Deformations of Lie Algebra Structures

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0. Introduction. A Lie algebra is a vector space with an alternating bilinear product which satisfies the Jacobi identity. In this paper we shall consider the set $\mathfrak{M}$ of all Lie algebra multiplications on a fixed (finite-dimensional) vector space V. If we choose a basis for V, then $\mathfrak{M}$ can be identified with the set (c_{ij}^k) of all Lie algebra structure constants. The group $G = GL(V)$ of all vector space automorphisms of V acts as a transformation group on $\mathfrak{M}$ and the orbits of G on $\mathfrak{M}$ are precisely the isomorphism classes of Lie algebra structures on V. In the study of "deformations" of Lie algebras, we are interested in geometric properties of the set $\mathfrak{M}$ considered as a transformation space for G.

Let $\mu \in \mathfrak{M}$ and let $L = (V, \mu)$ be the corresponding Lie algebra. We consider the following problems concerning deformations of L:

(i) Give sufficient conditions on L in order that every Lie algebra L' near L be isomorphic to L (we say, in this case, that L is rigid).

(ii) Describe in terms of algebraic properties of L the set of all Lie algebras in a neighborhood of L.

The solutions to both of these problems involve the Lie algebra cohomolgy space $H(L, L)$. Precisely, (i), L is rigid if $H^2(L, L) = 0$ and, (ii), a neighborhood of L (*i.e.*, μ) can be parametrized in the real or complex case by the zeros of an analytic map from $H^2(L, L)$ to $H^3(L, L)$. The computations involved are greatly simplified by the introduction of a graded Lie algebra $\mathrm{Alt}(V)$ associated with V.

Strictly speaking, once we establish the relation between Lie algebra structures on V and the graded Lie algebra $\mathrm{Alt}(V)$, all of our results could be obtained by simply quoting the general theorems on deformations in graded Lie algebras developed in [10]. Since so many purely technical details appear there, we have

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tried to make this paper as self-contained as possible. When we have to appeal to the results of [10], we sketch the ideas behind the proofs.

Our results on deformations of Lie algebras are analogous to earlier results of Gerstenhaber [4] on deformations of associative algebras.

The problems we consider are closely related to the notion of contraction of Lie algebras, which is of interest in modern physics. For a discussion, see several recent papers of R. Hermann [5].

As further applications of the ideas involved we briefly discuss deformations of modules over a Lie algebra (representations) and deformations of ideals. In another paper [11], we have discussed deformations of homomorphisms of a Lie algebra L (resp. Lie group G) into a second Lie algebra M (resp. Lie group H). The methods and results are similar to those of this paper.

1. Lie algebra structures on a vector space. We shall consider only finite-dimensional Lie algebras. A Lie algebra $L = (V, \mu)$ is given by a (finite-dimensional) vector space V, together with a bilinear map ("product map") $\mu : V \times V \to V$ which is alternating ($\mu(x, x) = 0$, this implies $\mu(x, y) = -\mu(y, x)$) and which satisfies the Jacobi identity:

$$\mu(x, \mu(y, z)) + \mu(y, \mu(z, x)) + \mu(z, \mu(x, y)) = 0. \tag{1}$$

Classically, the product $\mu(x, y)$ is denoted $[x, y]$; we shall, however, reserve square brackets for a different operation.

If we choose a basis $(e_1, \cdots, e_n)$ of V, then the multiplication is completely determined by the values of $\mu(e_i, e_j)$. We may set $\mu(e_i, e_j) = \sum_k c_{ij}^k e_k$, where c_{ij}^k are field elements, called the structure constants of the Lie algebra L. Clearly, $c_{ii}^k = 0$ and $c_{ij}^k = -c_{ji}^k$; the Jacobi identity is equivalent to

$$\sum_k (c_{ik}^l c_{jm}^k + c_{jk}^l c_{mi}^k + c_{mk}^l c_{ij}^k) = 0. \tag{1'}$$

The discussion of Lie algebras and their deformations could be carried out entirely in terms of structure constants, but preference will be given to the, usually simpler, intrinsic formulation.

Let $\mathfrak{M}$ denote the set of all alternating bilinear maps $\mu : V \times V \to V$ which satisfy the Jacobi identity (1). $\mathfrak{M}$ may be identified with the set of all $N = n\binom{n}{2}$-tuples (c_{ij}^k) of field elements which satisfy the conditions for structure constants. In view of (1′), this set is the intersection of a finite number of quadratic hypersurfaces in K^N; hence $\mathfrak{M}$ is an algebraic set. Here K denotes the base field of V. We shall assume char $K \neq 2$ throughout (otherwise, see [10] for details). The cases $K = \mathbf{R}$ or $\mathbf{C}$ (real or complex numbers) underlie most of the intuitive discussions.

2. Deformations of Lie algebras—a sketch. Let $\mu' : V \times V \to V$ be an alternating bilinear map and let μ be a Lie algebra multiplication on V. Set

$\varphi = \mu' - \mu$; then φ is also an alternating bilinear map. Now, μ' is a Lie algebra multiplication if it satisfies the Jacobi identity:

$$\begin{aligned} 0 &= \sum_{\text{cycl}} \mu'(x, \mu'(y, z)) \\ &= \sum_{\text{cycl}} (\mu + \varphi)(x, (\mu + \varphi)(y, z)) \\ &= \sum_{\text{cycl}} \mu(x, \mu(y, z)) + \sum_{\text{cycl}} \mu(x, \varphi(y, z)) \\ &\quad + \sum_{\text{cycl}} \varphi(x, \mu(y, z)) + \sum_{\text{cycl}} \varphi(x, \varphi(y, z)). \end{aligned}$$

Here $\sum_{\text{cycl}}$ denotes the sum over all cyclic permutations of $\{x, y, z\}$. The first term on the right vanishes since μ is a Lie product. The second and third terms take a form which is familiar in the cohomology theory of Lie algebras. (See Section 3.) Precisely, they give the coboundary $\delta\varphi$ of the 2-cochain φ:

$$\delta\varphi(x, y, z) = \sum_{\text{cycl}} \mu(x, \varphi(y, z)) + \sum_{\text{cycl}} \varphi(x, \mu(y, z)). \tag{2}$$

The last term in the computation is denoted $-\frac{1}{2}[\varphi, \varphi]$, for reasons to become clear later:

$$\tfrac{1}{2}[\varphi, \varphi](x, y, z) = \sum_{\text{cycl}} \varphi(\varphi(x, y), z). \tag{3}$$

Thus the "deformation equation" becomes

$$\delta\varphi - \tfrac{1}{2}[\varphi, \varphi] = 0, \tag{4}$$

and φ is a deformation of μ (*i.e.*, $\mu' = \mu + \varphi$ is a Lie product) if and only if φ is a solution of (4)).

