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Deformations of Principal Bundles on the Projective Line

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1. Introduction

The study of bundles on $\mathbb{P}^1$ apparently has a long history (see [22, Chap. I, Sect. 2.4]). Grothendieck proved that any principal bundle on $\mathbb{P}^1_{\mathbb{C}}$ with a complex reductive Lie groups as structure group admits a reduction of structure group to a maximal torus, unique up to Weyl group action [9]. Harder gave a simple proof of this result which works for $\mathbb{P}^1$ over arbitrary fields [11]. In this paper we study the deformations of principal bundles over $\mathbb{P}^1$.

Let G be a split reductive group over the field k. By the result of Grothendieck-Harder and Zariski locally trivial G-bundle on $\mathbb{P}^1$ is associated to the G_m-bundle $k^2-0\to\mathbb{P}^1$ by a 1-PS λ: $G_m\to G$. Let us denote this G-bundle by E_λ.

Let $\mathbf{E}\to S\times\mathbb{P}^1$ be a G-bundle with an isomorphism $\mathbf{E}_{s_0}=\mathbf{E}|s_0\times\mathbb{P}^1\simeq E_\lambda$. We then call $\mathbf{E}$ a deformation of E_λ parametrized by S, s_0. We say that the G-bundle E' tends or degenerates to the G-bundle E, and write $E'\rightsquigarrow E$, if there is a deformation $\mathbf{E}\to S\times\mathbb{P}^1$ of E such that in *every* neighbourhood of the base point $s_0\in S$, $(\mathbf{E}_{s_0}\simeq E)$, there is an s such that $\mathbf{E}_s\approx E'$.

We prove (Theorem 7.4) that if λ, μ are dominant 1-PS then $E_\mu\rightsquigarrow E_\lambda$ if and only if $\mu\leqq\lambda$, i.e. $\lambda-\mu$ is a positive integral combination of simple coroots (or, equivalently $(\lambda-\mu, \omega_i)\in\mathbb{Z}^+$ for every fundamental weight ω_i. See Sect. 2.5).

Note that the set of dominant μ such that $\mu\leqq\lambda$ is the same as the set of dominant weights occuring in the indecomposable (or irreducible, if $\operatorname{char} k=0$) representation of the dual group G^v (see Sect. 2.6) with highest weight λ (cf. [16, Sect. 21.3]). The deformation theory of G-bundles on $\mathbb{P}^1$ seems to be much the same as the representation theory of the dual group G^v (cf. [9, p. 123]). It would be interesting to find a more intrinsic connection between them.

The G-bundles E and E' are said to be algebraically equivalent if there is a G-bundle $\mathbf{E}\to S\times\mathbb{P}^1$, with S connected, such that $E\approx\mathbf{E}_s$ and $E'\approx\mathbf{E}_{s'}$ for some $s, s'\in S$. We prove (Theorem 7.7) that the algebraic equivalence classes of Zariski locally trivial G-bundles are classified by the fundamental group of G (i.e. the quotient of the lattice of 1-PS of G by the lattice of coroots). This result

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holds more generally for irreducible smooth projective curves of arbitrary genus (cf. [23, Sect. 5]; Sect. 7.10).

We also identify the rigid G-bundles as those E_λ such that λ is a dominant 1-PS which is minimal with respect to the ordering $\leq$ (Proposition 7.8).

Brieskorn has studied the equivalence of complex projective bundles [7] and Hulek that of complex orthogonal bundles [15].

In Sect. 5 we describe the automorphism group $\operatorname{Aut} E_\lambda$ (identity over the base) and prove the irreducibility of some spaces of B-reductions.

In Sect. 8 we construct and study the versal deformation $\mathbf{U} \to S \times \mathbb{P}^1$ of E_λ. We prove that $S_{\lambda\mu} = \{s \in S \mid \mathbf{U}_s \approx E_\mu\}$ are smooth locally closed subvarieties and give their dimensions. We have also indicated there how to deduce the existence of an (algebraic) versal deformation for a G-bundle over a curve of higher genus using the results of [1, 2].

In Sect. 9 we have given the modifications to be made when G is not connected (Theorem 9.2). We have also given there the specialisation of our results to the case of vector bundles and bundles with other classical groups as structure groups.

The results and proofs are often motivated by looking at what happens in the special case $G = GL(2)$ (i.e. vector bundles of rank 2) and giving it the right formulation so that it generalises. As in the theory of algebraic groups we often reduce inductively to this special case.

2. Algebraic Groups and Principal Bundles

We fix some notation to be adopted throughout this paper.

2.1. Let k be an arbitrary field. Let G be a connected reductive algebraic group defined and split over k (Chevalley group). Let T be a maximal split torus and B a Borel subgroup containing T. Let U be the unipotent radical of B. Then $B = T \cdot U$ (semidirect product). Let $i: T \hookrightarrow B$ and $j: B \hookrightarrow G$ be the inclusions and $p: B \to B/U = T$ be the projection. Let $W = N(T)/T$ be the Weyl group and $w_0 \in W$ the element of maximal length in W (cf. [4, 5]).

2.2. Let G_m be the 1-dimensional torus and G_a the additive group. We denote by $X_*(T)$ the group of homomorphisms of G_m into T. We write the group operation in $X_*(T)$ additively. We call elements of $X_*(T)$ 1-parameter subgroups (abbreviation: 1-PS). $X^*(T)$ denotes the group of characters of T. We have a natural perfect pairing $X_*(T) \otimes X^*(T) \to \operatorname{Hom}(G_m, G_m) = \mathbb{Z}$ given by composition. We denote this pairing by $(\ ,\ )$.

2.3. We refer to [29] for facts about root data (see also [5, 16]). Let $\Phi \subset X^*(T)$ be the system of roots of G, Φ^+ the set of positive roots and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots corresponding to B. For $\alpha \in \Phi$ let U_α be the root group corresponding to α [5, Sect. 2.3], T_α the connected component of $\ker \alpha$ and Z_α the centraliser of T_α in G. Then the derived group $[Z_\alpha, Z_\alpha]$ is of rank 1 and there is a unique 1-PS $\alpha^\vee: G_m \to T \cap [Z_\alpha, Z_\alpha]$ such that $T = (\operatorname{Im} \alpha^\vee) \cdot T_\alpha$ and $(\alpha^\vee, \alpha) = 2$ [29, Sect. 2]. This $\alpha^\vee$ is called the *coroot* corresponding to α. We denote by

Φ^v the set of coroots. The quadruple $\{X^*(T), \Phi, X_*(T), \Phi^v\}$ with the map $\Phi \to \Phi^v$ given by $\alpha \mapsto \alpha^v$ constitutes the root datum.

2.4. Let Q be the *root lattice*, i.e. the subgroup of $X^*(T)$ generated by Φ. Let $P = \{x \in X^*(T) \otimes \mathbb{Q} \mid (\alpha^v, x) \in \mathbb{Z}$ for every $\alpha \in \Phi\}$ be the *weight lattice*. Let $\omega_1, \ldots, \omega_l$ be the duals of the simple coroots $\alpha_1^v, \ldots, \alpha_l^v$, i.e. $(\alpha_i^v, \omega_j) = \delta_{ij}$ and $(\lambda, \omega_i) = 0$ for any λ in the centre of G. These elements of P are called the *fundamental weights*. Let X_0 be the subgroup of $X^*(T)$ othogonal to $\Phi^v \subset X_*(T)$. Then $X_0 = X^*(G)$ and G is semisimple if and only if $X_0 = 0$. We have $Q \cap X_0 = 0$ and $Q + X_0$ is of finite index in $X^*(T)$, [29]. Let Q^v, P^v and X_0^v be the corresponding objects for $X_*(T)$.

2.5. Let $P_+ = \{x \in P \mid (\alpha^v, x) \geqq 0$, for every $\alpha \in \Phi^+\}$. Elements of P_+ are called *dominant weights*. Dually $P_+^v = \{y \in P^v \mid (y, \alpha) \geqq 0$ for every $\alpha \in \Phi^+\}$. We denote by $X_*(T)_+$ the set of *dominant* 1-*PS*, i.e. $X_*(T)_+ = P_+^v \cap X_*(T)$. We have a partial ordering $\leqq$ in $X_*(T)$ (and dually on $X^*(T)$) defined as follows: $\mu \leqq \lambda$ if and only if $\lambda - \mu$ is a positive integral combination of $\alpha^v \in \Delta^v$ or equivalently $(\lambda - \mu, \omega_i) \in \mathbb{Z}^+$ and $(\lambda - \mu, \chi) = 0$ for $\chi \in X^*(G) = X_0$.

2.6. The group G^v whose root datum is the dual root datum $(X_*(T), \Phi^v, X^*(T), \Phi)$ is called the *dual group* of G [29]. The dominant 1-PS of G are the integral dominant weights of G^v. Hence a dominant 1-PS of G corresponds to an indecomposable representation of G^v, namely the one with highest weight λ (the so called Weyl module).

2.7. We illustrate the above notions by looking at the special case of $GL(n)$. The diagonal matrices $\operatorname{diag}[x_1, \ldots, x_n]$ form a maximal torus T and the upper triangular matrices form a Borel subgroup B. Any 1-PS of T is of the form $t \mapsto \operatorname{diag}[t^{a_1}, \ldots, t^{a_n}]$, $a_i \in \mathbb{Z}$. Therefore $X_*(T) = \mathbb{Z}^n$. Moreover $\operatorname{diag}[x_1, \ldots, x_n] \mapsto x_1^{b_1}, \ldots, x_n^{b_n}$ gives a typical character on T so that $X^*(T) = \mathbb{Z}^n$. The pairing $(\ ,\ )$ is $((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sum a_i b_i$. Let $\varepsilon_i = (0, \ldots, 0, 1, \ldots, 0)$ be the i^{th} coordinate vector. Then $\Phi = \Phi^v = \{\varepsilon_i - \varepsilon_j \mid i \neq j,\ 1 \leqq i, j \leqq n\}$ and $\Delta = \Delta^v = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leqq i \leqq n-1\}$. The root datum is $(\mathbb{Z}^n, \Phi, \mathbb{Z}^n, \Phi)$ and the dual group $GL(n)^v$ is $GL(n)$ itself. $Q = \{(b_1, \ldots, b_n) \mid \sum b_i = 0\}$ and $X_0 = \{(r, r, \ldots, r) \mid r \in \mathbb{Z}\}$. The maps induced by $(b_1, \ldots, b_n) \mapsto \sum b_i$ give isomorphisms $X^*(T)/Q \approx \mathbb{Z}$ and $X^*(T)/Q + X_0 \approx \mathbb{Z}_n$. The root datum of the derived group $SL(n)$ is $(X^*(T)/X_0, \Phi, Q, \Phi)$ (cf. [29]).

2.8. We usually use lower case bold face letters to denote the corresponding Lie algebras. Thus $\mathbf{g}$ denotes the Lie algebra of G, and $\mathbf{t}$, $\mathbf{b}$, $\mathbf{u}$ those of T, B, U. Let U_α be the root group corresponding to the root α. Then $\mathbf{u}_\alpha$ is isomorphic to U_α. The group multiplication gives an isomorphism $\prod_{\alpha \in \Phi^+} U_\alpha \to U$ of varieties [4, Sect. 14.4 Remark, p. 330].

2.9. *Principal Bundles*. By a *principal bundle* with structure group G (or a G-bundle) over X we mean a morphism π: $E \to X$ where G acts on E on the right and π is G-invariant and isotrivial (i.e. for every $x \in X$ there is an étale morphism φ: $Y \to X$, $x \in \varphi(Y)$, such that the pull back $\varphi^* E$ is isomorphic to $Y \times G$, G-equivariantly for the action of G on $Y \times G$ by right multiplication on the second factor). See [27].

2.10. If G operates on F (on the left) the associated bundle is denoted by $E(F)$. Recall that $E(F)$ is the quotient of $E\times F$ under the action of G given by $g(e,f)=(e\cdot g, g^{-1}\cdot f)$, $e\in E$, $f\in F$, $g\in G$, [27].

2.11. If G acts on F_1 and F_2 and $F_1\to F_2$ is a G-equivariant morphism then there is a natural morphism $E(F_1)\to E(F_2)$.

2.12. If $\rho\colon G\to H$ is a homomorphism of groups the associated bundle $E(H)$, for the action of G on H by left multiplication through ρ, is naturally a H-bundle. We denote this H-bundle sometimes by $\rho_* E$ and we say that $\rho_* E$ is obtained from E by *extension of structure group.*

2.13. A pair (E,φ), where E is a G-bundle and $\varphi\colon \rho_* E\to F$ is a H-bundle isomorphism, is said to give a *reduction of structure group* of F to G. We sometimes omit φ and call E a *G-reduction* of F. Two G-reductions of structure group (E_1,φ_1) and (E_2,φ_2) are equivalent or isomorphic if there is a G-bundle isomorphism $\psi\colon E_1\to E_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \rho_* E_1 & \xrightarrow{\rho_*\psi} & \rho_* E_2 \\ & {\scriptstyle\varphi_1}\searrow \quad \swarrow{\scriptstyle\varphi_2} & \\ & F & \end{array}$$

2.14. If $\rho\colon G\hookrightarrow H$ is a closed subgroup inclusion the quotient F/G is naturally isomorphic to the associated bundle $F(H/G)$. Further $F\to F/G$ is a G-bundle and a section $\sigma\colon X\to F/G$ of $F/G\to X$ gives the G-bundle $\sigma^* F$ on X with a natural isomorphism $\rho_*\sigma^* F\simeq F$. Thus equivalence classes of reductions of structure group of F to G are in bijective correspondence with sections of $F/G\to X$.

2.15. A $GL(n)$-bundle E is completely determined by the associated vector bundle $E(V)$ (where V is the canonical n-dimensional space on which $GL(n)$ acts) as its bundle of frames. Similarly a $PGL(n)$-bundle is equivalent to a projective bundle i.e. an isotrivial fibre bundle with $\mathbb{P}^n$ as fibre.

