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Deformations of singular symplectic varieties and termination of the log minimal model program

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Abstract

We generalize Huybrechts' theorem on deformation equivalence of birational irreducible symplectic manifolds to the singular setting. More precisely, under suitable natural hypotheses, we show that two birational symplectic varieties are locally trivial deformations of each other. As an application we show the termination of any log minimal model program for a pair (X, Δ) of a projective irreducible symplectic manifold X and an effective $\mathbb{R}$-divisor Δ. To prove this result we follow Shokurov's strategy and show that LSC and ACC for minimal log discrepancies hold for all the models appearing along any log MMP of the initial pair.

1. Introduction

In the theory of irreducible symplectic manifolds, an important result due to Huybrechts [Huy03, Theorem 2.5] ensures that two such manifolds X and X' that are birational, are deformation equivalent. (We will use the term birational instead of bimeromorphic also in the analytic context.) Even more is true, namely there exist smooth proper families $\pi\colon \mathcal{X} \to S$ and $\pi'\colon \mathcal{X}' \to S$ over a pointed disk $(S, 0)$ and a birational map between $\mathcal{X}$ and $\mathcal{X}'$ which is an isomorphism over $S\backslash 0$ and coincides with the given birational map between $X = \pi^{-1}(0)$ and $X' = \pi'^{-1}(0)$ over 0. In particular, X and X' have isomorphic Hodge structures. Huybrechts' result yields a characterization of non-separated points in the moduli space of marked irreducible symplectic manifolds. It not only is theoretically relevant, but has also been successfully applied to solve concrete problems (see, for example, [Bea99, Deb99, AL16, Leh15]).

Recently, there has been renewed interest in the study of singular symplectic varieties. For instance, there is a very interesting line of research started by Greb, Kebekus and Peternell on varieties with numerically trivial canonical divisor and singularities that appear in the minimal model program (MMP); see [GKP11]. On the other hand, singular symplectic varieties also play an important role in the study of smooth symplectic varieties; see, for example, [DV10, OG99, OG03]. Given the importance of Huybrechts' theorem to the theory of irreducible symplectic

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manifolds, it is natural to ask whether a *singular* version of this result holds true.

Before we state our result, recall that since the work of Beauville [Bea00] there is a well-established notion of singular symplectic variety (see Section 2). Let X and X' be singular symplectic varieties which are birational with each other. If they both admit crepant resolutions by irreducible symplectic manifolds, which is for example the case if X and X' show up in the log MMP of a given irreducible symplectic manifold (cf. Lemma 4.1), then their crepant resolutions are also birational and hence deformation equivalent thanks to Huybrechts' theorem. By work of Namikawa this implies that the two birational singular symplectic varieties will also be deformation equivalent, but by construction this deformation does not preserve the singularity type. The question arises under which circumstances we may find a deformation that connects X and X' and preserves the singularities. Our main result gives precise conditions under which this is possible.

THEOREM 1.1. *Let X and X' be $\mathbb{Q}$-factorial projective symplectic varieties having crepant resolutions by irreducible symplectic manifolds, and suppose that $\phi\colon X \dashrightarrow X'$ is a birational map which is an isomorphism in codimension 1. Then there exist proper families $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ of locally trivial deformations of X and X' over a pointed disk $(S, 0)$ and a birational map between $\mathcal{X}$ and $\mathcal{X}'$ which is an isomorphism over $S\backslash 0$ and coincides with ϕ over 0. In particular, X and X' are homeomorphic and their local analytic isomorphism type is the same.*

The theorem says that just as in the smooth case X and X' are non-separated points in the space of marked symplectic varieties with fixed underlying topological space. Note that we can drop neither the $\mathbb{Q}$-factoriality hypothesis, as every small contraction on a smooth symplectic manifold would give a counter-example, nor the hypothesis that ϕ is an isomorphism in codimension 1, as every divisorial contraction from a smooth X to a singular X' would give a counter-example. Thus, our result is optimal. To the best of our knowledge, it is the first result giving information on singularity preserving deformations of birational singular symplectic varieties. Of course one can still ask whether the conclusion holds for arbitrary symplectic varieties, that is, not necessarily possessing irreducible symplectic crepant resolutions. This is likely to be true, but is for the moment out of reach for technical reasons.

The proof of Theorem 1.1 relies heavily on Namikawa's foundational work on the deformation theory of singular symplectic varieties and on his comparison results of these deformations with those of crepant resolutions; cf. [Nam01, Nam06, Nam11]. Another important tool is Kaledin's local structure theorem of symplectic singularities; see [Kal06, Theorem 2.3]. These are the two pillars for the main new technical contribution of this work, which is the proof of the smoothness of certain deformation spaces (cf. Proposition 2.3). Together with Huybrechts' original strategy, whose simple geometric idea guides us through the technicalities, this is an essential ingredient in our proof of our main result, Theorem 1.1. The unobstructedness result is obtained by invoking Ran's T^1-lifting principle [Ran92, Kaw92, Kaw97]. More precisely, given an irreducible symplectic crepant resolution $\pi\colon \tilde{X} \to X$ of a symplectic variety X, thanks to the T^1-lifting principle we can relate the locally trivial deformations of X to those deformations of $\tilde{X}$ preserving the irreducible components of the exceptional locus of π (see Proposition 2.3 for the precise statement).