3. Lie algebra cohomology. We shall give only the basic definitions; for a detailed discussion see [3, 6].

Let $L = (V, \mu)$ be a Lie algebra and let ρ be a representation of L on a vector space W. Then W is said to be an L-module. If $x \in L$ and $w \in W$, then $\rho(x)(w)$ is denoted simply by $x \cdot w$. $C^n(L, W)$, the (vector) space of n-cochains of L with coefficients in W, is defined to be the vector space of all alternating n-linear maps of V into W and $C(L, W)$ denotes the direct sum $\bigoplus_{n \geq 0} C^n(L, W)$. The coboundary operator $\delta : C(L, W) \to C(L, W)$, a homogeneous linear map of degree 1, is defined as follows: if $\alpha \in C^n(L, W)$, then $\delta\alpha \in C^{n+1}(L, W)$ is given by

$$\begin{aligned} \delta\alpha(x_0, \cdots, x_n) = &\sum_{i=0}^{n} (-1)^i x_i \cdot \alpha(x_0, \cdots, \hat{x}_i, \cdots, x_n) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha(\mu(x_i, x_j), x_0, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n), \end{aligned}$$

where the sign $\wedge$ indicates that the argument below it has been omitted. It can be shown that $\delta \circ \delta = 0$ (a proof is given in Section 5), thus $(C(L, W), \delta)$

is a cochain complex. Let $Z^n(L, W) = \{\alpha \in C^n(L, W) \mid \delta\alpha = 0\}$ and $B^n(L, W) = \delta(C^{n-1}(L, W))$; $Z^n(L, W)$ (resp. $B^n(L, W)$) is the space of n-cocycles (resp. n-coboundaries). Since $\delta \circ \delta = 0$, $B^n(L, W)$ is included in $Z^n(L, W)$. The quotient space $H^n(L, W) = Z^n(L, W)/B^n(L, W)$ is the n-th cohomology space of L with coefficients in W and $H(L, W)$ denotes the direct sum $\bigoplus_n H^n(L, W)$.

If $x \in L$, then the linear map $y \rightarrow \mu(x, y)$ of L into L is denoted by $ad_L x$. It follows from the Jacobi identity that $x \rightarrow ad_L x$ is a representation of L on L, and hence L (or more precisely, V) becomes an L-module. The Lie algebra cohomology space $H(L, L)$ is the only one we shall have to consider in the main part of this paper.

4. Infinitesimal deformations. Let $L = (V, \mu)$ be a Lie algebra and let $\varphi \in C^2(L, L)$ be a 2-cochain. Then φ is a deformation of L if and only if φ is a solution of the deformation equation

$$\delta\varphi - \tfrac{1}{2}[\varphi, \varphi] = 0. \tag{4}$$

The solutions of the linearized version of (4)

$$\delta\varphi = 0 \tag{5}$$

will be called *infinitesimal deformations* of μ; thus the infinitesimal deformations of μ are just the 2-cocycles. Let $\mu_t = \mu + t\varphi_1 + t^2\varphi_2 + \cdots$ be a one-parameter family of Lie algebra multiplications on V where t may denote a real or complex variable, or an indeterminate. (In the latter case, K and V have to be replaced by $K[[t]]$ resp. $V[[t]]$, the sets of formal power series with coefficients in K resp. V; and various other technicalities have to be taken care of. We shall ignore them and refer the reader to Gerstenhaber [4].) One easily verifies that φ_1 is an infinitesimal deformation. It is not possible, in general, to find a one-parameter family which has a given infinitesimal deformation as its first order term.

Among the deformations of μ there will, in general, be equivalent ones. Two deformations, φ_1 and φ_2 , are called equivalent if the corresponding $\mu_1 = \mu + \varphi_1$ and $\mu_2 = \mu + \varphi_2$ define isomorphic Lie algebras $L_1 = (V, \mu_1)$ and $L_2 = (V, \mu_2)$. In that case we have $\mu_2(x, y) = g(\mu_1(g^{-1}x, g^{-1}y))$ for some $g \in GL(V)$, the group of vector space automorphisms of V.

Let $\alpha : V \rightarrow V$ be a linear map. Then $I + t\alpha$ is a family of invertible linear maps if $t \in \mathbf{R}$ or $\mathbf{C}$ and $|t|$ is small or if t is an indeterminate. (I is the identity map.) The Lie algebra multiplication μ_t given by

$$\mu_t(x, y) = (I + t\alpha)\mu((I + t\alpha)^{-1}x, (I + t\alpha)^{-1}y)$$

is equivalent to μ. If we expand $(I + t\alpha)^{-1}$ by the geometric series we find

$$\mu_t(x, y) = \mu(x, y) + t(\alpha(\mu(x, y)) - \mu(\alpha x, y) - \mu(x, \alpha y)) + \text{(higher order terms)}.$$

This shows that the alternating bilinear map

$$(x, y) \to \alpha\mu(x, y) - \mu(\alpha x, y) - \mu(x, \alpha y)$$

is an infinitesimal deformation. In fact we observe that α is a 1-cochain and that the above map is just $-\delta\alpha$. We note that $\delta\alpha$ is a special kind of infinitesimal deformation. It "deforms" μ into a structure isomorphic to it, and hence is called a *trivial* infinitesimal deformation. Thus the trivial infinitesimal deformations are precisely the 2-coboundaries. The above construction by means of $I + t\alpha$ shows that every trivial infinitesimal deformation is the first order term of some one-parameter family of deformations of μ. (It might have been more elegant to take $e^{t\alpha}$ instead of $I + t\alpha$; however, that would have compounded the difficulty for fields of prime characteristic.)

On the infinitesimal level, we now have two vector spaces which are of significance: (1) the space $Z^2(L, L)$ of all infinitesimal deformations of μ; (2) the space $B^2(L, L)$ of all trivial infinitesimal deformations of μ. The quotient space $Z^2(L, L)/B^2(L, L)$ is just the Lie algebra cohomology space $H^2(L, L)$. It "measures" the extent to which there exist non-trivial infinitesimal deformations. It would be reasonable to call L infinitesimally rigid if $H^2(L, L) = 0$. In fact the rigidity theorem of Section 7 states that if the base field is **R** or **C** or is algebraically closed, then $H^2(L, L) = 0$ is sufficient for L to be rigid, *i.e.* there is a neighborhood of μ in $\mathfrak{M}$ all of whose elements are equivalent to μ.

5. The graded Lie algebra Alt(V). In view of the complicated nature of expressions such as (1), (2) and (3), which will need to be manipulated in dealing with deformation problems, it becomes clear that a formidable amount of computation lies ahead unless we can find a systematic method for handling such expressions. The following will serve our purposes.

For each integer $n \geqq -1$, we define $\mathrm{Alt}^n(V)$ to be the vector space of all alternating $(n + 1)$-linear maps of V into itself. We denote by $\mathrm{Alt}(V)$ the direct sum $\bigoplus_n \mathrm{Alt}^n(V)$. For $\alpha \in \mathrm{Alt}^n(V)$ and $\beta \in \mathrm{Alt}^m(V)$, we define $\alpha \mathbin{\overline{\wedge}} \beta \in \mathrm{Alt}^{n+m}(V)$ by

$$(6) \qquad \alpha \mathbin{\overline{\wedge}} \beta(x_0, \cdots, x_{n+m}) = \sum_\sigma \mathrm{sgn}\,(\sigma)\alpha(\beta(x_{\sigma(0)}, \cdots, x_{\sigma(m)}), x_{\sigma(m+1)}, \cdots, x_{\sigma(m+n)}),$$

where the sum is taken over all permutations σ of $\{0, \cdots, n + m\}$ such that $\sigma(0) < \cdots < \sigma(m)$ and $\sigma(m + 1) < \cdots < \sigma(n + m)$. We now define $[\alpha, \beta] \in \mathrm{Alt}^{n+m}(V)$ by

$$(7) \qquad [\alpha, \beta] = \alpha \mathbin{\overline{\wedge}} \beta - (-1)^{mn}\beta \mathbin{\overline{\wedge}} \alpha.$$

It is now a matter of simple verification to verify that (1) becomes

$$(1'') \qquad \mu \mathbin{\overline{\wedge}} \mu = \tfrac{1}{2}[\mu, \mu] = 0,$$

that (2) becomes

$$(2') \qquad \delta\varphi = -\mu \mathbin{\overline{\wedge}} \varphi - \varphi \mathbin{\overline{\wedge}} \mu = -[\mu, \varphi],$$

and that (3) is consistent with the present definition. We note further that $\mathrm{Alt}^n(V) = C^{n+1}(L, L)$, hence that $\mathrm{Alt}(V)$ is identical with the cochain complex $C(L, L)$, except for a change in grading. Furthermore, the coboundary operator δ can be expressed in terms of the bracket product. We define $D_\mu : \mathrm{Alt}(V) \to \mathrm{Alt}(V)$, a homogeneous linear map of degree 1, by setting $D_\mu \alpha = [\mu, \alpha]$. A simple computation shows that $D_\mu \alpha = (-1)^n \delta\alpha$ for $\alpha \in \mathrm{Alt}^n(V)$. Thus we see that all formulae introduced so far can be expressed in terms of the newly introduced product on $\mathrm{Alt}(V)$. The main properties of this product are summarized in the following theorem.