2.16. If X is a projective smooth curve and $L\to X$ is a line bundle we mean by $\deg L$ the degree of the divisor associated to a rational section of L. If $W\to X$ is a vector bundle of rank n we denote by $\det W$ the line bundle $\Lambda^n W$, the n^{th} exterior power of W. We define $\deg W$ to be $\deg(\det W)$. The vector bundle W gives a locally free sheaf, namely the sheaf of sections. We will not differentiate between this sheaf and the vector bundle. If S is a subsheaf of W we call S a *subbundle* if the quotient sheaf is locally free. We call the minimal subbundle $\bar{S}$ containing a subsheaf S the subbundle *generated* by S (cf. [21, Sect. 4]).

3. *T*-bundles and *B*-bundles

We now describe the bundles on $\mathbb{P}^1$ (the projective line over k) with the split torus T as structure group.

3.1. On $\mathbb{P}^1$ we have the natural G_m-bundle $k^2-0\to\mathbb{P}^1$. If λ: $G_m\to T$ is a 1-PS we denote by T_λ the T-bundle obtained by the extension of structure group λ^{-1}: $G_m\to T$ (we take the inverse because we want the line bundle $\mathcal{O}(1)$ to be associated to $(G_m)_{\mathrm{id}}$ for the natural action of G_m on k). Note the $k^2-0\to\mathbb{P}^1$ is trivial on $\mathbb{P}^1-0$ and $\mathbb{P}^1-\infty$ and λ: $G_m=(\mathbb{P}^1-0)\cap(\mathbb{P}^1-\infty)\to T$ can be thought of as the transition function for the T-bundle T_λ.

3.2. Given a T-bundle E on $\mathbb{P}^1$ we get a homomorphism λ_E: $X^*(T)\to\mathbb{Z}$ by associating to $\chi\in X^*(T)$ the degree of the line bundle associated to E for the action of T on k through χ. By duality (Sect. 2.2) this homomorphism is given by a 1-PS $\lambda_E\in X_*(T)$: $\deg\chi_*E=(\lambda_E,\chi)$ for every $\chi\in X^*(T)$.

3.3. **Lemma.** *The mapping $\lambda\mapsto T_\lambda$ gives a bijective correspondence between $X_*(T)$ and isomorphism classes of T-bundles on $\mathbb{P}^1$, the inverse mapping being $E\mapsto\lambda_E$ described above.*

Proof. For $T=G_m$ the lemma is clear since $\mathcal{O}(d)\mapsto d$ gives a bijection $\operatorname{Pic}\mathbb{P}^1\to\mathbb{Z}$. When $T=G_m\times G_m\ldots$ is the product of r copies of G_m the lemma follows by noting that a T-bundle is nothing but an (ordered) r-tuple of line bundles.

3.4. Now we come to B-bundles. Let E be a B-bundle on $\mathbb{P}^1$. Then by the above lemma there is a unique $\lambda\in X_*(T)$ such that the T-bundle $p_*E(p$: $B\to B/U=T$, Sect. 2.1) is isomorphic to T_λ. We call λ the *T-type* or simply the *type* of the B-bundle E.

3.5. Let B act on U by inner conjugation. Since inner conjugation preserves the group structure of U the associated bundle $E(U)$ is a group scheme over $\mathbb{P}^1$ (i.e. the fibres are groups).

3.6. **Lemma.** *Let E be a B-bundle. Then the associated bundle $E(B/T)(=E/T)$ is a principal homogeneous space under the group scheme $E(U)$ over $\mathbb{P}^1$.*

Proof. Consider the action of U on B/T given by $U\times(B/T)\to B/T$, $(u,bT)\mapsto ubT$. This is simply transitive. Moreover if we make B act on U by inner conjugation and on B/T by left translation then $U\times B/T\to B/T$ is B-equivariant. Therefore this gives rise to the action $E(U)\times E(B/T)\to E(B/T)$ (see Sect. 2.11). Hence the lemma.

3.7. **Lemma.** *If the T-type λ of the B-bundle E is such that for every $\alpha\in\Phi^+\subset X^*(T)$ we have $(\lambda,\alpha)\geqq-1$ then $H^1(\mathbb{P}^1,E(U))=1$ and $E\approx i_*T_\lambda(i$: $T\hookrightarrow B$ is the inclusion, Sect. 2.1).*

Proof. The non-abelian cohomology group $H^1(\mathbb{P}^1,E(U))$ classifies the principal homogeneous spaces of the group scheme $E(U)$ over $\mathbb{P}^1$ (cf. [19, Chap. III, Sect. 4]). Therefore if $H^1(\mathbb{P}^1,E(U))=1$ then by Lemma 3.6 above $E(B/T)$ has a section and hence E has a reduction of structure group to T giving $i_*T_\lambda\approx E$.

To show $H^1(\mathbb{P}^1,E(U))=1$ note that U has a filtration $U\supset U_1\supset U_2\ldots$ by T-invariant normal subgroups such that the successive quotients are isomorphic to G_a with the T-action given by a positive root, (cf. [5 Sect. 2.3; 11]). From the exact cohomology sequence corresponding to $1\to U_1\to U\to G_a\to 1$ we have

$H^1(\mathbb{P}^1, E(U_1)) \to H^1(\mathbb{P}^1, E(U)) \to H^1(\mathbb{P}^1, E(G_a))$ (see [19, Chap. III, Sect. 4]). The last term is zero, from the hypothesis. Therefore it is enough to prove that $H^1(\mathbb{P}^1, E(U_1)) = 1$. Now proceed inductively with $U_2, \ldots$.

4. The Theorem of Grothendieck-Harder

In this section we give briefly Harder's proof of the theorem on the classification of Zariski locally trivial G-bundles on $\mathbb{P}^1$.

4.1. **Definition.** Let F be a B-bundle giving a B-reduction of the G-bundle E. We call the T-type of F (see Sect. 3.4) to be the *T-type*, or simply the *type*, of the B-reduction F. We say that F is a *split reduction* if F admits a T-reduction. (Note that if F is a split reduction of type λ then $F \approx i_* T_\lambda$.)

4.1.1. **Definition.** Let $q = j \cdot i$: $T \hookrightarrow G$ be the inclusion (Sect. 2.1). For $\lambda \in X_*(T)$ we denote by E_λ the G-bundle $q_* T_\lambda$ i.e. E_λ is the G-bundle obtained from the Hopf bundle $k^2 - 0 \to \mathbb{P}^1$ by the extension of structure group λ^{-1}: $G_m \to G$ (Sect. 3.1).

4.1.2. If one extends the structure group of a G-bundle E by an inner automorphism $\mathrm{Int}\, g$: $G \to G$ one gets an isomorphic G-bundle $(\mathrm{Int}\, g)_* E$ with the canonical isomorphism $(\mathrm{Int}\, g)_* E \simeq E$ induced by $E \times G \to E$, $(e, h) \mapsto egh$ (Sect. 2.10). For $w \in W$ the map w: $T \to T$ is induced by an inner conjugation of G, determined upto inner conjugation by an element of T. Therefore for $w \in W$, $q_* w_* T_\lambda \simeq q_* T_\lambda$, the isomorphism being determined upto inner conjugation of G by an element of T. Thus for each $w \in W$, $q_* T_\lambda = E_\lambda$ has the canonical T-reduction $w_* T_\lambda$ (unique, upto isomorphism, Sect. 2.13). This gives further the canonical split B-reduction $i_* w_* T_\lambda$ of the type $w\lambda$.

4.2. **Theorem** (Grothendieck-Harder). *Let $E \to \mathbb{P}^1$ be a G-bundle (k arbitrary field) which is locally trivial in the Zariski topology. Then $E \approx E_\lambda$ for some $\lambda \in X_*(T)$. For $\lambda, \mu \in X_*(T)$, $E_\lambda \approx E_\mu$ if and only if $\mu = w\lambda$ for some $w \in W$. Therefore the Zariski locally trivial G-bundles on $\mathbb{P}^1$ are classified by $X_*(T)/W$.*

Proof. To show that E *admits a reduction to* T we have only to find a reduction to B of T-type λ with $(\lambda, \alpha) \geqq 0$ for all $\alpha \in \Phi^+$ (Lemma 3.7). For a reduction σ: $\mathbb{P}^1 \to E/B$ and a character χ on B let $n(\chi, \sigma) = \deg \chi_* \sigma^* E$ (= the degree of the line bundle associated to the reduced B-bundle through the character χ). Let $\omega_1, \ldots, \omega_l$ be the fundamental weights. We can find an integer $s > 0$ such that $s\omega_1, \ldots, s\omega_l$ are characters of B. The number $n(s\omega_i, \sigma)$ are bounded from above as σ varies over all possible B-reductions (since $n(s\omega_i, \sigma)$ is the degree of a line subbundle of $E(V)$, where V is the irreducible representation of G with highest weight $s\omega_i$; cf. proof of Proposition 6.16 and [9, Lemma 2.2]).

Since E is locally trivial in the Zariski topology the set of B-reductions is nonempty. For we can take a generic section of E/B and it would extend to whole of $\mathbb{P}^1$ by properness criterion, G/B being complete. Let therefore σ be a reduction such that $n(s\omega_i, \sigma)$ is maximal in the sense that there exist no σ' with $n(s\omega_i, \sigma') \geqq n(s\omega_i, \sigma)$ for every i and for some i_0, $n(s\omega_{i_0}, \sigma') > n(s\omega_{i_0}, \sigma)$.

We claim that for such a maximal σ we have $n(\alpha, \sigma) \geqq 0$ for every $\alpha \in \Delta$. For a simple root α let P_α be the minimal parabolic subgroup corresponding to α generated by B and $U_{-\alpha}$. Let U' be the unipotent radical of P_α and $Z_\alpha = (\ker \alpha)^0 \subset T$. Then $P_\alpha/Z_\alpha \cdot U'$ is isomorphic to $SL(2)$ or $PSL(2)$ and the Borel subgroups of $P_\alpha/Z_\alpha \cdot U'$ are in bijective correspondence with those of G contained in P_α [5]. Thus a reduction of structure group of the $SL(2)$ or $PSL(2)$ bundle $\sigma^* E(P_\alpha/Z_\alpha \cdot U')$ to a Borel subgroup gives a reduction σ' of structure group of E to a Borel subgroup of G. Further since σ' is achieved within P_α it is easy to see that $n(s\omega_i, \sigma') = n(s\omega_i, \sigma)$ for all ω_i except ω_{i_0} corresponding to α. It follows immediately from the Riemann-Roch theorem that for any $SL(2)$ or (Zariski locally trivial) $PSL(2)$ bundle there exists a reduction $\bar{\sigma}$ to a Borel subgroup such that the corresponding $n(\bar{\alpha}, \bar{\sigma}) \geqq 0$ where $\bar{\alpha}$ is the simple root of $PSL(2)$. Let σ_1 be the corresponding reduction of E so that $n(s\omega_i, \sigma) = n(s\omega_i, \sigma_1)$, $i \neq i_0$. A simple computation shows that in the expression of ω_{i_0} in terms of α and ω_i, $i \neq i_0$, the coefficient of α is positive [11, p. 136]. Therefore if $n(\alpha, \sigma) < 0$ then $n(s\omega_{i_0}, \sigma_1) > n(s\omega_{i_0}, \sigma)$. This would contradict the maximality of σ and hence we have proved the claim that $n(\alpha, \sigma) \geqq 0$ for all $\alpha \in \Delta$. Hence by Lemma 3.7 $E \approx E_\lambda$ for some $\lambda \in X_*(T)$. The uniqueness statement follows from Corollary 6.17 in Sect. 6 below.

To complete the picture when $k = \bar{k}$ we have the following proposition.

4.3. **Proposition.** *Let X be a smooth projective curve over an algebraically closed field $\bar{k}$. Then any G-bundle E on X, with G connected reductive, is locally trivial in the Zariski topology.*

Proof. Let $K = k(X)$ be the function field of X. Then $E_K(G/B)$ is a principal homogeneous space under B_K over K. Since $\bar{k}$ is algebraically closed, by [30] it is trivial. Therefore $E(G/B)$ has a section over K and hence over an open subset of X and hence over the whole of X by the properness criterion. Thus E admits a reduction to B.

Now any T-bundle is Zariski locally trivial [19, Chap. III, Proposition 4.9]. Therefore it is enough to prove that any B-bundle F on X admits a T-reduction over any affine open subset A of X i.e. $H^1(A, F(U)) = 1$. This can be proved exactly as in the proof of Lemma 3.7 using $H^1(A, F(G_a)) = 0$, A being affine.

4.4. *Remark.* If $X = \mathbb{P}^1$ the assumption that G is connected can be dropped in Proposition 4.3. For, by applying the Riemann-Hurwitz formula the étale covering $E(G/G_0) \to \mathbb{P}^1$ has a section, where G_0 is the identity component of G ($\mathbb{P}^1$ is "simply connected"). Thus we get a reduction to the connected group G_0.

5. Automorphism Groups

5.1. Let λ be a dominant 1-PS. Let $P(\lambda)$ be the corresponding parabolic subgroup, generated by T and the root groups U_α with $(\lambda, \alpha) \geqq 0$ [20, Chap. II, Sect. 2]. Let $U(\lambda)$ be the unipotent radical of $P(\lambda)$. Let $Z(\lambda)$ be the centraliser of λ in G. Then $Z(\lambda)$ is a connected reductive group and $P(\lambda) = Z(\lambda) \cdot U(\lambda)$ and for the Lie algebras $\mathbf{z}(\lambda)=\mathbf{t}\oplus \sum_{(\lambda,\alpha)=0} \mathbf{u}_\alpha$, $\mathbf{u}(\lambda)=\sum_{(\lambda,\alpha)>0} \mathbf{u}_\alpha$ (cf. [5]). $Z(\lambda)$ and $\mathbf{z}(\lambda)$ are called Levi supplements (for the radicals).

