measurements are compared with linear and cnoidal results for the phase velocity. From linear theory we have

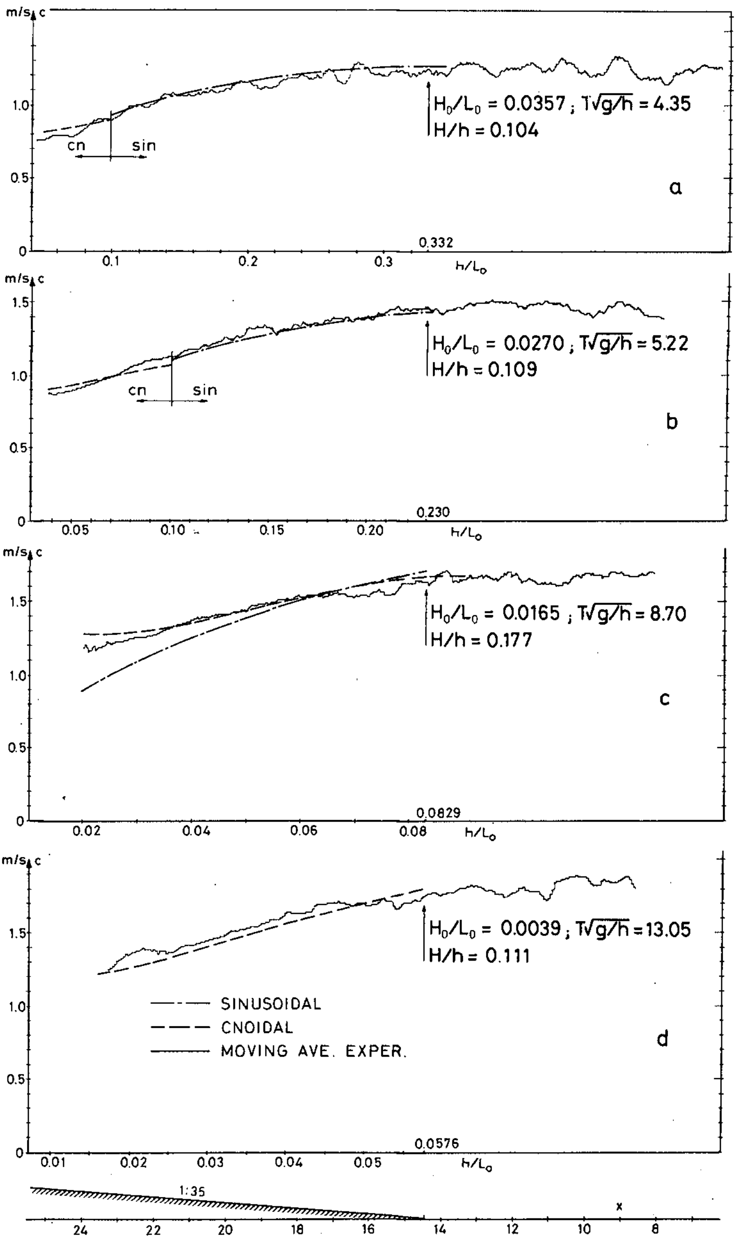

Fig. 9 Variation of phase velocity

$$c = \frac{g}{\omega} \tanh kh \tag{7}$$

which is used for all values of $h/L_o > 0.10$. As the cnoidal result for c is used (for $h/L_o < 0.10$)

$$c = \left[g\,h\left(1 + A\frac{H}{h}\right)\right]^{1/2} \tag{8}$$

$$A = \frac{2}{m} - 1 - \frac{3\,E}{m\,K} \tag{9}$$

where E is the complete elliptic integral of the second kind.

Since cnoidal theory assumes $H/h \ll 1$, (8) may also be written

$$c = \sqrt{g\,h}\left(1 + \frac{1}{2}A\frac{H}{h}\right) + O\left(\frac{H}{h}\right)^2 \tag{10}$$

which is equally valid. In the analogy with the wave height variation we realize that for waves near breaking the numerical results obtained for (8) and (10) differ appreciably. It turns out that the results obtained from (8) fit the measurements better. In (8) the theoretically determined wave height (i.e. from the shoaling process) has been used.