Theorem 5.1. *The graded vector space* $\mathrm{Alt}(V)$, *with the product* $[\,,\,]$ *defined by* (6) *and* (7) *is a graded Lie algebra. That is, if* $\alpha \in \mathrm{Alt}^m(V)$, $\beta \in \mathrm{Alt}^n(V)$ *and* $\gamma \in \mathrm{Alt}^p(V)$, *then*

(a) $[\alpha, \beta] \in \mathrm{Alt}^{m+n}(V)$; *and depends bilinearly on* α *and* β;

(b) $[\alpha, \beta] = (-1)^{mn}[\beta, \alpha]$;

(c) $(-1)^{mp}[\alpha, [\beta, \gamma]] + (-1)^{mn}[\beta, [\gamma, \alpha]] + (-1)^{np}[\gamma, [\alpha, \beta]] = 0.$

(*If m is odd, one also requires* $[\alpha, [\alpha, \alpha]] = 0$. *This follows from* (c) *except when the base field has characteristic* 3.)

The identity (c) is called the graded Jacobi identity. The proofs of (a) and (b) are immediate. The proof of the graded Jacobi identity follows by triple application (cyclic permutation of entries), *cf.* [10, p. 7], of the so-called commutative-associative law for $\overline{\wedge}$:

$$(\gamma \mathbin{\overline{\wedge}} \alpha) \mathbin{\overline{\wedge}} \beta - \gamma \mathbin{\overline{\wedge}} (\alpha \mathbin{\overline{\wedge}} \beta) = (-1)^{mn}\{(\gamma \mathbin{\overline{\wedge}} \beta) \mathbin{\overline{\wedge}} \alpha - \gamma \mathbin{\overline{\wedge}} (\beta \mathbin{\overline{\wedge}} \alpha)\}.$$

This latter identity may be proved by a direct (though rather demanding) write-out, *cf.* [9]. A more intrinsic proof is given in an appendix to this paper.

The graded Jacobi identity has a number of interesting consequences. For the present we shall mention only two, both related to Lie algebra cohomology. First it implies immediately that $\delta \circ \delta = 0$, where δ is the coboundary operator in $C(L, L)$. Since δ is (except for an irrelevant sign) equal to D_μ, it suffices to show that $D_\mu \circ D_\mu = 0$. But, if $\alpha \in \mathrm{Alt}\,(V)$, we have

$$D_\mu D_\mu \alpha = [\mu, [\mu, \alpha]] = \tfrac{1}{2}[[\mu, \mu], \alpha] = 0.$$

(If W is an arbitrary L-module, then essentially the same proof, applied to the semi-direct product $L + W$, shows that $\delta \circ \delta = 0$, where δ is the coboundary operator in $C(L, W)$.)

Since $D_\mu \circ D_\mu = 0$, the pair $(\mathrm{Alt}(V), D_\mu)$ is a cochain complex. We define $Z(\mathrm{Alt}(V), D_\mu)$ (resp. $B(\mathrm{Alt}(V), D_\mu)$) to be the kernel (resp. image) of D_μ and we set

$$H(\mathrm{Alt}(V), D_\mu) = Z(\mathrm{Alt}(V), D_\mu)/B(\mathrm{Alt}(V), D_\mu).$$

We similarly define $Z^n(\mathrm{Alt}(V), D_\mu)$, $B^n(\mathrm{Alt}(V), D_\mu)$ and $H^n(\mathrm{Alt}(V), D_\mu)$. Then we have $Z^n(\mathrm{Alt}(V), D_\mu) = Z^{n+1}(L, L)$, $B^n(\mathrm{Alt}(V), D_\mu) = B^{n+1}(L, L)$, and $H^n(\mathrm{Alt}(V), D_\mu) = H^{n+1}(L, L)$. The Jacobi identity implies that D_μ is a derivation of $\mathrm{Alt}(V)$. That is, if $\alpha \in \mathrm{Alt}^m(V)$ and $\beta \in \mathrm{Alt}^n(V)$, then $D_\mu[\alpha, \beta] = [D_\mu\alpha, \beta] + (-1)^m[\alpha, D_\mu\beta]$. It follows immediately that if $\alpha, \beta \in Z(\mathrm{Alt}(V), D_\mu)$, then $[\alpha, \beta] \in Z(\mathrm{Alt}(V), D_\mu)$. Similarly, if $\gamma \in B(\mathrm{Alt}(V), D_\mu)$, then $[\alpha, \gamma] \in B(\mathrm{Alt}(V), D_\mu)$. Thus $Z(\mathrm{Alt}(V), D_\mu)$ is a (graded Lie) subalgebra of the graded Lie algebra $\mathrm{Alt}(V)$ and $B(\mathrm{Alt}(V), D_\mu)$ is an ideal in the graded Lie algebra $Z(\mathrm{Alt}(V), D_\mu)$. It follows immediately that there is an induced structure of graded Lie algebra on the quotient space $H(\mathrm{Alt}(V), D_\mu)$. Thus we have proved:

Theorem 5.2. *The graded Lie algebra structure on* $\mathrm{Alt}(V)$ $(= C(L, L))$ *induces a structure of graded Lie algebra on the Lie algebra cohomology space* $H(L, L)$, *in which the usual grading is reduced by one.*

Remark. As noted above, the graded vector spaces $\mathrm{Alt}(V)$ and $C(L, L)$ are identical, except for a change of grading. However, (and this is a point we wish to emphasize) the product $[\,,\,]$ on $\mathrm{Alt}(V)$ is defined independently on any Lie algebra structure that V may have. This is the main reason we have chosen to distinguish between $\mathrm{Alt}(V)$ and $C(L, L)$.

6. The action of GL(V) on Alt(V). The properties of a graded Lie algebra imply that $\mathrm{Alt}^0(V)$ is a Lie algebra in the usual sense. In fact, a moment's reflection shows that it is the Lie algebra $\mathfrak{gl}(V)$ of all linear endomorphisms of V, supplied with the usual commutator product. Extending notation introduced earlier, we define, for each $\alpha \in \mathrm{Alt}^0(V)$, a homogeneous linear map $D_\alpha : \mathrm{Alt}(V) \to \mathrm{Alt}(V)$ of degree 0 by setting $D_\alpha\beta = [\alpha, \beta]$ for $\beta \in \mathrm{Alt}(V)$. It follows immediately from the graded Jacobi identity that the map $\alpha \to D_\alpha$ is a representation of the Lie algebra $\mathrm{Alt}^0(V)$ on $\mathrm{Alt}(V)$.