5.2. **Proposition.** *Let λ be a dominant* 1-PS *and E_λ the corresponding G-bundle on $\mathbb{P}^1$. Then $Z(\lambda)$ is naturally a subgroup of* $\operatorname{Aut} E_\lambda$, *the group of bundle automorphisms of E_λ (identity on the base). Further* $\operatorname{Aut} E_\lambda$ *is isomorphic as a variety to*

$$Z(\lambda)\times H^0(\mathbb{P}^1, T_\lambda(\mathbf{u}(\lambda))(=Z(\lambda)\times \prod_{(\lambda,\alpha)>0} H^0(\mathbb{P}^1, T_\lambda(\mathbf{u}_\alpha)).$$

Proof. We will write $E,P,Z,\ldots$ in place of $E_\lambda, P(\lambda), Z(\lambda), \ldots$. Let G act on itself by inner conjugation and $E(G)$ the associated bundle. It is a group scheme over $\mathbb{P}^1$ and $\operatorname{Aut} E=H^0(\mathbb{P}^1, E(G))$. Now $T_\lambda(\mathbf{p})$ is the sum of all lie line sub-bundles of $E(\mathbf{g})$ of degree $\geqq 0$. Hence any vector bundle endomorphism of $E(\mathbf{g})$ leaves it invariant. In particular $\operatorname{Aut} E$ leaves it invariant. This implies (since the normaliser of $\mathbf{p}$ in G is P) that any global section of $E(G)$ has values in $T_\lambda(P)$. Therefore $\operatorname{Aut} E=H^0(\mathbb{P}^1, T_\lambda(P))$. Now $P=ZU$ T-equivariantly and hence $H^0(\mathbb{P}^1, T_\lambda(P))=Z\times H^0(\mathbb{P}^1, T_\lambda(U)$. Again $U=\prod_{(\lambda,\alpha)>0} \mathbf{u}_\alpha$ T-equivariantly. Hence the result.

5.2.1. Let $B(E_\lambda,\mu)$ be the space of (isomorphism classes of) B-reductions of E_λ of type μ. Then $B(E_\lambda,\mu)$ is the space of certain sections of $E_\lambda/B\to\mathbb{P}^1$ (Sect. 2.14). To be more precise, consider the functor Γ from the category of schemes over k to the category of sets defined as follows. Γ associates to a scheme S the set of sections $\sigma\colon S\times\mathbb{P}^1\to S\times E_\lambda/B$ such that for every $s\in S$ the restriction $\sigma_s\colon s\times\mathbb{P}^1=\mathbb{P}^1\to s\times E_\lambda/B=E_\lambda/B$ is a section of type μ (i.e. gives a B-reduction of type μ). For a morphism $f\colon S'\to S$, $\Gamma(f)(\sigma)$ is the pull back section $f^*(\sigma)$. By [10, exposé 221] Γ is represented by an algebraic scheme (see also [11]). $B(E_\lambda,\mu)$ is this representing scheme. Note that for an arbitrary section $\sigma\colon S\times\mathbb{P}^1\to S\times E_\lambda/B$ the type of σ remains constant on the connected components of S.

5.2.2. There is a natural action of $\operatorname{Aut} E_\lambda$ on $B(E_\lambda,\mu)$: Let $g\in\operatorname{Aut} E_\lambda$ and $(F,\varphi)\in B(E_\lambda,\mu)$. Then $g(F,\varphi)=(F,g\varphi)$. (See Sect. 2.13.)

5.3. **Proposition.** *Let λ be a dominant* 1-PS *and $\lambda_0=w_0\lambda$ the opposite* 1-PS *(Sect. 2.1). Then* $\operatorname{Aut} E_\lambda$ *acts transitively on $B(E_\lambda,\lambda_0)$. Further $B(E_\lambda,\lambda_0)$ is smooth and irreducible.*

Proof. Let (F_0,φ_0) be the canonical B-reduction of E_λ of type λ_0 (Sect. 4.1.2). Since F_0 is a split reduction, i.e. a B-reduction which comes from a T-reduction (Sect. 4.1), any translate of it by $\operatorname{Aut} E_\lambda$ will also be a split reduction.

Conversely any split reduction (F,φ) of type λ_0 is an $\operatorname{Aut} E_\lambda$ translate of (F_0,φ_0). To see this first note that we have an isomorphism $\psi\colon F_0\to F$, both F_0 and F being isomorphic to $i_* T_{\lambda_0}$. Extending structure groups by $j\colon B\hookrightarrow G$ we get an isomorphism $j_*\psi\colon j_*F_0\to j_*F$. Define $g\in\operatorname{Aut} E_\lambda$ by $g=\varphi(j_*\psi)\varphi_0^{-1}$. Then clearly g takes (F_0,φ_0) to (F,φ).

So to prove the transitivity it is enough to show that any B-reduction of E_λ of type λ_0 is a split reduction.

First let us look at the $SL(2)$-case. Let $V=\mathcal{O}(n)\oplus\mathcal{O}(-n)$, $n\geqq 0$. Let L be a line subbundle of V of degree $-n$ (i.e. a B-reduction of type λ_0). Consider the composite $\mathcal{O}(n)\hookrightarrow V\to V/L$; since $\mathcal{O}(n)$ and V/L have the same degree the map is either zero or an isomorphism. It cannot be zero since $\mathcal{O}(n)\neq L$. Therefore it is an isomorphism so that $V=\mathcal{O}(n)\oplus L$. This proves that L corresponds to a split reduction. The proof in the general case is a natural generalisation of this, using the adjoint representation in the place of the canonical representation for $SL(2)$.

Let $(F,\varphi)\in B(E_\lambda,\lambda_0)$. We have to get a section of $F(B/T)$. To simplify notation let us write P, $\mathbf{p}$, Z, $\mathbf{z}$, U, $\mathbf{u}$ for $P(\lambda)$, $\mathbf{p}(\lambda)$, etc. and P_0 etc. for $P(\lambda_0)$ etc. Consider the Grassmannian X of subspaces of $\mathbf{p}$ of dimension that of $\mathbf{z}$. The Borel subgroup $B\subset P$ acts on X through the adjoint representation. The isotropy subgroup in B at $\mathbf{z}\in X$ is T [5]. Therefore B/T gets embedded B-equivariantly in X as the orbit of $\mathbf{z}$ under B. Further any Levisupplement of $\mathbf{p}$ is conjugated to $\mathbf{z}$ under the unipotent radical of P [5]. Therefore the B orbit of $\mathbf{z}$ consists precisely of the subspaces of $\mathbf{p}$ which are Levisupplements. Thus $F(B/T)\subset F(X)$ and a T-reduction of F is equivalent to a subbundle of the Lie algebra bundle $F(\mathbf{p})$ which at every fibre is a Levisupplement. We now proceed to produce such a subbundle.

We use the isomorphism $\varphi: j_* F\to E_\lambda$ and the inclusion $\mathbf{p}\subset\mathbf{g}$ to identify $F(\mathbf{p})$ as a subbundle Q_1 of $E_\lambda(\mathbf{g})$. Similarly the canonical T-reduction T_{λ_0} of type λ_0 (Sect. 4.1.2) and the inclusion $\mathbf{p}_0\subset\mathbf{g}$ give a subbundle $Q_2=T_{\lambda_0}(\mathbf{p}_0)$ of $E_\lambda(\mathbf{g})$. We will show that the subsheaf $Q_1\cap Q_2$ is actually a subbundle of Levisupplements of $F(\mathbf{p})$, thus getting a T-reduction for F.

If $\mathbf{p}_1$ and $\mathbf{p}_2$ are two parabolic subalgebras of $\mathbf{g}$ such that $\mathbf{g}=\mathbf{p}_1\oplus\mathbf{u}_2$, where $\mathbf{u}_2$ is the nilradical of $\mathbf{p}_2$ then one knows that $\mathbf{p}_1$ and $\mathbf{p}_2$ are opposite parabolic subalgebras with $\mathbf{p}_1\cap\mathbf{p}_2$ a Levi-supplement for both $\mathbf{p}_1$ and $\mathbf{p}_2$ [5]. Therefore to show that $Q_1\cap Q_2$ is a Levisupplement it is enough to show that the natural projection $T_{\lambda_0}(\mathbf{u}_0)\to E_\lambda(\mathbf{g})/F(\mathbf{p})$ is an isomorphism.

Now $\mathbf{u}_0$ has a T-invariant decomposition $\mathbf{u}_0=\bigoplus_{(\lambda,\beta)<0}\mathbf{u}_\beta=\bigoplus_{(\lambda_0,\beta)>0}\mathbf{u}_\beta$. Therefore we have $T_{\lambda_0}(\mathbf{u}_0)=\bigoplus_{(\lambda_0,\beta)>0}T_{\lambda_0}(\mathbf{u}_\beta)$.

This T-invariant decomposition of $\mathbf{u}_0$ can be suitably arranged to give a B-invariant filtration. For this introduce a total order $\prec$ in the set of roots as follows. Let $\alpha,\beta\in\Phi$.

i) If $(\lambda_0,\alpha)<(\lambda_0,\beta)$ define $\alpha\prec\beta$.

ii) If $(\lambda_0,\alpha)=(\lambda_0,\beta)$ define $\alpha\prec\beta$ if height of $\alpha>$height of β (where if $\alpha=\sum_{\alpha_i\in\Delta}a_i\alpha_i$, height of $\alpha=\Sigma a_i$).

iii) In the subsets where both $(\lambda_0,-)$ and height remain constant take arbitrary total orderings.

Let $\beta_1\prec\ldots\prec\beta_r$ be the total order induced on the subset $\{\beta\in\Phi|(\lambda_0,\beta)>0\}$. Let $V_j=\bigoplus_{i=1}^{j}\mathbf{u}_{\beta_i}$. Then the filtration $0=V_0\subset V_1\ldots\subset V_r=\mathbf{u}_0$ is B-invariant since for $\alpha\in\Phi^+$, $(\lambda_0,\alpha)\leqq 0$ and the adjoint action of U_α increases height. Since $\mathbf{p}$ and $\mathbf{p}_0$ are opposite $\mathbf{g}=\mathbf{p}\oplus\mathbf{u}_0$ and the above filtration induces a filtration $0=\bar{V}_0\subset\bar{V}_1\ldots\subset\bar{V}_r=\mathbf{g}/\mathbf{p}$. Forming associated bundles with respect to the B-bundle

F we get $0 \subset F(V_1) \ldots \subset F(\mathbf{g}/\mathbf{p})$. Since F is of type λ_0, $F(\bar{V}_i)/F(\bar{V}_{i-1}) \approx (G_m)_{\beta_i \lambda_0} \approx T_{\lambda_0}(\mathbf{u}_{\beta_i})$. Therefore the associated graded of the above filtration is isomorphic to $T_{\lambda_0}(\mathbf{u}_0)$.

Let $d_i = \deg(F(\bar{V}_i)/F(\bar{V}_{i-1}))$. Then $d_i = (\lambda_0, \beta_i)$. It follows easily from the definition of $\prec$ that $0 < d_1 \leqq d_2 \ldots \leqq d_r$. By similar considerations it is easy to see that $F(\mathbf{p})$ has a filtration whose successive quotients are line bundles of degree $\leqq 0$. Under these conditions the following lemma (part ii)) shows that $T_{\lambda_0}(\mathbf{u}_0) \to E_\lambda(\mathbf{g})/F(\mathbf{p})$ is an isomorphism as was to be shown.

5.3.1. **Lemma.** *Let X be a smooth projective irreducible curve.*

i) Let $V = L_1 \oplus \ldots \oplus L_r$ where $L_1, \ldots, L_r$ are line bundles on X of the same degree d. Let W be a vector bundle on X with a filtration $0 = W_0 \subset W_1 \ldots \subset W_s = W$ such that W_i/W_{i-1} is a line bundle of degree d. Let $f\colon V \to W$ be a homomorphism. Then $\ker f$ *is a direct summand of V and, if nonzero, is itself a direct sum of line bundles of degree d.*

ii) Let V be a vector bundle of rank n on X. Let W, W' be subbundles of rank r and $n-r$ respectively. Suppose that there is a filtration $W' = V_0 \subset V_1 \ldots \subset V_r = V$ such that $V_i/V_{i-1} = L_i$ is a line bundle of degree d_i with $d_1 \leqq d_2 \leqq \ldots$. Further suppose that $W = M_1 \oplus \ldots \oplus M_r$ such that degree $M_i = d_i$ and that W' has a filtration whose successive quotients are line bundles of degree $< d_1$. Then the natural projection $W \to V/W'$ is an isomorphism.

Proof. i) We can assume without loss of generality that $f(V) \not\subset W_{s-1}$ and that the natural map $L_r \to W_s/W_{s-1}$ induced by f is nonzero. It is then an isomorphism. Therefore $W = W_{s-1} \oplus f(L_r)$. Now consider $p \circ f\colon V \to W_{s-1}$ (where p is the projection $W \to W_{s-1}$) and use induction on rank W.

ii) Let $M_r, M_{r-1} \ldots M_{r-i}$ be the set of M_j with $\deg M_j = d_r$. Suppose $M_r \oplus \ldots \oplus M_{r-i} \subset V_{r-1}$. Then by part i) the kernel of $M_r \oplus \ldots \oplus M_{r-i} \to V_{r-1}/V_{r-1-i}$ will contain a line subbundle L of degree d_r. Then $L \subset V_{r-1-i}$. But by assumption V_{r-1-i} admits a filtration with successive quotients line bundles of degree $< d_r$. Therefore there can be no nonzero homomorphism from L into V_{r-1-i}. This contradiction shows that $M_r \oplus \ldots \oplus M_{r-i} \not\subset V_{r-1}$. Therefore we can assume that the composite $M_r \hookrightarrow W \to V/W' \to V/V_{r-1} = L_r$ is nonzero and hence an isomorphism. We then have $V = V_{r-1} \oplus M_r$; consider V_{r-1}, W' and $W \cap V_{r-1}$ in place of V, W' and W and use induction on rank V/W'.