In general the conclusion is positive. The two theories predict the phase velocity to within a few per cent, the linear theory for $h/L_o >$ 0.10, the cnoidal shoreward of that point. The only exception is close to breaking, where the cnoidal theory overestimates the finite amplitude effect and yields results somewhat above the measurements. In Fig. 9 c is for comparison given the linear curve even though $h/L_o < 0.10$ everywhere.

7. WAVE BREAKING

The last topic to be discussed in this paper is the characteristics of the waves at the breaking point, including the prediction of the position of this point, e.g. in terms of the water depth where breaking is initiated.

Even though cnoidal theory seems to predict the wave height variation reasonably well, no information can be deduced from that theory (or any other known theory) about where the breaking occurs. In that question we must rely entirely upon empirical data.

One of the problems is to define exactly where the breaking has started. Often breaking is defined to start 'where the front of the wave becomes vertical', though in the case of a spilling breaker there is no such point. Also the initiation of foam may be a very uncertain definition in small scale experiments where the surface tension will cause scale effects for the foam production.

In consequence of these arguments we have chosen to define the breaking point as the point where H/h is maximum. Since the wave height has a maximum close to the point where the energy dissipation starts and h is decreasing, H/h appears to have a rather sharp maximum. In the evaluation of h the set-down is incorporated. The choice of H/h to identify the breaking point has the advantage that from an engineering point of view the maximum of H/h is one of the primary information about the wave breaking.

Part of the large scatter in breaking data for earlier experiments is believed to be due to free second harmonics in the waves (Battjes, 1974). And even when this irrelevant effect is removed, as in our experiments, each breaking wave will generate wavelets which influence the breaking of the next wave etc. This is particularly pronounced for plunging breakers and represents an effect which must be expected also in the nature.

In the attempt to find coherence in the data obtained, many different plots and relationships among the parameters have been tried. One of the most promising is shown in Fig. 10.

It shows the value of the wave length to water depth ratio $(L/h)_B$ at the point where H/h is maximum. Since all the experiments were performed with a slope 1:35, Iversen's (1952) data for slopes, 1:10, 1:20, 1:30 and 1:50, and those reported by Iwagaki and Sakai (1976) for slopes, 1:10, 1:20 and 1:30 have been included, too. For all the points, L has been determined from the cnoidal wave theory using the wave period, and the wave height and water depth at the breaking point.

The abscissa in Fig. 10 is the theoretically determined deep water wave steepness H_o/L_o. It appears from the figure that within the accuracy expected by the experiments (and the more advanced ISVA-experiments show a smaller variation, as they should) the data can be described by the relationship

$$(L/h)_B = 2.30\,(H_o/L_o)^{-1/2} \tag{11}$$

From this relationship several deductions follow. Since L/h is increasing monotonously shorewards, this relation means that breaking starts when the wave length to water depth ratio grows to a value which depends only on the deep water wave steepness.

It is of particular interest to note that $(L/h)_B$ does not depend on the bottom slope h_x, and that the scattering is considerably smaller than for any other correlation between breaking parameters. The first suggests that shoaling conditions are satisfied in most of the experiments (Iversen's 1:10-data showing a weak tendency to larger values of $(L/h)_B$). Since L/h varies rapidly with the position in the breaking zone, the small scattering indicates that the initiation of breaking depends strongly on the value of L/h.

Once the relation (11) has been established, the wave height at the breaking point H_B is actually *theoretically* fixed too for a wave with a given deep water steepness H_o/L_o. From (11) we get $(L/h)_B$, and since L/h is a monotonous function of h/L_o for given H_o/L_o (Svendsen and Brink-Kjær, 1972) this means that $(L/h)_B$ corresponds to one particular value of h_B/L_o. This value can in principle be determined, though none of the tables published so far are suited for this purpose.* Finally, from h_B/L_o and H_o/L_o the value of $(H/h)_B$ can be determined (using e.g. Table 3 in Skovgaard et al., 1974).