There is a natural representation ρ of the group $G = GL(V)$ on $\mathrm{Alt}(V)$. If $g \in G$ and $\beta \in \mathrm{Alt}^n(V)$, then $\rho(g)\beta \in \mathrm{Alt}^n(V)$ is given by

$$(\rho(g)\beta)(x_0, \cdots, x_n) = g(\beta(g^{-1}(x_0), \cdots, g^{-1}(x_n))). \tag{8}$$

In fact, ρ is a rational representation of the algebraic group $GL(V)$ on $\mathrm{Alt}(V)$ (*i.e.*, $\rho(g)\beta$ depends rationally on g for fixed β).

Now let $\alpha \in \mathfrak{gl}(V)$, and $g = I + t\alpha$. Expanding $(I + t\alpha)^{-1}$ by the geometric series, we find

$$\begin{aligned}
&(\rho(g)\beta)(x_0, \cdots, x_n)\\
&\quad = (I + t\alpha)(\beta((I + t\alpha)^{-1}x_0, \cdots, (I + t\alpha)^{-1}x_n))\\
&\quad = \beta(x_0, \cdots, x_n) + t\alpha(\beta(x_0, \cdots, x_n)) - t\sum_{i=0}^{n} \beta(x_0, \cdots, \alpha x_i, \cdots, x_n) + \cdots\\
&\quad = \beta(x_0, \cdots, x_n) + t(\alpha \overline{\wedge} \beta - \beta \overline{\wedge} \alpha)(x_0, \cdots x_n) + \cdots,
\end{aligned}$$

where the dots indicate higher order terms. It follows that

$$\rho(I + t\alpha)\beta = \beta + t[\alpha, \beta] + \cdots .$$

This formula gives the relationship between the (finite) action of G on $\mathrm{Alt}(V)$ and the "infinitesimal" action of $\mathfrak{gl}(V)$ on $\mathrm{Alt}(V)$. More precisely, $\mathfrak{gl}(V)$ is the Lie algebra of the linear algebraic group $GL(V)$ and the above formula shows that the differential $d\rho$ of the rational representation ρ is just the representation $\alpha \to D_\alpha$ of $\mathfrak{gl}(V)$ on $\mathrm{Alt}(V)$. (For the appropriate definitions concerning linear algebraic groups, see [1, 2]. A detailed proof of the above result on $d\rho$ follows easily from the general results of [2, Chapitre III]. If V is a real or complex vector space, then $GL(V)$ is a Lie group, $\mathfrak{gl}(V)$ is its Lie algebra, ρ is a representation of Lie groups, and $d\rho$ is just the representation $\alpha \to D_\alpha$.)

It follows easily from (6) and (8) that $\rho(g)(\beta \overline{\wedge} \gamma) = \rho(g)\beta \overline{\wedge} \rho(g)\gamma$ if $g \in G$ and β, γ are homogeneous elements of $\mathrm{Alt}(V)$. This implies that $\rho(g)[\beta, \gamma] = [\rho(g)\beta, \rho(g)\gamma]$, thus that each $\rho(g)$ is an automorphism of the graded Lie algebra $\mathrm{Alt}(V)$.

Let $\mathfrak{M} = \{\mu \in \mathrm{Alt}^1(V) \mid [\mu, \mu] = 0\}$. Then $\mathfrak{M}$ is just the set of all Lie algebra multiplications on V. Since G acts on $\mathrm{Alt}(V)$ by automorphisms, it follows immediately that $\mathfrak{M}$ is stable under the action of G. Let μ_1 , $\mu_2 \in \mathfrak{M}$ and let $L_1 = (V, \mu_1)$ and $L_2 = (V, \mu_2)$ be the corresponding Lie algebras. Then it follows immediately from the definitions that L_1 and L_2 are isomorphic if and only if there exists $g \in G$ such that $\rho(g)\mu_1 = \mu_2$. Thus the orbits of G on $\mathfrak{M}$ correspond precisely to isomorphism classes of Lie algebra structures on V.

In summary, we have shown that the triple $(\mathrm{Alt}(V), G, \rho)$ satisfies the conditions for an algebraic, graded Lie algebra given in [10, p. 11]; when the base field is either **R** or **C** it is also an analytic graded Lie algebra. Thus we may apply the general theorems on deformations in graded Lie algebras given in [10] to study deformations of Lie algebra structures on V. If $\mu \in \mathfrak{M}$ and $L = (V, \mu)$, then the cohomology space $H^n(\mathrm{Alt}(V), D_\mu)$ is identical with the Lie algebra cohomology space $H^{n+1}(L, L)$.

7. Rigid Lie algebras. Let V be a finite-dimensional real vector space and let $\mathfrak{M}$ be the real algebraic set of all Lie algebra multiplications on V; we consider $\mathfrak{M}$ as a topological space with the topology induced as a subset of $\mathrm{Alt}^1(V)$. ($\mathrm{Alt}^1(V)$ is given the usual (Hausdorff) topology of a finite-dimensional real vector space.) A Lie algebra $L = (V, \mu)$ is *rigid* if the orbit $G(\mu) = \{\rho(g)\mu \mid g \in G\}$ is an open subset of $\mathfrak{M}$. The following "rigidity theorem" is a special case of [10, Theorem 18.1 and Corollary 18.2]. (The parenthetical remark in the last sentence should read "(resp. is a (finite) union of components of a Zariski open subset of $\mathfrak{M}$)."

Theorem 7.1. *Let V be a finite-dimensional real vector space, let $\mathfrak{M}$ be the algebraic set of all Lie algebra multiplications on V, and let $L = (V, \mu)$ be a Lie algebra such that $H^2(L, L) = 0$. Then L is rigid. More precisely, the orbit $G(\mu)$ is a (finite) union of components of a Zariski-open subset of $\mathfrak{M}$. Furthermore, there exist only a finite number of isomorphism classes of Lie algebras L with underlying vector space V such that $H^2(L, L) = 0$.*

The idea behind the proof of Theorem 7.1 is simple. The form of the "deformation equation" (4) shows that the "tangent space" to $\mathfrak{M}$ at μ is a subspace of $Z^2(L, L)$ (to be precise, one must use the Zariski tangent space, since $\mathfrak{M}$ may have a singularity at μ). The argument given in Section 4 shows that the tangent space to the orbit $G(\mu)$ at μ is $B^2(L, L)$. If $H^2(L, L) = 0$, *i.e.*, if $Z^2(L, L) = B^2(L, L)$, then an elementary differential-geometric argument shows that the orbit $G(\mu)$ is an open subset of $\mathfrak{M}$, hence that L is rigid. The other statements of the Theorem depend upon a technical lemma [10, Proposition 17.1] and properties of the Zariski topology.

The rigidity theorem above actually holds for Lie algebras over an algebraically closed field of arbitrary characteristic. Let V be a finite-dimensional vector space over an algebraically closed field K and let $\mathfrak{M}$ be the algebraic set of all Lie algebra multiplications on V. We consider $\mathfrak{M}$ as a topological space supplied with the Zariski topology (induced by the Zariski topology of $\text{Alt}^1(V)$). A Lie algebra $L = (V, \mu)$ is rigid if the orbit $G(\mu)$ is an open subset of $\mathfrak{M}$. The following theorem is a special case of [10, Theorem 22.1 and Corollary 22.2].