This proves the lemma and with it we have completed the proof of the transitivity of the action of $\operatorname{Aut} E_\lambda$ on $B(E_\lambda, \lambda_0)$.

Since $\operatorname{Aut} E_\lambda$ is irreducible by Proposition 5.2 it follows that $B(E_\lambda, \lambda_0)$ is irreducible. The smoothness of $B(E_\lambda, \lambda_0)$ follows from the infinitesmal criterion of [10, exposé 221]. See Lemma 6.11 below.

5.4. *Remark.* Propositions 5.2 and 5.3 can be suitably generalised to T-bundles on curves of higher genus. If X is of genus g the proofs go through if we assume $E \to X$ is a G-bundle admitting a T-reduction of type λ with $(\lambda, \alpha) \geqq g \,\forall \alpha \in \Phi^+$. If $E \to X$ is an arbitrary G-bundle we can degenerate it to a T-bundle (cf. Lemma 6.9). Therefore if we could functorially embed (compactify) the space of B-reductions as a dense subset (at least in the irreducible components of maximal dimension) of a complete space then from the analogue of Proposition 5.3 we would get another proof for Harder's result [12] that the space of B-reductions of E (of suitable T-type) is irreducible.

6. Deformations and B-reductions

In this section we show that E_μ degenerates to E_λ if and only if E_μ admits a B-reduction of type $w(\lambda)$ for some $w \in W$. Thus we convert the problem of studying deformations into one of studying B-reductions. Proposition 6.16 gives the possible T-types of B-reductions of E_λ.

6.1. Let E be a G-bundle on $\mathbb{P}^1$. A G-bundle $\mathbf{E} \to S \times \mathbb{P}^1$ together with an isomorphism $\mathbf{E}_{s_0} = \mathbf{E}|s_0 \times \mathbb{P}^1 = E$ at the base point $s_0 \in S$ is called a *deformation* of E parametrized by S, s_0. We sometimes refer to $\mathbf{E} \to S \times \mathbb{P}^1$ as a *family of G-bundles* parametrized by S.

6.2. A *morphism* of two deformations $\mathbf{E} \to S \times \mathbb{P}^1$ and $\mathbf{E}' \to S' \times \mathbb{P}^1$ of E (with base points $s \in S$, $s' \in S'$) is a G-bundle morphism $\mathbf{E} \to \mathbf{E}'$ inducing identity: $E = \mathbf{E}_s \to E'_{s'} = E$. A morphism $\mathbf{E} \to \mathbf{E}'$ is equivalent to an isomorphism of $\mathbf{E}$ with the pull back of $\mathbf{E}'$ by the map $S \to S'$ on the base.

6.3. We can define in the obvious way the functor D_E of deformations of E from the category of pointed schemes over k to the category of sets by associating to S, s_0 the set of isomorphism classes of deformations of E parametrized by S, s_0.

6.4. It follows from Theorem 4.2 and Proposition 4.3 that there is a unique dominant 1-PS such that E and E_λ become isomorphic over a finite extension of the base field. We then say that E is of type λ. If k is algebraically closed then E is of type λ if and only if $E \approx E_\lambda$.

6.5. Let $\mathbf{E} \to S \times \mathbb{P}^1$ be a G-bundle and S irreducible. Let $K = k(S)$ be the function field of S. Let $\mathbf{E}_K \to \mathbb{P}^1_K$, the base change of $\mathbf{E} \to S \times \mathbb{P}^1$ by $\operatorname{spec} K \to S$, be of type $\lambda \in X_*(T)_+$. This implies that for all s in some nonempty open subset of S, $\mathbf{E}_s$ is of type λ. We then call λ the *generic type* of the deformation $\mathbf{E}$.

6.6. **Definition.** *Let λ, μ be* 1-PS. *We say E_μ tends or degenerates to E_λ and write $E_\mu \rightsquigarrow E_\lambda$ if there exists a deformation $\mathbf{E} \to S \times \mathbb{P}^1$ of E_λ, with S an irreducible variety, whose generic type is μ (see Sect.* 6.5 *above).*

6.7. It is easy to see that $E_\mu \rightsquigarrow E_\lambda$ if and only if there is a deformation $\mathbf{E} \to (\operatorname{Spec} A) \times \mathbb{P}^1 = \mathbb{P}^1_A$ of E_λ where A is a discrete valuation ring with residue field a finite extension of k and quotient field K, such that $\mathbf{E}_K \approx E_\mu$. (We can even assume the residue field to be k, since as we shall show later there is a versal deformation space for E_λ over k itself; cf. Sect. 8.5.)

6.8. **Definition.** *We say that E_λ is rigid if $E_\mu \rightsquigarrow E_\lambda$ implies $E_\mu \approx E_\lambda$.*

6.9. **Proposition.** *Let λ, μ be dominant* 1-PS. *If E_μ has a B-reduction of type $w(\lambda)$ for some $w \in W$ then $E_\mu \rightsquigarrow E_\lambda$.*

Proof. The notion of extension of structure group can be slightly generalised as follows. Let $\mathbf{E} \to S \times X$ be a family of G-bundles on X (Sect. 6.1). Let $\rho\colon S \times G \to S \times G$ be a family of homomorphisms i.e. ρ is a morphism over S such that for every $s \in S$ the restriction ρ_s of ρ to $s \times G = G$ is a homomorphism of groups. Then we can form in the obvious way a G-bundle $\rho_*\mathbf{E}\to S\times X$ which when restricted to $s\times X=X$ corresponds to the extension of structure group by ρ_s.

We shall show below that there is a family of homomorphisms $\rho\colon \mathbb{A}^1\times B\to\mathbb{A}^1\times B$, where $\mathbb{A}^1$ is the affine line, such that for $0\neq z\in\mathbb{A}^1$, $\rho : B\to B$ is inner conjugation by an element of B and ρ_0 is the composite $B\xrightarrow{p} T\xrightarrow{i} B$ (Sect. 2.1). Now assume that such a ρ exists. Let F be a B-reduction of E_μ of type $w(\lambda)$. Consider the product family $\mathbb{A}^1\times F\to\mathbb{A}^1\times\mathbb{P}^1$. Extend the structure group by ρ as above. Since extension of structure group by an inner conjugation does not change the isomorphism class (Sect. 4.1.2) we see that $\rho_*(\mathbb{A}^1\times F)$ gives a degeneration of the B-bundle F to the split bundle $i_*T_{w(\lambda)}$. By further extending the structure group to G (and noting that $(ji)_*T_{w(\lambda)}\approx(ji)_*T_\lambda=E_\lambda$) we get a degeneration $E_\mu\rightsquigarrow E_\lambda$.

So it only remains to construct ρ. Start with a 1-PS v such that $(v,\alpha)>0$ for every $\alpha\in\Phi^+$. Since $G_m=\mathbb{A}^1-0$, v gives rise to a map $(\mathbb{A}^1-0)\times B\to(\mathbb{A}^1-0)\times B$ defined by $(z,tu)\mapsto(z,t\,v(z)\,u\,v(z)^{-1})$ where $z\in\mathbb{A}^1-0$, $t\in T$ and $u\in U$. For a root α we have the isomorphism $\theta_\alpha\colon G_a=\mathbb{A}^1\to U_\alpha$ such that $\theta_\alpha(\alpha(t)\,x)=t\,\theta_\alpha(x)\,t^{-1}$ [5, Sect. 2.3]. Therefore $\rho(z,t\theta_\alpha(x))=(z,t\theta_\alpha(z^{(v,\alpha)}\cdot x))$. Since $(v,\alpha)>0$ for $\alpha\in\Phi^+$ this shows that ρ can be extended to $\mathbb{A}^1\times U_\alpha$ as a morphism by setting $\rho(0,\theta_\alpha(x))=(0,e_\alpha)$, e_α the unit element of U_α. Since U is the product of the U_α's ρ extends to $\mathbb{A}^1\times B$ with the required properties.

6.10. Let $\mathbf{E}\to S\times\mathbb{P}^1$ be a family of G-bundles. By [10, exposé 221], we have an S-scheme $\mathbf{B}(\mathbf{E}|S,\lambda)\to S$ whose fibre over $s\in S$ is the space of B-reductions of $\mathbf{E}_s$ of type λ, with the natural universal property (cf. Sect. 5.2.1).

6.11. **Lemma.** *Let $\mathbf{E}\to S\times\mathbb{P}^1$ be a family of G-bundles. Let μ be a* 1-PS *such that $(\mu,\alpha)\leqq 1$ for $\alpha\in\Phi^+$. Then the morphism $\mathbf{B}(\mathbf{E}|S,\mu)\to S$ is smooth.*

Proof. We think of a $\sigma\in\mathbf{B}(\mathbf{E}|S,\mu)_s$ as a section of $\mathbf{E}_s(G/B)\to\mathbb{P}^1$. The infinitesimal criterion for the smoothness of the space of sections at σ is that $H^1(\mathbb{P}^1,\sigma^*N)=0$ where N is the normal bundle of $\sigma(\mathbb{P}^1)$ in $\mathbf{E}_s(G/B)$ (cf. [10, exposé 221]; also [11]). It is easy to see that $\sigma^*N=\sigma^*\mathbf{E}_s(\mathbf{g}/\mathbf{b})$. But $\sigma^*\mathbf{E}_s(\mathbf{g}/\mathbf{b})$ has a filtration (given by the negative root spaces) whose associated graded is the direct sum of line bundles $\sum_{\beta\in\Phi^-} T_\mu(\mathbf{u}_\beta)$ (cf. [11]). Since $(\mu,\beta)\geqq -1$ for $\beta\in\Phi^-$, $H^1(\mathbb{P}^1,T_\mu(\mathbf{u}_\beta))=0$. Hence the lemma.

6.12. **Corollary.** *If $E_\mu\rightsquigarrow E_\lambda$ and E_λ has a B-reduction of type w_0v, $v\in X_*(T)_+$, then so does E_μ. More generally if $E_{\mu_1}\rightsquigarrow E_{\mu_2}\rightsquigarrow\ldots\rightsquigarrow E_{\mu_n}$ and E_{μ_n} has a B-reduction of type w_0v so does E_{μ_1}.*

Proof. The second assertion follows from the first by induction. To prove the first let $\mathbf{E}\to S\times\mathbb{P}^1$ be a family giving the degeneration $E_\mu\rightsquigarrow E_\lambda$ with $\mathbf{E}_{s_0}\simeq E_\lambda$. By Lemma 6.11, $\mathbf{B}(\mathbf{E}|S,w_0(v))\to S$ is smooth and by assumption its image contains s_0. Since a smooth map is an open map its image contains a neighbourhood of s_0. It follows that $B(E_\mu,w_0(v))$, which is a neighbouring fibre, is nonempty.

6.13. **Proposition.** *Let* $\lambda, \mu \in X_*(T)_+$. *Then* $E_\mu \rightsquigarrow E_\lambda \Leftrightarrow E_\mu$ *has a B-reduction of type* $w(\lambda)$ *for some* $w \in W$.

Proof. $\Rightarrow$ follows from Corollary 6.12 above (since E_λ has a reduction of type $w_0\lambda$, cf. Sect. 4.1.2). $\Leftarrow$ follows from Proposition 6.9.

6.14. **Corollary.** *If* $E_{\mu_1} \rightsquigarrow E_{\mu_2}$ *and* $E_{\mu_2} \rightsquigarrow E_{\mu_3}$ *then* $E_{\mu_1} \rightsquigarrow E_{\mu_3}$.

Proof. Follows immediately from Proposition 6.13 and Corollary 6.12.

6.15. *Remark.* The above corollary can also be deduced from Sect. 8.6 and 8.7.

6.16. **Proposition.** *Let* λ *be a dominant* 1-PS. *If* $\mu \in X_*(T)$ *is the T-type of a B-reduction of E then* $\mu \leqq \lambda$.

Proof. Let μ be the T-type of a B-reduction of E_λ. Then we first show that $(\mu, \omega) \leqq (\lambda, \omega)$ for every dominant integral ω (Sect. 2.5).

Let $G \to GL(V)$ be the irreducible representation with highest weight ω (or $n\omega$, $n \in \mathbb{Z}^+$). Let $V = \Sigma V_l$ be the direct sum decomposition of V into 1-dimensional spaces V_l on which T acts by the character $l \in X^*(T)$. Then $E_\lambda(V) = \Sigma T_\lambda(V_l)$ and $T_\lambda(V_l)$ is a line bundle of degree $= (\lambda, l)$ (Sect. 3.2). Since ω is the highest weight any other weight l is of the form $\omega - \sum_{\alpha \in \Phi^+} n_\alpha \alpha^v$, $n_\alpha \geqq 0$. Since λ is dominant, $\deg V_l \leqq (\lambda, \omega)$. Therefore $E_\lambda(V)$ is a direct sum of line bundles each of which has degree $\leqq (\lambda, \omega)$.

Let F be a B-bundle giving a B-reduction of E_λ of type μ. Since the highest weight space V_ω of V is B-invariant $F(V_\omega)$ is a line subbundle of $F(V) = E_\lambda(V)$. Now $\deg F(V_\omega) = (\mu, \omega)$ and since $F(V_\omega)$ admits a nonzero homomorphism into at least one of $T_\lambda(V_l)$ we must have $(\mu, \omega) \leqq (\lambda, \omega)$.

From Proposition 6.9 it follows that $E_\lambda \rightsquigarrow E_\mu$. Let $\mathbf{E} \to S \times X$ be a family of G-bundles giving the degeneration $E_\lambda \rightsquigarrow E_\mu$. Then if $\chi \in X^*(G)$, extending structure group by χ: $G \to G_m$, the family of line bundles $\chi_* \mathbf{E}$ has constant degree over S and hence $(\lambda, \chi) = (\mu, \chi)$.