Notice that if only H_o/L_o is given we cannot determine the absolute value of h_B and H_B, only the ratios described above. Often, however, the wave period will be given too, and then $L_o = g/(2\pi)\,T^2$ yields the

*The table required should have the entries L/h and H_o/L_o and yield values of h/L_o.

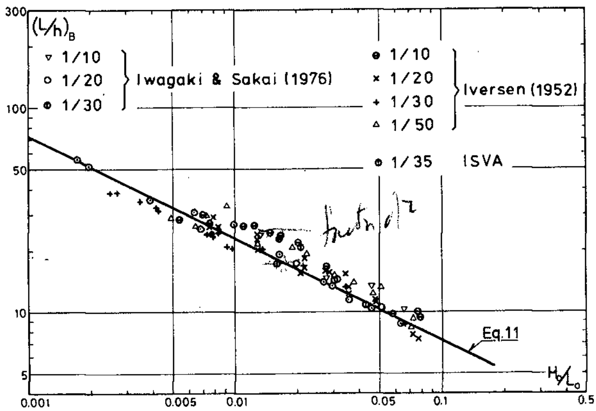

Fig. 10 Wave length to water depth ratio $(L/h)_B$ at the breaking point versus deep water wave steepness H_o/L_o

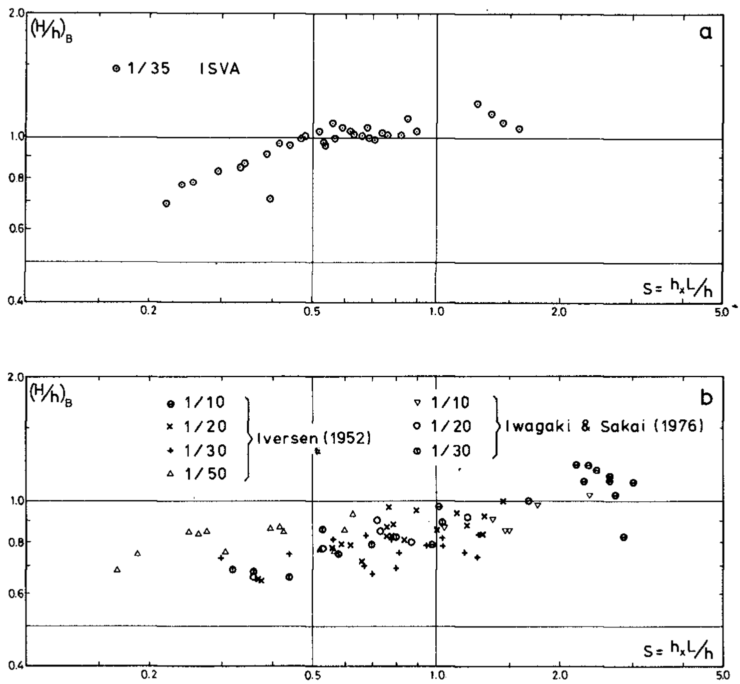

Fig. 11 Maximum value of wave height to water depth ratio $(H/h)_B$ versus slope parameter $S = h_x(L/h)_B$

length required to specify the absolute values from the dimensionless ratios.

In Fig. 11 a the values of $(H/h)_B$ are plotted versus $S = h_x(L/h)_B$. A comparison between theoretical and experimental results here would yield no information, which could not be drawn from Figs. 4 and 5.

The measurements of Iversen, and Iwagaki and Sakai (Fig. 11 b) do not quite fit into the pattern of the present investigation but the tendency is the same. Their results all correspond to smaller values of $(H/h)_B$ and the scattering is considerable. As mentioned before, this is probably to a large extent due to the free second harmonics generated by their wave generator. The scattering, however, is considerably decreased by using S instead of $(L/h)_B$. Thus the value of H/h at the breaking point is actually a function of the bottom slope h_x.