Theorem 7.2. *Let V be a finite-dimensional vector space over an algebraically closed field and let $L = (V, \mu)$ be a Lie algebra. If $H^2(L, L) = 0$, then L is rigid. Furthermore, there exist only a finite number of isomorphism classes of Lie algebras L with underlying vector space V such that $H^2(L, L) = 0$.*

Let L be a semi-simple Lie algebra over a field K of characteristic 0. Then it follows from [3, Theorem 24.1] that $H^2(L, L) = 0$. Hence L is rigid if $K = \mathbf{R}$ or if K is algebraically closed. More generally, if L' is the direct sum of L and a 1-dimensional Lie algebra, then it follows from the Hochschild-Serre spectral sequence [6, Theorem 13] that $H^2(L', L') = 0$. In particular, if $K = \mathbf{R}$ or if K is algebraically closed of characteristic 0 and if V is a finite-dimensional vector space over K, then the Lie algebra $\mathfrak{gl}(V)$ is rigid.

The following example shows that $H^2(L, L) = 0$ is not a necessary condition that a Lie algebra L be rigid. Let S denote the (unique to within isomorphism) three-dimensional simple Lie algebra over $\mathbf{C}$ and, for each positive integer n, let σ_n denote the irreducible representation of weight n of S on $\mathbf{C}^{2n+1}$. We consider $\mathbf{C}^{2n+1}$ as an abelian Lie algebra and form the corresponding semi-direct product $S +_{\sigma_n} \mathbf{C}^{2n+1}$, which we denote by L_n. Then it is shown in [12] that L_n is rigid if $n \neq 1, 2, 3, 5$ and that $H^2(L_n, L_n) \neq 0$ if n is odd. Thus we obtain a family $L_7, L_9, \cdots$ of rigid Lie algebras with $H^2 \neq 0$.

8. Interpretation of $\mathbf{H^3(L, L)}$ as obstructions. Let $L = (V, \mu)$ be a Lie algebra. In this section we shall show how the elements of $H^3(L, L)$ may be interpreted as "obstructions" to "expanding" an infinitesimal deformation of μ into a one-parameter family of deformations of μ. We shall deal throughout only with formal one-parameter families, *i.e.*, one-parameter families given by a formal power series. A more precise, although less intuitive, interpretation will be given in the following section.

Let $\mu_t = \mu + t\varphi_1 + t^2\varphi_2 + \cdots$ ($\varphi_i \in \mathrm{Alt}^1(V)$) be a (formal) one-parameter family of alternating bilinear maps of $V \times V$ into V. In order that μ_t be a one-parameter family of Lie algebra multiplications on V, we must have $[\mu_t, \mu_t] = 0$. This leads to an infinite sequence of conditions on μ and the φ_i's. The first three conditions are

$$[\mu, \mu] = 0, \qquad D_\mu\varphi_1 = 0, \quad \text{and} \quad D_\mu\varphi_2 + \tfrac{1}{2}[\varphi_1, \varphi_1] = 0.$$

The first condition is automatically satisfied and the second implies that φ_1 must be an infinitesimal deformation. If the first two conditions are satisfied, it follows from the fact that D_μ is a derivation that $\frac{1}{2}[\varphi_1, \varphi_1]$ is a cocycle. In order that φ_2 can be found satisfying the third condition, $\frac{1}{2}[\varphi_1, \varphi_1]$ must be a coboundary. Thus the class of $\frac{1}{2}[\varphi_1, \varphi_1]$ in $H^2(\mathrm{Alt}(V), D_\mu)$ must vanish. This class is the *first obstruction* to forming a one parameter family of deformations whose first order term is φ_1. If it vanishes, another obstruction may show up at the next level, *etc.* All of these obstructions will be shown to lie in $H^2(\mathrm{Alt}(V), D_\mu) = H^3(L, L)$. If $H^3(L, L) = 0$, then each infinitesimal deformation is the first order term of a one-parameter family.

A simple proof that all obstructions lie in $H^2(\mathrm{Alt}(V), D_\mu)$ is the following. Let $\mu_t = \mu + t\varphi_1 + \cdots + t^n\varphi_n$ satisfy the Jacobi identity through order n, *i.e.*, $[\mu_t, \mu_t] = t^{n+1}\gamma_{n+1} + \cdots$, where three dots denote higher order terms. If, now, $\mu_t' = \mu + t\varphi_1 + \cdots + t^{n+1}\varphi_{n+1}$, then

$$[\mu_t', \mu_t'] = t^{n+1}\alpha_{n+1} + \cdots,$$

where $\alpha_{n+1} \in \mathrm{Alt}^2(V)$ is given by

$$\alpha_{n+1} = 2D_\mu\varphi_{n+1} + \sum_{i=1}^{n} [\varphi_i, \varphi_{n+1-i}].$$

In order that φ_{n+1} exist such that μ_t' satisfies the Jacobi identity through order $n + 1$, it is necessary and sufficient that $\beta_{n+1} = \sum_{i=1}^{n} [\varphi_i, \varphi_{n+1-i}]$ be a coboundary. We show that β_{n+1}, or equivalently α_{n+1}, is a cocycle in any case, so that its cohomology class in $H^2(\mathrm{Alt}(V), D_\mu)$ is then an obstruction. By the graded Jacobi identity, we have $[\mu_t', [\mu_t', \mu_t']] = 0$. In view of the above formulas, this implies that

$$0 = [\mu_t', [\mu_t', \mu_t']] = t^{n+1}[\mu, \alpha_{n+1}] + \cdots.$$

Hence, $[\mu, \alpha_{n+1}] = D_\mu\alpha_{n+1} = 0$, which was to be proved.

The power series method outlined above was used by Kodaira, Nirenberg and Spencer [7, p. 453] in a problem concerning deformations of complex structures on a compact manifold. A more involved method was used by Gerstenhaber [4] in studying deformations of associative algebras. In the following section, we shall give preference to a study of all local solutions to the deformation equation at once. The successive obstructions are then replaced by one obstruction map Ω with values in $H^2(\mathrm{Alt}(V), D_\mu)$.

9. Locally complete families of deformations. We sketch the solution of the deformation equation

$$D_\mu \varphi + \tfrac{1}{2}[\varphi, \varphi] = 0 \tag{9}$$

for the case when the base field is **R** or **C**, so analytic methods may be used. For details we refer of course to [10]. As a preliminary we choose in each $Z^n = Z^n(\mathrm{Alt}(V), D_\mu)$ a subspace H^n complementary to $B^n = B^n(\mathrm{Alt}(V), D_\mu)$; we shall identify H^n with $H^n(\mathrm{Alt}(V), D_\mu) = H^{n+1}(L, L)$ by means of the canonical isomorphism. We also choose in each $\mathrm{Alt}^n(V)$ a subspace C^n complementary to Z^n. Then $\mathrm{Alt}^n(V) = B^n \oplus H^n \oplus C^n$. Denote by π_B, π_H, π_C the projections on these subspaces. To solve (9) we first solve π_B times it, which is a sub-system of (9):

$$D_\mu \varphi + \tfrac{1}{2}\pi_B[\varphi, \varphi] = 0, \tag{10}$$

and set $\varphi = z + c$; $z \in Z^1$, $c \in C^1$. Then $D_\mu z = 0$, while $D_\mu c = 0$ if and only if $c = 0$,

$$D_\mu c + \tfrac{1}{2}\pi_B[z + c, z + c] = 0. \tag{11}$$

The left side denotes a map of $Z^1 \times C^1$ into B^2. For $z = 0$ the map $C^1 \to B^2$ is just D_μ, hence is an isomorphism. By the implicit function theorem it is therefore possible to express c as an analytic function of $z : c = \Phi(z)$ in a neighborhood of $(z, c) = (0, 0)$. Hence the "small" solutions of (11), or of (10), are of the form $\varphi = z + c = z + \Phi(z)$. We now apply π_H to the left side of (9) and substitute the solution of (11) into it:

$$\Omega(z) = \tfrac{1}{2}\pi_H[z + \Phi(z), z + \Phi(z)],$$

so Ω is an analytic map defined in a neighborhood of 0 in Z^1, with values in H^2. It is called the *obstruction map*. We now consider the expression obtained by applying π_C to the left side of (9): $\tfrac{1}{2}\pi_C[\varphi, \varphi]$, which equals $\tfrac{1}{2}\pi_C[\mu + \varphi, \mu + \varphi]$. Denote it γ for short. We claim that, if $\Omega(z) = 0$ and $\varphi = z + \Phi(z)$, and if φ is sufficiently small (*i.e.*, z sufficiently small), then $\gamma = 0$. Observe that, under the hypotheses, $\tfrac{1}{2}[\mu + \varphi, \mu + \varphi] = \gamma \in C^2$, and that by the Jacobi identity

$$0 = \tfrac{1}{2}[\mu + \varphi, [\mu + \varphi, \mu + \varphi]] = [\mu + \varphi, \gamma] = D_\mu \gamma + [\varphi, \gamma].$$

Now D_μ sends C^1 injectively into $\mathrm{Alt}^2(V)$; hence so does any map close to D, such as $D + [\varphi, \cdot]$ for small φ. Hence $D\gamma + [\varphi, \gamma] = 0$ implies $\gamma = 0$ for small φ (*i.e.*, small z) since $\gamma \in C^1$. Thus there is a neighborhood N of 0 in Z^1 such that the $\varphi = z + \Phi(z)$ with $\Omega(z) = 0$ and $z \in N$ fill out a neighborhood of μ in $\mathfrak{M}$.

A locally complete family of deformations of μ is a connected subset W of $\mathfrak{M}$ with $\mu \in W$ such that the orbit $G(W)$ is a neighborhood of μ in $\mathfrak{M}$. Thus, intuitively, every element of $\mathfrak{M}$ near μ lies on the orbit of an element of W near μ.

Theorem 9.1. *Let V be a finite-dimensional real or complex vector space and let $\mathfrak{M}$ be the algebraic set of all Lie algebra multiplications on V. Let $\mu \in \mathfrak{M}$ and* *let* $L = (V, \mu)$. *Then there exists a neighborhood* N *of* 0 *in* $H^2(L, L)$ *and analytic maps* $\Phi : N \to \mathrm{Alt}^2(V, V)$ *and* $\Omega : N \to H^3(L, L)$ *such that*

$$\mathcal{K} = \{\mu + z + \Phi(z) \mid z \in H^2(L, L) \cap N \text{ and } \Omega(z) = 0\}$$

is a locally complete family of deformations of μ.

(We recall that $H^2(L, L)$ (resp. $H^3(L, L)$) has been identified with a vector subspace of $\mathrm{Alt}^1(V)$ (resp. $\mathrm{Alt}^2(V)$).)

For the proof we note first that the tangent space at μ to the orbit $G(\mu)$ consists of all elements of $\mathrm{Alt}^1(V)$ of the form $[\alpha, \mu] = D_\mu(-\alpha)$ with $\alpha \in \mathrm{Alt}^0(V)$; *i.e.*, this tangent space is exactly B^1. Since $\mathrm{Alt}^1(V) = B^1 \oplus (H^1 \oplus C^1)$, it follows that the orbit $G(\mu)$ meets the linear variety $\mu + H^1 + C^1$ transversally at μ. An easy application of the inverse function theorem shows that there exists a neighborhood N_1 of μ in the linear variety $\mu + H^1 + C^1$ such that the orbit $G(N_1)$ is a neighborhood of μ in $\mathrm{Alt}^1(V)$. Since $\mathfrak{M}$ is stable under the action of G, it follows from the above results that $\mathcal{K}$ is a locally complete family of deformations of μ.

The locally complete family of deformations $\mathcal{K}$ is an exact analogue of the locally complete family of complex structures on a compact manifold constructed by Kuranishi [8].

For the case of a Lie algebras $L = (V, \mu)$ over an algebraically closed field, a similar result holds. For details see [10]. In particular, if $H^3(L, L) = 0$, we have:

Theorem 9.2. *Let* V *be a finite-dimensional vector space over an algebraically closed field and let* $\mathfrak{M}$ *be the algebraic set of all Lie algebra multiplications on* V. *Let* $\mu \in \mathfrak{M}$ *be such that, setting* $L = (V, \mu)$, *we have* $H^3(L, L) = 0$. *Then there is precisely one irreducible component* $\mathfrak{M}_1$ *of* $\mathfrak{M}$ *which contains* μ. *Furthermore,* μ *is a simple point of* $\mathfrak{M}_1$ *and the tangent space to* $\mathfrak{M}_1$ *at* μ *is* $Z^2(L, L)$.

10. Deformations of modules. The methods developed in the preceding can be applied to the problem of deforming a representation σ of a Lie algebra $L = (V, \mu_0)$ on a vector space W. The crucial observation to be made is that the conditions that μ_0 satisfy the Jacobi identity (1) and that σ be a homomorphism of L into $\mathfrak{gl}(W)$ can be expressed in *one* condition. We consider the alternating bilinear map μ of $V \times W$ into $V \times W$ defined by

$$\mu(x, y) = \mu_0(x, y) \quad \text{if} \quad x, y \in V,$$
$$\mu(x, y) = -\mu(y, x) = \sigma(x)y \quad \text{if} \quad x \in V, y \in W,$$
$$\mu(x, y) = 0 \quad \text{if} \quad x, y \in W.$$

Then the above conditions on μ_0 and σ are satisfied if and only if μ satisfies (1), *i.e.*, if $(V \times W, \mu)$ is a Lie algebra. Conversely, if μ is a Lie algebra product on $V \times W$ and if $\mu(V, V) \subset V$, $\mu(V, W) \subset W$, $\mu(W, W) = \{0\}$, then the restriction μ_0 of μ to V satisfies (1), and the linear map $\sigma : L \to \mathfrak{gl}(W)$ obtained from the restriction of μ to (V, W) is a homomorphism. From here on we assume that μ is a Lie algebra product of the type discussed, and μ_0 and σ are as above.

By restricting attention to the deformations of μ into product maps of the type discussed we may thus study deformations of σ, or also simultaneous deformations of μ_0 and σ. Each of these problems can be formulated by means of its own graded Lie algebra. They will be denoted F resp E.

We set $E = \bigoplus E^n$ and define $E^{-1} = V \times W$, while for $n \geqq 0$ we take

$$E^n = \left\{ \alpha \varepsilon \operatorname{Alt}^n (V \times W) \left| \begin{array}{lll} \alpha(x_0, \cdots, x_n) \varepsilon V & \text{if} & x_0, \cdots, x_n \varepsilon V \\ \alpha(x_0, \cdots, x_n) \varepsilon W & \text{if} & \exists i \text{ with } x_i \varepsilon W \\ \alpha(x_0, \cdots, x_n) = 0 & \text{if} & \exists i \neq j \text{ with } x_i, x_j \varepsilon W \end{array} \right. \right\}.$$

A moment's reflection on (6) shows that if $\alpha, \beta \varepsilon E$ then $\alpha \mathbin{\overline{\wedge}} \beta \varepsilon E$. Hence $[\alpha, \beta] \varepsilon E$, and E is a subalgebra of $\operatorname{Alt}(V \times W)$. The multiplication maps μ under consideration are exactly those in E^1, and hence E is stable under these D_μ. Multiplication maps μ_1 and μ_2 on $V \times W$ in the present context are equivalent if they are related by a map $g \varepsilon GL(V \times W)$ (*cf.* §4) which sends V and W onto themselves. The group G of these maps has as its Lie algebra the set of all maps $V \times W \to V \times W$ under which V and W are stable; *i.e.*, the Lie algebra of G is E^0. A look at (8) confirms that E is stable under the action of G.