It only remains to prove that $\lambda - \mu$ is an *integral* combination of the simple coroots α_i^v. For this we can assume without loss of generality that G is semisimple and that the base field is algebraically closed. Let $\tilde{G} \to G$ be the simply connected covering group of G [8, exposé 23]. Let $\tilde{T}, \tilde{B}$ be respectively the maximal torus and the Borel subgroup of $\tilde{G}$ which are the inverse images of T, B respectively. We have the commutative diagram of exact sequences:

$$(*) \qquad \begin{array}{ccccccccc}
1 & \longrightarrow & Z & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \| & & \uparrow \tilde{j} & & \uparrow j & & \\
1 & \longrightarrow & Z & \longrightarrow & \tilde{B} & \longrightarrow & B & \longrightarrow & 1 \\
 & & \| & & \tilde{i} \uparrow \downarrow \tilde{p} & & i \uparrow \downarrow p & & \\
1 & \longrightarrow & Z & \longrightarrow & \tilde{T} & \longrightarrow & T & \longrightarrow & 1
\end{array}$$

where Z is a finite commutative group scheme (not necessarily reduced when $\operatorname{char} k \neq 0$). The above exact sequences give rise to exact sequences of sheaves in the flat topology (cf. [19, Chap. III] and [13]). We have the following cohomology diagram [19, 13]:

$$
(**)\qquad
\begin{array}{ccccc}
H^1(\mathbb{P}^1,\tilde{G}) & \longrightarrow & H^1(\mathbb{P}^1,G) & \xrightarrow{\delta_1} & H^2(\mathbb{P}^1,Z)\\
\uparrow{\scriptstyle \tilde{j}_*} & & \uparrow{\scriptstyle j_*} & & \|\\
H^1(\mathbb{P}^1,\tilde{B}) & \longrightarrow & H^1(\mathbb{P}^1,B) & \xrightarrow{\delta_2} & H^2(\mathbb{P}^1,Z)\\
{\scriptstyle \tilde{i}_*}\uparrow\ \downarrow{\scriptstyle \tilde{p}_*} & & {\scriptstyle i_*}\uparrow\ \downarrow{\scriptstyle p_*} & & \|\\
H^1(\mathbb{P}^1,\tilde{T}) & \longrightarrow & H^1(\mathbb{P}^1,T) & \xrightarrow{\delta_3} & H^2(\mathbb{P}^1,Z)
\end{array}
$$

We have $H^1(\mathbb{P}^1,\tilde{T})=$ isomorphism classes of $\tilde{T}$-bundles on $\mathbb{P}^1=X_*(\tilde{T})$ by Lemma 3.3 (cf. [19, Chap. III, Propn. 4.9]). Similarly $H^1(\mathbb{P}^1,T)=X_*(T)$ and the map $H^1(\mathbb{P}^1,\tilde{T})\to H^1(\mathbb{P}^1,T)$ is the natural map $X_*(\tilde{T})\to X_*(T)$.

By the commutativity of (**) we have for $F\in H^1(\mathbb{P}^1,B)$,

$$\delta_3 p_* F=\delta_1 j_* F \tag{1}$$

Since F is of type μ, $p_*F=\mu\in X_*(T)$ so that the left hand side of (1) is $\delta_3(\mu)$. Since $j_*F=j_*i_*\lambda$ (both being E_λ) we have $\delta_1 j_*F=\delta_1 j_* i_*\lambda$. But $\delta_1 j_* i_*=\delta_3$. Therefore the right hand side of (1) is $\delta_3(\lambda)$. We thus have proved $\delta_3(\mu)=\delta_3(\lambda)$ which implies that $\delta_3(\lambda-\mu)\in X_*(\tilde{T})$. Since $X_*(\tilde{T})$ is precisely the lattice generated by $\alpha_i^\vee$ we are done.

6.17. **Corollary.** *Let λ, μ be* 1-PS. *Then $E_\lambda\approx E_\mu$ if and only if $\mu=w(\lambda)$ for some $w\in W$.*

Proof. We can assume, without loss of generality, that λ and μ are dominant. E_λ has the canonical B-reduction of type λ and E_μ has one of type μ (Sect. 4.1.2). If $E_\lambda\approx E_\mu$ the above proposition gives $\lambda\leqq\mu$ and $\mu\leqq\lambda$ so that $\lambda=\mu$.

6.18. *Remark.* A G-bundle is *semistable* (resp. *stable*) if and only if for any B-reduction the T-type μ satisfies $(\mu,\omega)\leqq 0$ (resp. <0) for all the dominant weights ω (see [23, 24]). From Proposition 6.16 it follows easily that there are no stable bundles on $\mathbb{P}^1$ and that E_λ is semistable $\Leftrightarrow\lambda$ is in the centre of G. Further by Proposition 5.2 E_λ is semistable $\Leftrightarrow\operatorname{Aut}E_\lambda$ is reductive $\Leftrightarrow\operatorname{Aut}E_\lambda=G$. From Proposition 7.3 below we get that E_λ semistable $\Rightarrow E_\lambda$ rigid. When λ is noncentral and dominant the Harder-Narasimhan flag of the unstable E_λ is the reduction to $P(\lambda)$ obtained from the canonical T-reduction T_λ by the extension of structure group $T\to P(\lambda)$ (cf. [24]).

7. Deformations of E_λ

In this section, given a dominant 1-PS λ, we determine the set of 1-PS μ such that $E_\mu\rightsquigarrow E_\lambda$ (Theorem 7.4). We also classify algebraic equivalence classes (Theorem 7.7).

7.1. **Proposition.** *Let A be a discrete valuation ring with residue field k and quotient field K. Let $\mathbf{E}\to\mathbb{P}^1_A$ be a family of G-bundles parametrized by* spec A

such that $\mathbf{E}_K \approx E_\mu$ *and* $\mathbf{E}_k \approx E_\mu$ *with* λ *a dominant* 1-PS *and* μ *an arbitrary* 1-PS, *then* $\mu \leqq \lambda$.

Proof. Let σ be the canonical B-reduction of E_μ (§4.1.2). We consider σ as a section of $\mathbf{E}_K(G/B) \to \mathbb{P}^1_K$. Since $\mathbb{P}^1_A$ is nonsingular and G/B is complete it follows from the valuative criterion that we can extend σ to a section $\tilde{\sigma}$ over an open subset U of $\mathbb{P}^1_A$ with codimension of $\mathbb{P}^1_A - U$ in $\mathbb{P}^1_A \geqq 2$, i.e. $\mathbb{P}^1_A - U$ is only a finite set of closed points of $\mathbb{P}^1_k$. Restricting $\tilde{\sigma}$ to $\mathbb{P}^1_K \cap U$ we get a section of $\mathbf{E}_k(G/B) \to \mathbb{P}^1_k$ over the nonempty open set $\mathbb{P}^1_k \cap U$. Again by the properness criterion we can extend $\tilde{\sigma} | \mathbb{P}^1_k \cap U$ to a section σ_0 over the whole of $\mathbb{P}^1_k$. Thus the reduction σ of $\mathbf{E}_K$ gives in the limit a reduction σ_0 of $\mathbf{E}_k$.

We claim that $\mu \leqq \lambda'$, where λ' is the T-type of σ_0. To prove this let ω be a dominant integral weight of G and $G \to GL(V)$ be the irreducible representation with ω as the highest weight. Let V_ω be the 1-dimensional weight space of V of weight ω. Since V_ω is B-invariant the reduction $\tilde{\sigma}$ gives the line subbundle $L = \tilde{\sigma}^*(\mathbf{E}|U)(V_\omega)$ of $\mathbf{E}(V)$ over U. Since the codimension of U in $\mathbb{P}^1_A$ is $\geqq 2$ L extends uniquely to a line bundle $\bar{L}$ on the whole of $\mathbb{P}^1_A$. In fact the sections of $\bar{L}$ over an open subset $S \subset \mathbb{P}^1_A$ are by definition the same as the sections of L over $S \cap U$. Therefore the map $L \to \mathbf{E}(V)|U$ over U extends naturally to $\bar{L} \to \mathbf{E}(V)$ over $\mathbb{P}^1_A$ as a sheaf map, though $\bar{L}$ may not be a subbundle of $\mathbf{E}(V)$. On the other hand $L|\mathbb{P}^1_k \cap U$, a line subbundle of $\mathbf{E}_k(V)$ over $\mathbb{P}^1_k \cap U$, extends as a subbundle L_0 of $\mathbf{E}_k(V)$ on the whole of $\mathbb{P}^1_k$ (by properness criterion by considering the subbundle as a section of a Grassmann bundle). Clearly L_0 is also the associated bundle $\sigma_0^*(\mathbf{E}_k)(V_\omega)$.

Over $\mathbb{P}^1_k \cap U$ the sheaf map $\bar{L}|\mathbb{P}^1_k \to \mathbf{E}(V)|\mathbb{P}^1_k$ factors through $L_0|\mathbb{P}^1_k \cap U$. Therefore the composite $\bar{L}|\mathbb{P}^1_k \to \mathbf{E}_k(V) \to \mathbf{E}_k(V)/L_0$ is zero on $\mathbb{P}^1_k \cap U$ and hence on $\mathbb{P}^1_k$. Therefore we have a generic isomorphism $\bar{L}|\mathbb{P}^1_k \to L_0$. Therefore $\deg \bar{L}|\mathbb{P}^1_k \leqq \deg L_0$. But $\deg(\bar{L}|\mathbb{P}^1_k) = \deg(\bar{L}|\mathbb{P}^1_K) = (\mu, \omega)$ and $\deg L_0 = (\lambda', \omega)$ where λ' is the T-type of σ_0. Therefore $(\mu, \omega) \leqq (\lambda', \omega)$.

If ω_i is a fundamental weight then for some $s > 0$, $s\omega_i$ is the highest weight of a representation of G. Hence $(\lambda' - \mu, \omega_i) = \frac{1}{s}(\lambda' - \mu, s\omega_i) \geqq 0$. By Proposition 6.16 $\lambda' \leqq \lambda$. Therefore $(\lambda - \mu, \omega_i) \geqq 0$.

It remains to prove that $(\lambda - \mu, \omega_i)$ is an integer and $(\lambda - \mu, \chi) = 0$ for $\chi \in X^*(G)$ i.e. $\lambda - \mu \in Q^v$. Since E_λ has a B-reduction of type $w_0 \lambda$ so does E_μ, by Corollary 6.12. Therefore by Proposition 6.16 $w\mu - w_0 \lambda \in Q^v$, where $w \in W$ is such that $w\mu$ is dominant. But $\mu - w\mu \in Q^v$ (since $s_\alpha \mu - \mu = -(\mu, \alpha)\alpha^v$). Therefore $\mu - w_0 \lambda \in Q^v$. Since $\lambda - w_0 \lambda \in Q^v$ it follows that $\lambda - \mu \in Q^v$.

7.2. **Corollary.** *Let* $\mathbf{E} \to S \times \mathbb{P}^1$ *be a family of* G*-bundles and* μ *a dominant* 1-PS. *Then the set* $S_\mu = \{s \in S | \mathbf{E}_s$ *is of type* $E_\mu\}$ *is an open subset of its closure* $\bar{S}_\mu$ *in S. Further* $\bar{S}_\mu = \{s \in S | \mathbf{E}_s$ *is of type* $v \in X_*(T)_+$ *with* $\mu \leqq v\}$.

Proof. Follows from the above proposition and the valuation criterion.

7.3. **Corollary.** *If* λ *is a* 1-PS *of the centre* Z *of* G *then* E_λ *is rigid (cf. Sect.* 6.8*).*

Proof. Note that the characters of G and the fundamental weights of G together generate $X^*(T) \otimes \mathbb{Q}$. If χ is a character of G then we have the line bundle $\chi_* \mathbf{E} \to \mathbb{P}^1_A$ so that $\deg \chi_* \mathbf{E}_K = \deg \chi_* \mathbf{E}_k$ which gives $(\mu, \chi) = (\lambda, \chi)$. Further if ω is a fundamental weight $(\lambda, \omega) = 0$ since $\lambda \subset Z$. Therefore $\mu = \lambda$.

7.3.1. *Remark.* The above corollary also follows from deformation theory: since in this case λ acts trivially on $\mathbf{g}$, the infinitesimal deformation space $H^1(\mathbb{P}^1, E(\mathbf{g})) = 0$.

We will now prove the main result.

7.4. **Theorem.** *Let λ, μ be dominant* 1-PS. *Then $E_\mu \rightsquigarrow E_\lambda$ (Sects.* 6.6, 6.7*) if and only if $\mu \leqq \lambda$ (Sect.* 2.5*).*

Proof. We have already proved that $E_\mu \rightsquigarrow E_\lambda$ implies $\mu \leqq \lambda$ (Proposition 7.1). We have only to prove the converse. The idea of the proof is to get a sequence $\mu_1, \ldots, \mu_n$ of 1-PS starting from μ and going to λ such that μ_{i+1} is got from μ_i by a simple process (Lemma 7.4.1 below) and then to construct a degeneration $E_{\mu_i} \rightsquigarrow E_{\mu_{i+1}}$ by $SL(2)$-theory. Then the transitivity of $\rightsquigarrow$ (Corollary 6.14) shows $E_\mu \rightsquigarrow E_\lambda$.

7.4.1. **Lemma.** *If $\lambda, \mu \in X_*(T)_+$ and $\mu \leqq \lambda$ then we can find a sequence $\mu = \mu_1, \ldots, \mu_n = \lambda$ of elements of $X_*(T)$ (not necessarily dominant) such that $\mu_{i+1} = \mu_i + \alpha_i^v$ for some $\alpha_i \in \Delta$ with $(\mu_i, \alpha_i) \geqq 0$.*

Proof. This is well known. See [16, p. 70].