It will be seen that the values of the $(H/h)_B$ are in general somewhat larger than the height 0.827 h considered the largest possible for a solitary wave on a horizontal bottom (Longuet-Higgins and Fenton, 1974), and other results usually quoted for the maximum possible height of periodic waves. Also, the largest values of $(H/h)_B$ correspond to the largest values of S.

In both these respects, the results seem to fit into the pattern found by Camfield and Street (1969) who for solitary waves (i.e. theoretically infinitely long waves) found $(H/h)_B$-values up to 2.

8. CONCLUSION

Linear ('sinusoidal') and cnoidal wave theories are compared with experimental results obtained with waves without free second harmonic disturbance on a plane slope $h_x = 1{:}35$. It is shown that:

(a) Linear theory can predict the shoaling as long as the wave height to water depth ratio H/h is small (Fig. 4 a).

(b) Cnoidal theory, which can only be used for $h/L_o < 0.10$ (L_o being deep water wave length), predicts the variation of the wave height quite well even close to breaking (Fig. 4 b and c).

(c) Linear theory is used for $h/L_o > 0.10$, and wave height is matched with cnoidal theory at that point. For waves with large deep water steepness H_o/L_o (> 3 - 4%) the value of H/h is not small for $h/L_o >$ 0.10. Hence linear theory fails (Fig. 5 b and c). Second or higher order Stokes theory is recommended in this case for $h/L_o > 0.10$.

(d) The skew shape of the wave profiles is well predicted by a theory taking into account the effect of the bottom slope (Figs. 7 and 8). The theory cannot predict breaking.

(e) Both the linear and the cnoidal formulae for phase velocity c fit remarkably well to the data for $h/L_o > 0.10$ and < 0.10, respectively (Fig. 9).

(f) At the breaking point the wave length to water depth ratio $(L/h)_B$ appears to be independent of bottom slope, for bottom slopes less than 1:10, i.e. $(L/h)_B = f(H_o/L_o)$, Fig. 10, whereas $(H/h)_B$ is primarily a function of the slope parameter $S = h_x(L/h)_B$ (Fig. 11).

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APPENDIX

Given a wave with period T = 12 s, and $H_o/L_o = 0.01$.

Find the wave height H at h = 7 m.

Since $L_o = g\,T^2/2\pi = 225$ m, we have $h/L_o = 0.031 > 0.10$ so that cnoidal wave theory is appropriate to use for this wave at h = 7 m.

Continuity in energy flux at the matching point, Table 3 in Skovgaard et al., yields directly (using h/L_o and H_o/L_o as entry) that $H/H_o = 1.133$, or H = 2.55 m, since $H_o = 2.25$ m.

Continuity in wave height at the matching point requires that we stage the calculation through that point. Linear theory yields (h_m denoting the depth at the matching point)

$$h_m/L_o = 0.10 \Rightarrow H/H_o = 0.933 \qquad H = 0.933 \cdot 0.01 \cdot 225 = 2.10 \text{ m}$$

$$h_m = 0.10 \cdot 225 = 22.5 \text{ m}$$

At this point the table for linear (i.e. sinusoidal) waves yields $H_{sin}/H_o = 0.933$, whereas the cnoidal result for $H_o/L_o = 0.01$ is $H_{cn}/H_o = 0.867$. If we now require that at the matching point $H_{cn} = H_{sin}$, we find that in the cnoidal calculation we must formally use a deep water wave height $H_{o,cn} = H_o \cdot 0.933/0.867 = 2.43$ m, i.e. in the cnoidal computation the deep water steepness must be $H_{o,cn}/L_o = 0.01076 \sim 0.0108$. The wave height at h = 7 m can then be found from Table 3 (Skovgaard et al.) using $h/L_o = 0.031$ and $H_o/L_o = 0.0108$ as entry. We get

$$H/H_o = 1.140 \qquad H = 1.140 \cdot 2.43 = 2.78 \text{ m}$$

against 2.55 m obtained by the other matching procedure.

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