In summary, we thus see that (E, D_μ) is a cochain complex, and that (E, G, ρ) is an algebraic resp. analytic graded Lie algebra in the sense of [10]. Therefore, Theorems 7.1, 9.1 and 9.2 hold for simultaneous deformations of μ_0 and σ, provided the groups $H^n(L, L)$ in these statements are replaced by $H^{n-1}(E, D_\mu)$.

For the deformations of σ in which μ_0 is kept fixed we may use the graded Lie algebra $F = \bigoplus F^n$, with $F^{-1} = W$, and, for $n \geqq 0$,

$$F^n = \{\alpha \varepsilon E^n \mid \alpha(x_0, \cdots, x_n) = 0 \quad \text{if} \quad x_0, \cdots, x_n \varepsilon V\}.$$

A look at (6) now shows that $\alpha \mathbin{\overline{\wedge}} \beta \varepsilon F$ if $\alpha \varepsilon E, \beta \varepsilon F$ or if $\alpha \varepsilon F, \beta \varepsilon E$. It follows that F is an ideal in the graded Lie algebra E. In particular, F is stable under D_μ, with $\mu \varepsilon E^1$. Representations σ_1, σ_2 are equivalent if they differ by an element g of $GL(W) : \sigma_2(x) = g \cdot \sigma_1(x) \cdot g^{-1}$. The Lie algebra, $\mathfrak{gl}(W)$ of $GL(W)$ is just F^0. Clearly, F is stable under the action of $GL(W)$ (considered as a subgroup of $GL(V \times W)$) on $\operatorname{Alt}(V \times W)$ *via* ρ, (*cf.* (8)). Thus (F, D_μ) is a cochain complex, and $(F, GL(W), \rho)$ is an algebraic resp. analytic graded Lie algebra. Theorems 7.1, 9.1 and 9.2 hold for deformations of σ, with $H^n(L, L)$ replaced by $H^{n-1}(F, D_\mu)$.

It is intuitively clear that the infinitesimal deformations of σ are also infinitesimal deformations of μ, though of a special kind, since μ_0 is not changed. Similarly, any infinitesimal deformation of μ in E^1 gives rise to an infinitesimal deformation of μ_0, by restriction to V. These facts, and several more, are brought out by considering the quotient graded Lie algebra E/F. Brief contemplation of the definitions of E^n and F^n shows $(E/F)^n$, which is the same as

E^n/F^n, to be naturally isomorphic to $\mathrm{Alt}^n(V)$, and it also shows that $\overline{\wedge}$ (*cf.* (6)) commutes with the isomorphism. Furthermore, the composition j of the canonical map $E \to E/F$ and the isomorphism $E/F \to \mathrm{Alt}(V)$ maps μ into μ_0. We observe that

$$0 \to F \xrightarrow{i} E \xrightarrow{j} \mathrm{Alt}(V) \to 0,$$

where i is the injection, is an exact sequence of graded Lie algebra homomorphisms which commute with the coboundary maps D_μ, D_μ and $D_{\mu 0}$, respectively.

A standard result in homological algebra yields an exact triple of vector space homomorphisms

$$\begin{array}{ccc} H^*(F, D_\mu) & \xrightarrow{i^*} & H^*(E, D_\mu) \\ D^* \uparrow & & \downarrow j^* \\ & H^*(\mathrm{Alt}(V), D_{\mu_0}) & \end{array}$$

where, in fact, i^* and j^* are graded Lie algebra homomorphisms, while D^* is of degree 1. An interesting portion of the triangle is the following sequence

$$\to H^0(\mathrm{Alt}(V)) \xrightarrow{D^*} H^1(F) \xrightarrow{i^*} H^1(E) \xrightarrow{j^*} H^1(\mathrm{Alt}(V)) \xrightarrow{D^*} H^2(F) \to.$$

The exactness of maps between the middle three terms was formulated at the beginning of this discussion. Exactness at $H^1(F)$ shows that a non-trivial infinitesimal deformation of σ is trivial as a deformation of μ if it is the image of an element of $H^0(\mathrm{Alt}(V))$, *i.e.* of an outer derivation (infinitesimal outer automorphism) of $L = (V, \mu_0)$. This reflects the fact that representations σ_1, σ_2 obtainable from each other *via* an automorphism of L were, in general, not considered as equivalent. The portion of the sequence:

$$\to H^1(\mathrm{Alt}(V)) \xrightarrow{D^*} H^2(F) \xrightarrow{i^*} H^2(E) \xrightarrow{j^*} H^2(\mathrm{Alt}(V)) \to$$

involving the obstruction spaces, can be similarly discussed.

In §3 it was shown that $H^n(\mathrm{Alt}(V), D_\mu)$ can be identified with the more familiar $H^{n+1}(L, L)$. We now show that $H^n(F, D_\mu)$ can be identified with $H^n(L, \mathfrak{gl}(W))$, where $\mathfrak{gl}(W)$ is an L-module *via* σ and the adjoint representation. The exact triple then becomes

$$\begin{array}{ccc} H^*(L, \mathfrak{gl}(W)) & \xrightarrow{i^*} & H^*(E, D_\mu) \\ D^* \uparrow & & \downarrow j^* \\ & H^*(L, L) & \end{array}$$

where now i^* and D^* are of degree 0 and j^* of degree 1.

To each $\alpha \in F^n$ $(n \geqq 0)$ is associated a uniquely determined alternating n-linear map $\bar{\alpha}$ of V into $\mathfrak{gl}(W)$, by

$$\bar{\alpha}(x_1, \cdots, x_n)y = \alpha(x_1, \cdots, x_n, y),$$

and conversely. The representation τ of $L = (V, \mu_0)$ on $\mathfrak{gl}(W)$ is given by

$$(\tau(x)\beta)y = \sigma(x)\beta y - \beta\sigma(x)y = \mu(x, \beta y) - \beta\mu(x, y),$$

where $x \in V$, $y \in W$, $\beta \in \mathfrak{gl}(W)$. For $\beta = \bar{\alpha}(x_1, \cdots, x_n)$ this becomes

$$\begin{aligned}(\tau(x)\bar{\alpha}(x_1, \cdots, x_n))y &= \mu(x, \bar{\alpha}(x_1, \cdots, x_n)y) - \bar{\alpha}(x_1, \cdots, x_n)\mu(x, y)\\ &= -\mu(\alpha(x_1, \cdots, x_n, y), x) - (-1)^n\alpha(\mu(x, y), x_1, \cdots, x_n).\end{aligned}$$

We now replace $(x, x_1, \cdots, x_n)$ by $(x_i, x_0, \cdots, \hat{x}_i, \cdots, x_n)$, multiply by $(-1)^i$ and sum on i; thus finding the first sum in the expression for $\delta\bar{\alpha}$; *cf.* (3).

$$\begin{aligned}(\sum_i (-1)^i\tau(x_i)\bar{\alpha}(x_0, \cdots, \hat{x}_i, \cdots, x_n))y &\\ = -\sum_i (-1)^i\mu(\alpha(x_0, \cdots, \hat{x}_i, \cdots, x_n, y), x_i)&\\ + \sum_i (-1)^{i+n+1}\alpha(\mu(x_i, y), x_0, \cdots, \hat{x}_i, \cdots, x_n).&\end{aligned}$$