7.4.2. **Lemma.** *Let $\mu_1, \mu_2 \in X_*(T)$, not necessarily dominant, such that $\mu_2 = \mu_1 + \alpha^v$ with $\alpha \in \Delta$ and $(\mu_1, \alpha) \geqq 0$. Then E_{μ_1} has a reduction to a Borel subgroup of type $s_\alpha \mu_2$, where s_α is the reflection corresponding to α.*

Proof. Let P_α be the minimal parabolic subgroup corresponding to the simple root α. Then P_α is generated by B and $U_{-\alpha}$ and $P = M \cdot U'$ where U' is the unipotent radical and M is the reductive part generated by T and $U_{\pm\alpha}$. Let $Z_\alpha = (\ker \alpha)^0$, which is the connected component of the centre C of M, $M/C \approx PSL(2)$. Further the root α induces $\bar{\alpha} \in X^*(T/Z_\alpha)$ which is the simple root of the rank 1 group $M/Z_\alpha = \bar{M}$. The coroot $(\bar{\alpha})^v$ is $\overline{\alpha^v}$, the image of α^v under $X_*(T) \to X_*(T/Z_\alpha)$. The simple reflection s_α induces the simple reflection $\bar{s}_\alpha$ [5, 29].

The projection $P_\alpha \to M/Z_\alpha = \bar{M}$ induces an isomorphism $P_\alpha/B \to \bar{M}/\bar{B} (\approx \mathbb{P}^1)$ where $\bar{B}$ is the image of B. Therefore the B-reductions of the P_α-bundle $T_{\mu_1}(P_\alpha)$ are in bijective correspondence with $\bar{B}$-reductions of the $\bar{M}$-bundle $T_{\mu_1}(\bar{M})$. Let λ be the T-type of a B-reduction of $T_{\mu_1}(P_\alpha)$ and $\bar{\lambda}$ the $\bar{T}$-type of the corresponding $\bar{B}$-reduction of $T_{\mu_1}(\bar{M})$. Clearly for any $\chi \in X^*(T)$ which extends to P_α we have $(\lambda, \chi) = (\mu_1, \chi)$. Now $X^*(P_\alpha) \otimes \mathbb{Q} = X^*(Z_\alpha) \otimes \mathbb{Q}$ and $X^*(T) \otimes \mathbb{Q} = X^*(Z_\alpha) \otimes \mathbb{Q} \oplus X^*(\operatorname{Im} \alpha^v) \otimes \mathbb{Q}$ (cf. Sect. 2.3). Hence it follows that $\lambda = \mu_1 + a\alpha^v$. Further to determine a we have only to look at the $\bar{B}$-reduction which must be of the form $\bar{\lambda} = \bar{\mu}_1 + a \cdot \bar{\alpha}^v$.

Therefore it follows that the P_α-bundle $T_{\mu_1}(P_\alpha)$ (and hence the G-bundle E_{μ_1}) admits a B-reduction of type $s_\alpha \mu_2$ if and only if the $\bar{M}$-bundle $T_{\mu_1}(\bar{M})$ admits a $\bar{B}$-reduction of type $\bar{s}_\alpha \bar{\mu}_2$. Thus we are reduced to proving the lemma for $SL(2)$ or (Zariski locally trivial) $PSL(2)$-bundles. Since any Zariski locally trivial projective bundle comes from a vector bundle (cf. [27] and Sect. 9.4 below) it is easy to see that this is equivalent to proving the following for rank 2 vector bundles: The vector bundle $\mathcal{O}(m)\oplus\mathcal{O}(n)$ with $m-n\geqq 0$ has a line subbundle isomorphic to $\mathcal{O}(n-1)$. To prove this note that we can find sheaf morphisms $s_1: \mathcal{O}(n-1)\to\mathcal{O}(m)$ (which is a section of $\operatorname{Hom}(\mathcal{O}(n-1), \mathcal{O}(m))\approx\mathcal{O}(m-n+1)$) vanishing only at the fibre over $P\in\mathbb{P}^1$ and $s_2: \mathcal{O}(n-1)\to\mathcal{O}(n)$ vanishing only at the fibre over $Q\neq P$ (by Riemann-Roch). Then $s_1\oplus s_2: \mathcal{O}(n-1)\to\mathcal{O}(m)\oplus\mathcal{O}(n)$ makes $\mathcal{O}(n-1)$ a subbundle.

Now we have everything we need to complete the proof of Theorem 7.4. So suppose $\mu\leqq\lambda$. Choose $\mu=\mu_1,\ldots,\mu_n=\lambda$ as in Lemma 7.4.1. Then by Lemma 7.4.2. E_{μ_i} admits a B-reduction of type $w(\mu_{i+1})$ for some $w\in W$ and hence by Proposition 6.13, $E_{\mu_i}\rightsquigarrow E_{\mu_{i+1}}$ which implies $E_\mu\rightsquigarrow E_\lambda$ by transitivity (Corollary 6.14).

7.5. **Definition.** *We call the bundles E_λ, E_μ to be algebraically equivalent if there is a family of bundles $\mathbf{E}\to S\times\mathbb{P}^1$ parametrized by a connected variety S such that $\mathbf{E}_{s_1}\approx E_\lambda$ and $\mathbf{E}_{s_2}\approx E_\mu$ for some $s_1, s_s\in S$.*

7.6. *Remark.* Algebraic equivalence is symmetric in E_λ, E_μ and does not imply $E_\mu\rightsquigarrow E_\lambda$, for s_1 may have a neighbourhood in S where E_μ does not occur. Of course $E_\mu\rightsquigarrow E_\lambda$ implies E_μ and E_λ are algebraically equivalent.

Recall (§2.4) that Q^v is the subgroup of $X_*(T)$ generated by the coroots Φ^v.

7.7. **Theorem.** *Let λ,μ be* 1-PS. *Then E_μ is algebraically equivalent to E_λ if and only if μ and λ have the same image in $X_*(T)/Q^v$. Therefore $X_*(T)/Q^v$ (which is the "fundamental group" of G) classifies the algebraic equivalence classes of (Zariski locally trivial) G-bundles on $\mathbb{P}^1$.*

Proof. If s_α is the reflection corresponding to $\alpha\in\Delta$, $s_\alpha(\lambda)-\lambda=-(\lambda,\alpha)\cdot\alpha^v$. Therefore W operates trivially on $X_*(T)/Q^v$ and we can assume that both λ and μ are dominant.

Suppose E_μ is algebraically equivalent to E_λ. Let $\mathbf{E}\to S\times\mathbb{P}^1$ be a family in which both E_λ and E_μ occur. Assume S to be irreducible. Let $\nu\in X_*(T)_+$ be the generic type of this family. Then by Proposition 7.1 $\nu\leqq\lambda$ and $\nu\leqq\mu$. Therefore λ,μ and ν have the same image in $X_*(T)/Q^v$. If S is not irreducible argue as above with irreducible components and use the connectedness of S.

To prove the converse we need the following lemma. We call $\nu\in X_*(T)_+$ *minimal* if $\nu'\in X_*(T)_+$ and $\nu'\leqq\nu$ then $\nu'=\nu$. (See [8, exposé 20, Sect. 2].)

7.7.1. **Lemma.** *Each coset of $X_*(T)/Q^v$ contains a unique minimal element. Further if ν is the minimal element in a coset and ν' is any dominant* 1-PS *in the same coset then ν and ν' are comparable: $\nu\leqq\nu'$.*

Proof. See [16, Sect. 13.2, Lemma B and Exercise 13, p. 72]. (There the root system is assumed to be semisimple. It is easy to see that the same works for reductive systems as well.) See also [8, exposé 20].

Suppose λ and μ belong to the same coset and let λ_0 be the minimal element of that coset. Then by the above lemma $\lambda_0\leqq\lambda$ and $\lambda_0\leqq\mu$. Therefore by Theorem 7.4 (and Proposition 6.13 and the proof of Proposition 6.9) there is a family $\mathbf{E}\to\mathbb{A}^1\times\mathbb{P}^1$ parametrized by the affine line such that $\mathbf{E}_0\approx E_\lambda$ and $\mathbf{E}|(\mathbb{A}^1-0)\approx E_{\lambda_0}\times(\mathbb{A}^1-0)$. Similarly there is a family $\mathbf{E}'$ with $\mathbf{E}'_0\approx E_\mu$ and

$\mathbf{E}'|(\mathbb{A}^1-0)\approx E_{\lambda_0}\times(\mathbb{A}^1-0)$. So by patching $\mathbf{E}$ and $\mathbf{E}'$ along $\mathbb{A}^1-0$ we get a family $\mathbf{E}''\to\mathbb{P}^1\times\mathbb{P}^1$ parametrized by $\mathbb{P}^1$ such that $\mathbf{E}''_0\approx E_\lambda$ and $\mathbf{E}''_\infty\approx E_\mu$.

7.7.2. *Remark.* Note that we have actually connected E_λ and E_μ by an irreducible parameter variety, viz., $\mathbb{P}^1$.

7.8. **Proposition.** *Let λ be a dominant* 1-PS. *Then E_λ is rigid if and only if λ is minimal. In each algebraic equivalence class of G-bundles there is precisely one rigid bundle.*

Proof. Follows immediately from Theorem 7.4 and Lemma 7.7.1.

7.8.1. *Remark.* It follows from [16, Exercise 13, p. 72] that λ is minimal $\Leftrightarrow$ $(\lambda,\alpha)=0,1,-1$ for all $\alpha\in\Phi$. Therefore E_λ is rigid $\Leftrightarrow$ λ is minimal $\Leftrightarrow$ $H^1(\mathbb{P}^1,E_\lambda(\mathbf{g}))=0$ since $H^1(\mathbb{P}^1,\mathcal{O}(n))=0$ if and only if $n\geqq -1$.

7.9. *Remark.* For $GL(n)$, $X_*(T)/Q^v=\mathbb{Z}$ (Sect. 2.7). The algebraic equivalence class of a vector bundle of rank n is determined by its degree.

7.10. *Remark.* Let X be a smooth projective curve of arbitrary genus. Let $E\to X$ be a Zariski locally trivial G-bundle. Then E admits some B-reduction (cf. proof of Proposition 4.3). Let its type be λ. Then one can show, by similar methods as for $\mathbb{P}^1$, that the image of λ in $X_*(T)/Q^v$ is independent of the chosen B-reduction of E and characterizes the algebraic equivalence class of E. When $k=\mathbb{C}$ the same result holds for topological equivalence (see [23, Sect. 5]).

8. Versal Deformation Space

We will now construct a versal deformation space for $E=E_\lambda$, $\lambda\in X_*(T)_+$ (See Remark 8.11 below for G-bundles on curves of higher genus.)

Let G act on the Lie algebra $\mathbf{g}$ by adjoint representation and $E(\mathbf{g})$ the associated vector bundle (adjoint bundle).

Suppose $k=\mathbb{C}$. Since $H^2(\mathbb{P}^1,E(\mathbf{g}))=0$ we have a family of bundles parametrized by an (analytic) neighbourhood of 0 in $H^1(\mathbb{P}^1,E(\mathbf{g}))$ which is complete (or versal) in the sense that any deformation of E is induced locally from this [17, 21]. We indicate below how the results of Artin [1, 2] imply the analogous results for an arbitrary field k.

8.1. By [26] it is quite easy to see that there exists a *formal versal deformation*, that is, there is a complete local ring A, with maximal ideal m and residue field k, and a compatible family of deformations of E over $\mathbb{P}^1_{A/m^n}$ such that any compatible family of deformations over a complete local ring is induced from this. In fact there exists a *hull (local moduli)* i.e. a formal versal deformation whose infinitesimal deformation map is an isomorphism. See [26] and [28, Theorem 2.2, Remark 2.3 and Theorem 2.3].

8.2. This is a rather weak result and we will need that there is an (algebraic) *versal deformation* [2, Sect. 3] i.e. a deformation $\mathbf{U}\to S\times\mathbb{P}^1$ parametrised by an algebraic scheme S over k such that if $\mathbf{E}\to R\times\mathbb{P}^1$ is any deformation, with $\mathbf{E}_{r_0}$

$=E, r_0 \in R$, R/k, then there is an étale neighbourhood of r_0, i.e. an étale morphism $\varphi\colon R' \to R$ over k with $r_0 \in \varphi(R)$, such that $\varphi^* \mathbf{E}$ is induced from $\mathbf{U}$ by a morphism $R' \to S$. The existence of such a versal deformation follows from the existence of an effective versal deformation, by the algebraization theorems of Artin. However in this case as we shall show, it is easy to directly construct an algebraic family which is a formal versal deformation and check Artin's conditions for versality for the deformation functor D_E.

8.3. *Construction.* There are natural trivialisations of $k^2 - 0 \to \mathbb{P}^1$ on $A_0 = \mathbb{P}^1 - \infty$ and $A_\infty = \mathbb{P}^1 - 0$ and $k^2 - 0 \to \mathbb{P}^1$ is given by the transition function id: $A_0 \cap A_\infty = G_m \to G_m$. Using this trivialisation, for any line bundle L we can represent an element of $H^1(\mathbb{P}^1, L)$, as a Cech cocycle, by a function on $A_0 \cap A_\infty$. By choosing representative cocycles for a basis of $S_\alpha = H^1(\mathbb{P}^1, T_\lambda(U_\alpha))$ we identify it with a finite dimensional vector space of functions from $A_0 \cap A_\infty$ to $U_\alpha (\approx G_a)$. Let $S = \prod_{\beta \in \Phi^-} S_\beta$. We define $\mathbf{U} \to S \times \mathbb{P}^1$ to be the G-bundle obtained by patching up the trivial G-bundles on $S \times A_0$ and $S \times A_\infty$ by the transition function

$$\varphi_{0\infty}\colon S \times G_m \to B^- \subset G$$

defined by $\varphi_{0\infty}(s_{\beta_1}, \ldots, s_{\beta_N}, z) = \lambda(z) \cdot \varphi(s_{\beta_1}(z), \ldots, s_{\beta_N}(z))$ where $\{\beta_1, \ldots, \beta_N\} = \Phi^-$, $\varphi\colon \Pi U_\alpha \to U^-$ is the group multiplication (cf. 2.8) and B^- is the Borel opposite to B and U^- its unipotent radical. Note that at $0 \in S$, $\varphi_{0\infty}(0, z) = \lambda(z)$ so that at the base point 0 we have E_λ.