The first term on the right equals $(-1)^n(\mu \,\overline{\wedge}\, \alpha)$ $(x_0, \cdots, x_n, y)$. The second term on the right combines with

$$\sum_{i<j} (-1)^{i+j}\alpha(\mu(x_i, x_j), x_0, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n, y)$$

(which is the second sum in the expression for $\delta\bar{\alpha}$, *cf.* (3)) to give $(\alpha \,\overline{\wedge}\, \mu)$ $(x_0, \cdots, x_n, y)$. Hence we find

$$\begin{aligned}(\delta\bar{\alpha}(x_0, \cdots, x_n))y &= ((-1)^n(\mu \,\overline{\wedge}\, \alpha + \alpha \,\overline{\wedge}\, \mu)(x_0, \cdots, x_n, y)\\ &= (-1)^n(D_\mu\alpha)(x_0, \cdots, x_n, y).\end{aligned}$$

Thus the coboundary maps of (F, D_μ) and $(C^n(L, \mathfrak{gl}(W)), \delta)$ commute under $\alpha \to \bar{\alpha}$ up to a non-vanishing factor. The correspondence $\alpha \to \bar{\alpha}$ thus gives an isomorphism between $H^n(F, D_\mu)$ and $H^n(L, \mathfrak{gl}(W))$.

The complex (E, D_μ) and its cohomology take on a more natural form if $(V \times W, \mu)$ is considered as a graded Lie algebra Λ in which $V = \Lambda^0$, $W = \Lambda^1$ and all other summands vanish. Then $H^n(E, D_\mu)$ is the same as $H^{n+1,0}(\Lambda, \Lambda)$, the $(n + 1)^{st}$ cohomolgy space based on maps $\alpha \in \mathrm{Alt}(\Lambda)$ of degree 0. In particular, $H^1(E, D_\mu) = H^{2,0}(\Lambda, \Lambda)$ which measures the non-trivial infinitesimal degree-preserving deformations of the product map of Λ.

The study of deformations of $\sigma : L \to \mathfrak{gl}(W)$ may be considered as a special case of the general problem of deforming a Lie algebra homomorphism $\sigma : L \to M$. This problem also fits into the general pattern of deformation theory, but uses a different graded Lie algebra. It is discussed in [11].

11. Deformations of ideals. Let $L = (V, \mu)$ be a Lie algebra and U a subspace of V which is an ideal: $\mu(V, U) \subset U$. One may ask for deformations of μ such that U remains an ideal. Such deformations yield deformations of the Lie algebra structure on U, as a subalgebra, deformations of the induced Lie algebra structure on V/U, and deformations of the structure of U as an L-module. They are also related to deformations of (V, μ) as an extension of V/U by U. So clearly, the range of problems to be investigated here is even richer than that sketched in §10.

We confine ourselves to the construction of the appropriate graded Lie algebra $I = \bigoplus I^n$ for deformations of μ. We set $I^{-1} = V$, and for $n \geqq 0$

$$I^n = \{\alpha \in \operatorname{Alt}^n(V) \mid \alpha(x_0, \cdots, x_n) \in U \quad \text{if} \quad \exists\, i \quad \text{with} \quad x_i \in U\}.$$

Observations of the usual kind show that I is indeed a subalgebra, that $\mu \in I^1$, hence I is stable under D_μ, and that the group G of equivalences consists of those $g \in GL(V)$ under which U is stable. Its Lie algebra is exactly I^0. Also, I is stable under G *via* ρ, (*cf.* (8)). Hence (I, D_μ) is a cochain complex, and (I, G, ρ) is an algebraic resp. analytic graded Lie algebra. Theorems 7.1, 9.1 and 9.2 give results on deformations of μ, provided $H^n(L, L)$ is replaced by $H^{n-1}(I, D_\mu)$.

12. Appendix: Proof of the commutative-associative law. The identity in question is

$$(12) \qquad (\gamma \mathbin{\overline{\wedge}} \alpha) \mathbin{\overline{\wedge}} \beta - \gamma \mathbin{\overline{\wedge}} (\alpha \mathbin{\overline{\wedge}} \beta) = (-1)^{mn}((\gamma \mathbin{\overline{\wedge}} \beta) \mathbin{\overline{\wedge}} \alpha - \gamma \mathbin{\overline{\wedge}} (\beta \wedge \alpha)),$$

where $\alpha, \beta, \gamma \in \operatorname{Alt}(V)$, of respective degrees m, n, p.

Since V is finite-dimensional, $\operatorname{Alt}^n(V)$ is canonically isomorphic to $V \otimes A^{n+1}$, where A^{n+1} is the vector space of all alternating $(n + 1)$-forms on V (*i.e.*, all alternating $(n + 1)$-linear maps of V into the base field). Hence, it is sufficient to prove (12) for $\alpha = x \otimes \omega$, $\beta = y \otimes \pi$ and $\gamma = z \otimes \tau$, where $x, y, z \in V$ and where ω, π, and τ are alternating forms on V of respective degrees $m + 1$, $n + 1$, and $p + 1$. Then

$$\gamma \mathbin{\overline{\wedge}} \alpha = z \otimes \omega \wedge (\tau \mathbin{\overline{\wedge}} x),$$
$$(\gamma \mathbin{\overline{\wedge}} \alpha) \mathbin{\overline{\wedge}} \beta = z \otimes \pi \wedge ((\omega \wedge (\tau \mathbin{\overline{\wedge}} x)) \mathbin{\overline{\wedge}} y),$$
$$\gamma \mathbin{\overline{\wedge}} (\alpha \mathbin{\overline{\wedge}} \beta) = z \otimes \pi \wedge (\omega \mathbin{\overline{\wedge}} y) \wedge (\tau \mathbin{\overline{\wedge}} x).$$

These expressions contain only $\overline{\wedge}$-products of forms with a vector, which means contraction with a vector. This operation satisfies the evident rules.

$$(\sigma \wedge \omega) \mathbin{\overline{\wedge}} y = (\sigma \mathbin{\overline{\wedge}} y) \wedge \omega + (-1)^{\deg \sigma} \sigma \wedge (\omega \mathbin{\overline{\wedge}} y),$$
$$(\tau \mathbin{\overline{\wedge}} x) \mathbin{\overline{\wedge}} y = -(\tau \mathbin{\overline{\wedge}} y) \mathbin{\overline{\wedge}} x.$$

The first rule, with $\sigma = \tau \mathbin{\overline{\wedge}} x$, applied in $(\gamma \mathbin{\overline{\wedge}} \alpha) \mathbin{\overline{\wedge}} \beta$ yields $\gamma \mathbin{\overline{\wedge}} (\alpha \mathbin{\overline{\wedge}} \beta)$ plus the term

$$(-1)^{m+1} z \otimes \pi \wedge \omega \wedge ((\tau \mathbin{\overline{\wedge}} x) \mathbin{\overline{\wedge}} y).$$

The same computation, with α, β interchanged would yield

$$(-1)^{n+1} z \otimes \omega \wedge \pi \wedge ((\tau \,\overline{\wedge}\, y) \,\overline{\wedge}\, x).$$

By the second rule for $\overline{\wedge}$ and the "commutativity" of $\wedge$ we see that the first of these results equals $(-1)^{m+1}(-1)^{n+1}(-1)(-1)^{(m+1)(n+1)} = (-1)^{mn}$ times the second. That proves (12).

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