8.4. **Proposition.** *$\mathbf{U} \to S \times \mathbb{P}^1$, constructed as above, is an (algebraic) versal deformation (cf. 8.2) of E_λ whose infinitesimal deformation map is an isomorphism.*

Proof. First we check that this is a formal versal deformation (cf. 8.1). The functor D_E (Sect. 6.3) satisfies the conditions H_1, H_2, H_3 of [26]. Hence a formal versal deformation exists. The tangent space to D_E (the infinitesimal deformation space) is $H^1(\mathbb{P}^1, E_\lambda(\mathbf{g}))$ and the versal deformation is formally smooth since the obstruction space $H^2(\mathbb{P}^1, E_\lambda(\mathbf{g})) = 0$.

In fact any deformation parametrized by a smooth variety with the infinitesimal deformation map an isomorphism gives a formal versal deformation which is a hull. Thus we have only to check that the infinitesimal deformation map $T_0(S) \to H^1(\mathbb{P}^1, E_\lambda(\mathbf{g}))$, where $T_0(S)$ is the tangent space to S at 0, is an isomorphism. Since λ is dominant $(\lambda, \alpha) \geqq 0$ for $\alpha \in \Phi^+$. Therefore $H^1(\mathbb{P}^1, T_\lambda(U_\alpha)) = 0$ for $\alpha \in \Phi^+$. Therefore $H^1(\mathbb{P}^1, E_\lambda(\mathbf{g})) = \sum_{\beta \in \Phi^-} H^1(\mathbb{P}^1, T_\lambda(U_\beta))$. Since the differential of the map φ is an isomorphism, in fact identity if we make the obvious identifications, it follows that

$$\frac{\partial}{\partial t} \{\varphi_{0\infty}(ts, z)\}_{t=0} = s$$

which shows that the infinitesimal deformation map is an isomorphism. Thus we have proved that $\mathbf{U}$ is a formal versal deformation.

To prove $\mathbf{U}$ is actually versal, by Artin [2, Theorem 3.7], we have only to check that the map $D_E(\hat{A}) \to \varprojlim D_E(A/m^n)$, is injective, where A is the local ring of an algebraic scheme with residue field k. If E is a vector bundle this is guarantied by the existence theorems of Grothendieck [EGA III 5]. We can reduce the general case to that case as follows. The idea is to look upon a G-bundle as a vector bundle with additional structure, analogous to considering on $O(n)$-bundle as a vector bundle with quadratic forms on the fibres.

Let $\varphi: G \to GL(V)$ be a faithful representation. By a theorem of Chevalley [4] there is an element $l \in \mathbb{P}(W)$, where $W = V^r \otimes V^{*s}$ is a suitable tensor space, whose isotropy for the action of $GL(V)$ on W is precisely G. Let $C = GL(V)/G$ be the orbit of l under $GL(V)$. Then given a G-bundle E we have a section of $\varphi_*(E)(C)$. Conversely if we have a $GL(V)$-bundle F (equivalently, a vector bundle) and a section of $F(W)$ with values in $F(C)$ we have a reduction of structure group of F to G and hence a G-bundle. Thus we can view a G-bundle as a vector bundle together with a section of an associated vector bundle. Looked at this way the fact that $D_E(\hat{A}) \to \varprojlim D_E(A/m^n)$ is injective follows from the existence theorems of [EGA III 5].

We now note some properties of the versal family $\mathbf{U} \to S \times \mathbb{P}^1$.

8.5. *Zariski local triviality.* Since by construction $\mathbf{U}_s$ is Zariski locally trivial for $s \in S$, by the versality of $\mathbf{U}$ it follows that any small deformation of E_λ is Zariski locally trivial over the base field k itself.

8.6. *Versality in a neighbourhood.* Since we are in the "unobstructed" case it is easy to see that the conditions of [2, Sect. 4.1] are satisfied by D_E. Therefore $\mathbf{U} \to S \times \mathbb{P}^1$ which is versal at $0 \in S$ remains so in a neighbourhood of 0 in S.

8.7. *Homogeneity.* The maximal torus T acts on the associated line bundle $T_\lambda(U_\beta)$ and on $H^1(\mathbb{P}^1, T_\lambda(U_\beta))$ by the character β. Thus we can make T operate on S. For $t \in T$,

$$\varphi_{0\,\infty}(t \cdot s, z) = \lambda(z) \cdot \varphi(t \cdot s_{\beta_1}(z), \ldots) = t \cdot \lambda(z) \cdot \varphi(s_{\beta_1}(z), \ldots)\, t^{-1}$$

(since φ is T-equivariant). Therefore we have a lift of the action of T on S to $\mathbf{U}$ and $\mathbf{U}_s \simeq \mathbf{U}_{ts}$ for $s \in S$. For a $\mu \in X_*(T)$ with $(\mu, \beta) > 0$, for every $\beta \in \Phi^-$, the μ-orbit of $s \in S$ is a "ray" in S tending to 0. Thus the family S is "homogeneous" and is determined by any "small" neighbourhood of 0.

8.8. **Lemma.** *Let $s \in S$. Then $\mathbf{U}_s \approx E_\lambda$ if and only if $s = 0$.*

Proof. Suppose $\mathbf{U}_s \approx E_\lambda$. Let C be the T-orbit of s and $\bar{C}$ its closure. Consider the restriction $\mathbf{U} | \bar{C} \times \mathbb{P}^1$. Since for any $x \in \bar{C}$, $\mathbf{U}_x \approx E_\lambda$ (Corollary 7.2) it is easy to see that there is an étale neighbourhood $f: C' \to \bar{C}$ of $0 \in \bar{C}$ such that $f^*\mathbf{U}$ becomes a trivial family. Hence any tangent to $\bar{C}$ at 0 will go to zero under the infinitesimal deformation map. But we know that the infinitesimal deformation map is an isomorphism. Therefore $s = 0$.

8.9. **Proposition.** *Let λ, μ be dominant* 1-PS *such that $\mu \leqq \lambda$. Let $S_{\lambda\mu} = \{s \in S \mid \mathbf{U}_s \approx E_\mu\}$. Then $S_{\lambda\mu}$ is a locally closed smooth subvariety of S and*

$$\begin{aligned}\dim S_{\lambda\mu} &= \sum_{\beta \in \Phi^-} \{\dim H^1(\mathbb{P}^1, T_\lambda(U_\beta)) - \dim H^1(\mathbb{P}^1, T_\mu(U_\beta))\} \\ &= \dim(\operatorname{Aut} E_\lambda) - \dim(\operatorname{Aut} E_\mu).\end{aligned}$$

Proof. $S_{\lambda\mu}$ is open in its closure by Corollary 7.2. Because of homogeneity (Sect. 8.7) and openness of versality (Sect. 8.6) we see that for $s \in S_{\lambda\mu}$, $\mathbf{U}$ is versal at s. Therefore the morphism inducing $\mathbf{U}$, in a neighbourhood of s, from the versal family $\mathbf{U}_\mu \to S_\mu \times \mathbb{P}^1$ for E_μ is smooth at s (since the tangent map of the inducing map, being the infinitesimal deformation map must be surjective). By Lemma 8.8, in an étale neighbourhood of s, $S_{\lambda\mu}$ is the fibre of such an inducing map $S \to S_\mu$, over $0 \in S_\mu$. Hence $S_{\lambda\mu}$ is smooth. To get the $\dim S_{\lambda\mu}$ we have only to note that

$$\dim S = \sum_{\beta \in \Phi^-} \dim H^1(\mathbb{P}^1, T_\lambda(U_\beta))$$

and

$$\dim S_\mu = \sum_{\beta \in \Phi^-} \dim H^1(\mathbb{P}^1, T_\mu(U_\beta)).$$

8.9.1. *Remark.* One can show that $S_{\lambda\mu} \times \operatorname{Aut} E_\mu$ is locally isomorphic to $B(E_\mu, w_0\lambda)$. The smoothness of $S_{\lambda\mu}$ can be deduced from this.

8.10. *Remark.* The varieties $S_{\lambda\mu}$ are irreducible. For a given μ, $S_{\lambda\mu}$ are irreducible when λ is sufficiently large (i.e. $(\lambda, \alpha) \gg 0, \forall \alpha \in \Phi^+$) by a result of Harder [12]. For rank 2 vector bundles the varieties $S_{\lambda\mu}$ can easily be seen to be irreducible. In fact $\bar{S}_{\lambda\mu}$ are defined, in this case, by rank conditions on matrices and are determinantal varieties. See Remark 8.13, in general.

8.11. *Remark.* Let X be a smooth projective curve of arbitrary genus. Then any G-bundle $E \to X$ has an (algebraic) versal deformation (Sect. 8.2). We can see this as follows. First the conditions H_1, H_2 and H_3 of [26] are easily verified for D_E. Hence a formal versal hull exists (Sect. 8.1).

To see the existence of a versal deformation we first deal with the vector bundle case. Let $V \to X$ be a vector bundle. Let L be an ample line bundle on X such that $H^1(X, V \otimes L) = 0$ and $H^0(X, V \otimes L)$ generates $V \otimes L$. Clearly it is enough to construct a versal family for $V \otimes L$. So we assume the above conditions for V. Let $\dim H^0(X, V) = n$. Let I be the trivial bundle of rank n. Let Q be the Quot scheme of quotients of I whose rank and degree are those of V. From the exact sequence $0 \to K \to I \to V \to 0$ applying $\operatorname{Hom}(-, V)$ and taking the cohomology sequence we get a map $H^0(X, K^* \otimes V) \to H^1(X, V^* \otimes V) \to 0$ and $H^1(X, K^* \otimes V) = 0$. Now $H^0(X, K^* \otimes V)$ is the tangent space to Q at $I \to V \to 0$, and the above map is the infinitesimal deformation map for the universal quotient family $\mathbf{V} \to Q \times X$ (thought of as a deformation of V). Further $H^1(X, K^* \otimes V) = 0$ implies Q is smooth at V. Hence $\mathbf{V} \to Q \times X$ is a formal versal family for V. But it is not a hull since it may have a higher dimension. To get a hull pull up $\mathbf{V}$ to a smooth subvariety in an étale neighbourhood of $V \in Q$ which has tangent space at V supplementary to the kernel of the infinitesimal deformation map. This family is versal by [2], the verification of condition (ii) of [2, Theorem 3.7] being same as in the proof of Proposition 8.4.

To deal with G-bundles we again have only to view a G-bundle as a vector bundle with additional structure. Take a faithful representation $G \hookrightarrow GL(V)$, form the universal quotient family $\mathbf{V} \to Q \times X$ as above for $E(V)$. Form the associated bundle $\mathbf{V}(GL(V)/G)\to Q\times X$ and let $\Sigma\to Q\times X$ be the space of sections [10, exposé 221]. One can check that Σ is smooth at E [25]. Then one proceeds as in the case of vector bundles to take a subvariety in an étale neighbourhood of $\mathbf{E}$ to get a versal deformation hull for E.

8.12. *Remark.* For constructing versal deformations of bundles on $\mathbb{P}^1$ (and on curves of higher genus as well) we can avoid using Artin's general results. For, any B-bundle on $S\times\mathbb{A}^1$ becomes a T-bundle after pulling up by an étale $S'\to S$ and thus we would get transition functions for any family of G-bundles $\mathbf{E}\to S\times\mathbb{P}^1$ for the covering $S'\times A_0$, $S'\times A_\infty$ and this transition function could be induced from $\varphi_{0\,\infty}$.

8.13. *Remark.* Our results can be interpreted in terms of the generalised Schubert varieties introduced by Kazhdan-Lusztig in [31]. This will be done elsewhere.

9. Non-connected Groups and Classical Groups

9.1. *Bundles with non-connected structure group.* In this section G is a not necessarily connected group with its identity component G^0 a reductive split group. Maximal torus, 1-PS, ... etc. of G are the same as those of G^0. However the Weyl group W of G is $N_G(T)/T$ where $N_G(T)$ is the normaliser of T in G. The Weyl group $W^0=N_{G^0}(T)/T$ is a subgroup of W of finite index.

9.2. **Theorem.** *Let $E\to\mathbb{P}^1$ be a Zariski locally trivial G-bundle (G not necessarily connected). Then*

i) *E admits a reduction of structure group to G^0, the identity component of G.*

ii) *$E\approx E_\lambda$ for a* 1-PS.

iii) *The G-bundles E_λ and E_μ are isomorphic if and only if $\mu=w(\lambda)$ for some $w\in W$.*

iv) *Suppose λ, μ are dominant* 1-PS. *Then $E_\mu\rightsquigarrow E_\lambda$ if and only if for some $w\in W$ $w(\mu)$ is dominant and $w(\mu)\leqq\lambda$.*

v) *The versal deformation for $\mathbf{E}$ is the same as that for a G^0-reduction.*

vi) *The Weyl group W of G acts naturally on $X_*(T)/Q^v$ (with W^0 acting trivially). The algebraic equivalence classes of Zariski locally trivial G-bundles on $\mathbb{P}^1$ are in bijective correspondence with the elements of $(X_*(T)/Q^v)/W$.*

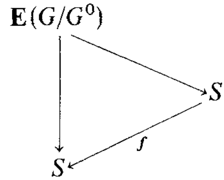
Proof. i) See Remark 4.4.

ii) Use i) and the connected structure group case (Theorem of Grothendieck-Harder, Sect. 4.2).

iii) From the exact (nonabelian) cohomology sequence (in the étale topology) corresponding to $1\to G^0\to G\to G/G^0\to 1$ (cf. [19, Chap. III, Sect. 4]) we see that $E_\lambda\approx E_\mu$ implies that λ and μ differ by an action of G/G^0 (by inner conjugation). Since representatives in G for the cosets of G/G^0 can be chosen in $N_G(T)$ it follows that $\mu=w(\lambda)$ for some $w\in W$ (cf. [9, p. 136]).

iv) Use iii) and the connected structure group case (Theorem 7.4).

v) Let $\mathbf{E}\to S\times\mathbb{P}^1$ be a deformation of E. Let



be the Stein factorisation of the composite $\mathbf{E}(G/G^0) \to S \times \mathbb{P}^1 \to S$. Then $f\colon S' \to S$ is étale and $(f \times \mathrm{id})^* \mathbf{E}$ admits reduction of structure group to G^0.

vi) Suppose λ, μ have the same image in $(X_*(T)/Q^v)/W$. Then for some $w \in W$, λ and $w(\mu)$ are in the same coset in $X_*(T)/Q^v$. Therefore by the connected structure group case (Theorem 7.7) E_λ, $E_{w(\mu)}$ are algebraically equivalent as G^0-bundles and hence as G-bundles (since $E_{w(\mu)} \approx E_\mu$ as G-bundles).

Conversely, suppose E_λ is algebraically equivalent to E_μ and $\mathbf{E} \to S \times \mathbb{P}^1$ a family of G-bundles with S irreducible and $\mathbf{E}_{s_1} \approx E_\lambda$, $\mathbf{E}_{s_2} \approx E_\mu$, $s_1, s_2 \in S$. Then in an étale neighbourhood of s_1 (resp. s_2) $\mathbf{E}$ admits a G^0-reduction $\mathbf{E}^1$ (resp. $\mathbf{E}^2$) with generic type v_1 (resp. v_2). Since S is irreducible clearly $v_1 = w(v_2)$ for some $w \in W$. Therefore the G^0-bundles E^0_λ and $E^0_{w(\mu)}$ are algebraically equivalent and hence by the connected structure group case (Theorem 7.7) λ and $w(\mu)$ have the same image in $X_*(T)/Q^v$. Clearly the case when S is only connected can be reduced to the irreducible case.

9.2.1. *Remark.* It follows that the deformations of a G-bundle are the "same" as those of the corresponding ad G^0-bundle, ad G^0 being the adjoint group.

We will now indicate the particular form taken by our results when G is one of the classical groups. From the standard description of maximal tori, Weyl groups (cf. [8, exposés 20–22]) and the root data (cf. [6, Tables]) it is easy to read off the results (cf. Sects. 2.5, 7.8.1, 9.2.1).

9.3. *Vector Bundles.* A $GL(n)$-bundle is always Zariski locally trivial (cf. [27]) and in fact a $GL(n)$-bundle is equivalent to a vector bundle. From the description of maximal torus etc. given in Sect. 2 the following result is easily deduced from our general results.

i) Any vector bundle of rank n on $\mathbb{P}^1$ is isomorphic to a (unique) direct sum of line bundles $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n)$, with $a_1 \geqq \ldots \geqq a_n$ integers.

Denote this vector bundle by $A(a_1, \ldots, a_n)$.

ii) For integers $b_1 \geqq \ldots \geqq b_n$, $A(b_1, \ldots, b_n) \rightsquigarrow A(a_1, \ldots, a_n)$ if and only if $a_1 + \ldots + a_n = b_1 + \ldots + b_n$ and $a_1 + \ldots + a_i \geqq b_1 + \ldots + b_i$, $1 \leqq i \leqq n-1$.

iii) The rigid vector bundles of rank n are $A(m+1, \ldots, m+1, m, \ldots, m)$, $m \in \mathbb{Z}$.

iv) $A(b_1, \ldots, b_n)$ is algebraically equivalent to $A(a_1, \ldots, a_n)$ if and only if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ i.e. the two vector bundles have the same degree.

9.4. *Projective Bundles.* Let $\bar{T}$ be the image in $PGL(n)$ of the standard torus T of $SL(n)$ (Sect. 2.7). Then

$$X_*(\bar{T}) = \{[a_1, \ldots, a_n] \in X_*(T) \otimes \mathbb{Q} \mid n a_i \in \mathbb{Z},\ a_i \equiv a_j (\mathrm{mod}\, \mathbb{Z})\}.$$

Any 1-PS of $PGL(n)$ therefore lifts to $GL(n)$ (though not to $SL(n)$). Therefore any Zariski locally trivial projective bundle comes from a vector bundle (cf. Sect. 2.15). Note that if V, W are vector bundles then $\mathbb{P}(V)\approx\mathbb{P}(W)$ if and only if there is a line bundle L such that $V\otimes L\approx W$ (use the exact sequence $1\to G_m\to GL(n)\to PGL(n)\to 1$).

i) Any projective bundle is uniquely of the form $\mathbb{P}A(a_1,\dots,a_n)$ with $a_i\in\mathbb{Z}$, $a_1\geqq\dots\geqq a_n$ and $0\leqq\Sigma a_i<n$ (where $A(a_1,\dots,a_n)$ is the vector bundles defined in Sect. 9.3 i)). Denote this bundle by $\mathbb{P}(a_1,\dots,a_n)$.

ii) $\mathbb{P}(b_1,\dots,b_n)\rightsquigarrow\mathbb{P}(a_1,\dots,a_n)\Leftrightarrow A(b_1,\dots,b_n)\rightsquigarrow A(a_1,\dots,a_n)$ (see Sect. 9.3).

iii) The rigid bundles are $\mathbb{P}(1,\dots,1,0,\dots,0)$.

iv) $\mathbb{P}(b_1,\dots,b_n)$ is algebraically equivalent to $\mathbb{P}(a_1,\dots,a_n)$ if and only if $\Sigma a_i=\Sigma b_i$. (Note that we have normalised Σa_i to be between 0 and n.)

9.5. *Orthogonal Bundles.* The orthogonal group $O(2l+1)$(resp. $O(2l)$) is the subgroup of $GL(2l+1)$(resp. $GL(2l)$) leaving invariant the quadratic form

$$Q\left(\sum_{i=0}^{2l} x_i e_i\right)=x_0^2+\sum_{i=1}^{l} x_i x_{i+l}\left(\text{resp. }\sum_{i=1}^{l} x_i\cdot x_{i+l}\right).$$

$SO(2l+1)$(resp. $SO(2l)$) is the connected component of $O(2l+1)$ (resp. $O(2l)$) defined by the determinant or, if $\operatorname{char} k=2$, by the Dickson invariant. The maximal torus T consists of diagonal elements and a 1-PS is of the form $t\mapsto\operatorname{diag}[1,t^{a_1},\dots,t^{a_l},t^{-a_1},\dots,t^{-a_l}]$ (for $SO(2l)$ the initial 1 is dropped). The Weyl group of $O(2l+1)$ is the same as that of $SO(2l+1)$. The Weyl group of $SO(2l)$ is of index two in that of $O(2l)$ (the "sign changes" need not be even cf. [6, p. 257, item (X)]).

9.5.1. *Type* $B_l(l\geqq 2)$.

i) A Zariski locally trivial $O(2l+1)$-bundle has the underlying vector bundle $\mathcal{O}\oplus\sum_{i=1}^{l}\{\mathcal{O}(a_i)\oplus\mathcal{O}(-a_i)\}$ with $a_1\geqq\dots\geqq a_l\geqq 0$ integers. The quadratic form is the orthogonal sum of the constant quadratic form "x_0^2" on $\mathcal{O}$ and the hyperbolic form on $\mathcal{O}(a_i)\oplus\mathcal{O}(-a_i)$ given by the duality $\mathcal{O}(a_i)^*=\mathcal{O}(-a_i)$.

Denote this bundle by $B(a_1,\dots,a_l)$. *Assume* $l\geqq 2$.

ii) $B(b_1,\dots,b_l)\rightsquigarrow B(a_1,\dots,a_l)\Leftrightarrow a_1+\dots+a_i\geqq b_1+\dots+b_i$, $1\leqq i\leqq l$ and $\sum_{i=1}^{l} a_i\equiv\sum_{i=1}^{l} b_i(\operatorname{mod} 2)$.

iii) The rigid bundles are $B(0,\dots,0)$(=trivial bundle) and $B(1,0,\dots,0)$.

iv) $B(b_1,\dots,b_l)$ is algebraically equivalent to $B(a_1,\dots,a_l)\Leftrightarrow$ $\sum_{i=1}^{l} a_i\equiv\sum_{i=1}^{l} b_i(\operatorname{mod} 2)$.

9.5.2. *Type* $D_l(l\geqq 3)$

i) A Zariski locally trivial $O(2l)$-bundle is uniquely of the form

$$D(a_1,\dots,a_l)=\sum_{i=1}^{l}\{\mathcal{O}(a_i)\oplus\mathcal{O}(-a_i)\}$$

with $a_1\geqq\dots\geqq a_l\geqq 0$ integers and the quadratic form as in 9.5.1 i).

Assume $l\geqq 3$.

ii) $D(b_1,\dots,b_l)\rightsquigarrow D(a_1,\dots,a_l)$ if and only if

$a_1+\dots+a_i\geqq b_1+\dots+b_i$, $1\leqq i\leqq l-2$, $a_1+\dots+a_{l-1}+a_l\geqq b_1+\dots+b_{l-1}+b_l$,

$a_1+\dots+a_{l-1}-a_l\geqq b_1+\dots+b_{l-1}-b_l$ and $\sum_{i=1}^{l}a_i\equiv\sum_{i=1}^{l}b_i \pmod 2$.

iii) and iv) same as in 9.5.1.

9.5.3. $O(1)$. Since $O(1)=\mathbb{Z}_2$ the only $O(1)$-bundle is the trivial bundle.

9.5.4. $O(2)$. It is easily seen that $SO(2)\simeq G_m$, and that $O(2)$ is the semidirect product $\mathbb{Z}_2\times G_m$, $\mathbb{Z}_2$ acting on G_m by inversion: $z\mapsto z^{-1}$. Using the notation of 9.5.2 i), $D(a)=\mathcal{O}(a)\oplus\mathcal{O}(-a)\mapsto\mathcal{O}(a)$, $a\geqq 0$, gives an identification of $O(2)$-bundles with line bundles of degree $\geqq 0$. Since line bundles on $\mathbb{P}^1$ are rigid it follows that all $O(2)$-bundles are rigid.

9.5.5. $O(3)$. We have $SO(3)\simeq PGL(2)$ (One way of seeing this is to note that a nonsingular conic is isomorphic to $\mathbb{P}^1$). So the map sending $B(a)=\mathcal{O}\oplus\mathcal{O}(a)\oplus\mathcal{O}(-a)$ to $\mathbb{P}(a/2,-a/2)$ if a ($\geqq 0$) is even or to $\mathbb{P}([a/2]+1,-[a/2])$ if a is odd gives a bijection between $O(3)$-bundles and $PGL(2)$-bundles. Moreover $B(b)\rightsquigarrow B(a)\Leftrightarrow a\geqq b$. $B(b)$ is algebraically equivalent to $B(a)$ if and only if $a\equiv b \pmod 2$. The rigid bundles are $B(0)$ and $B(1)$.

9.5.6. $O(4)$. We have a two sheeted covering $SO(4)\to PGL(2)\times PGL(2)$. (This can be seen from the isomorphism of a nonsingular quadratic surface with $\mathbb{P}^1\times\mathbb{P}^1$). The map which sends $D(a_1,a_2)=\sum_{i=1}^{2}\{\mathcal{O}(a_i)\oplus\mathcal{O}(-a_i)\}$, $a_1\geqq a_2\geqq 0$, to $\mathbb{P}((a_1+a_2)/2, -(a_1+a_2)/2)\times\mathbb{P}((a_1-a_2)/2,-(a_1-a_2)/2)$ if a_1+a_2 is even or to

$$\mathbb{P}([(a_1+a_2)/2]+1,-[(a_1+a_2)/2])\times\mathbb{P}([(a_1-a_2)/2]+1,-[(a_1-a_2)/2])$$

if a_1+a_2 is odd gives a bijection of $O(4)$-bundles with unordered pairs of $PGL(2)$-bundles of the same parity (i.e. belonging to the same algebraic equivalence class). $D(b_1,b_2)\rightsquigarrow D(a_1,a_2)$ if and only if $a_1+a_2\geqq b_1+b_2$, $a_1-a_2\geqq b_1-b_2$ and $a_1+a_2\equiv b_1+b_2 \pmod 2$. $D(b_1,b_2)$ is algebraically equivalent to $D(a_1,a_2)$ if and only if $a_1+a_2\equiv b_1+b_2 \pmod 2$. The rigid bundles are $D(0,0)$ and $D(1,0)$.

9.6. *Remark.* For orthogonal bundles the condition $\Sigma a_i\equiv\Sigma b_i \pmod 2$ arises from the integrality for the fundamental weights corresponding to the spin representations (ω_{l-1} and ω_l in the notation of [6, Tables]). In [15] $\Sigma a_i \bmod 2$ is called the Mumford invariant and its relation with algebraic equivalence (item iv)) is proved when $k=\mathbb{C}$. This result of [15] also follows from [23, Proposition 4.2 and Sect. 5].

9.7. *Symplectic Bundles.*

i) Any $Sp(2n)$-bundle ($n \geqq 2$) is Zariski locally trivial [27]. The underlying vector bundle is $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n) \oplus \mathcal{O}(-a_1) \ldots \oplus \mathcal{O}(-a_n)$, with $a_1 \geqq \ldots \geqq a_n$ integers, the symplectic form being the standard one. Denote this bundle by $C(a_1, \ldots, a_n)$.

ii) $C(b_1, \ldots, b_n) \rightsquigarrow C(a_1, \ldots, a_n)$ if and only if $a_1 + \ldots + a_i \geqq b_1 + \ldots + b_i$, $i \leqq 1 \leqq n$.

iii) The trivial bundle $C(0, \ldots, 0)$ is the only rigid bundle.

iv) All $Sp(2n)$-bundles are algebraically equivalent (since $Sp(2n)$ is simply connected).

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