The second part of the paper is devoted to presenting our main application of Theorem 1.1; it is concerned with the minimal model program (or strictly speaking, rather with its logarithmic version (log MMP); we will however mostly use the term MMP for simplicity). Though there has been much progress, its most important goals, finding good representatives, so-called *minimal models*, in every birational equivalence class of algebraic varieties, and connecting a given variety X to one of its minimal models by elementary birational transformations, have not yet been completely accomplished. The existence of minimal models as well as the termination of certain special MMPs have been established in many cases in the seminal paper [BCHM10]; see also [CL12]. What is missing in general is the termination of flips.

We show here that irreducible symplectic manifolds behave as well as possible with respect to the MMP. To put our result into perspective, notice that the termination of log flips has been shown for irreducible symplectic manifolds by Matsushita–Zhang [MZ13] (see also [Mat14]) following a strategy due to Shokurov [Sho04] (see below). However, the termination of log flips does not imply that every MMP on a symplectic manifold terminates, for smoothness plays a crucial role in Matsushita–Zhang's argument. For example, if the MMP produces not only flips but also divisorial contractions, the resulting variety will acquire singularities and then there could still be an infinite sequence of flips. As an application of Theorem 1.1, we show that this does not happen.

Theorem 1.2. *Let X be a projective irreducible symplectic manifold, and let Δ be an effective $\mathbb{R}$-Cartier divisor on X such that the pair (X, Δ) is log canonical. Then every log MMP for (X, Δ) terminates in a minimal model (X', Δ'), where X' is a symplectic variety with canonical singularities and Δ' is an effective, nef $\mathbb{R}$-Cartier divisor.*

It is well known that from the previous result one derives the following (see [Bir12] for the relevant definitions and further developments).

Corollary 1.3. *Let X be a projective irreducible symplectic manifold, and let Δ be an effective $\mathbb{R}$-Cartier divisor on X. Then birationally Δ has a Zariski decomposition in the sense of Fujita and in the sense of Cutkosky–Kawamata–Moriwaki.*

The proof of Theorem 1.2 follows Shokurov's strategy. Let us go a little more into detail. To show the termination of flips, Shokurov introduced the so-called *minimal log discrepancy* (mld for short), which is a local invariant associated to (X, Δ) and which increases under flips. It is nowadays interpreted as an invariant of the singularity of (X, Δ) at a given point. Ambro and Shokurov have made two strong conjectures about the behavior of mlds. These are the lower semi-continuity conjecture (LSC) and the ascending chain condition conjecture (ACC); see Section 3.2. Shokurov proved that these two conjectures imply the termination of flips [Sho04]. For smooth varieties, LSC holds by the fascinating paper [EMY03] and if all varieties in a sequence of flips are smooth, ACC holds for trivial reasons. However, even if we start with a smooth variety X, the MMP easily carries us out of the class of smooth varieties. Matsushita–Zhang's key point is that a flip of a smooth symplectic variety remains smooth by deep results of Namikawa [Nam06]; see Section 4 for more details.

The ACC and LSC conjectures seem to be out of reach for arbitrary varieties. Starting with some variety X and running an MMP might a priori produce a huge variety of different singularities. Nevertheless, if we can bound the class of singularities of varieties that show up in intermediate steps of the MMP, then there is hope that Shokurov's strategy can be used. In our case, as recalled before, this class of varieties will be the class of proper varieties with symplectic singularities which have a crepant resolution by an irreducible symplectic manifold. We first prove the following result.

Theorem 3.6. *Let Y be a normal projective $\mathbb{Q}$-Gorenstein variety, and let Δ be an effective $\mathbb{R}$-Cartier divisor on Y such that (Y, Δ) is log canonical. If $\pi \colon X \to Y$ is a crepant morphism and LSC holds on X, then LSC holds for (Y, Δ).*

Then we use Theorem 1.1 in a crucial way to show that in a sequence of flips of singular symplectic varieties the singularities, more precisely, their local analytic isomorphism type, *does not change*. This allows us to invoke a result of Kawakita [Kaw14] to deduce that ACC holds along any log MMP of an irreducible symplectic manifold and conclude.

The paper is organized as follows: in Section 2 we prove Theorem 1.1. Then we turn to the application and prove Theorem 3.6 in Section 3, after having recalled the basic definitions and results on minimal log discrepancies. Finally, we put all the ingredients together and show how to deduce termination in Section 4.

2. Deformations

We work over the field of complex numbers. A deformation of a variety Y is a flat morphism $\mathscr{Y} \to S$ of complex spaces to a pointed space $(S, 0)$ such that the fiber $\mathscr{Y}_0$ over $0 \in S$ is (isomorphic to) Y. We will mostly work with space germs, that is, equivalence classes of deformations where two deformations $\mathscr{Y} \to S$ and $\mathscr{Y}' \to S'$ are equivalent if S and S' are isomorphic in some small neighborhoods of their distinguished points and moreover $\mathscr{Y}$ and $\mathscr{Y}'$ are isomorphic in neighborhoods of $\mathscr{Y}_0$ and $\mathscr{Y}'_0$ and in a way which is compatible with the maps to the base. We will often use representatives of these equivalence classes and shrink them if necessary without mention. A *symplectic variety* is a normal projective variety X admitting an everywhere non-degenerate closed 2-form ω on the regular locus X_{reg} of X such that, for any resolution $f\colon \tilde{X} \to X$ with $f^{-1}(X_{\mathrm{reg}}) \cong X_{\mathrm{reg}}$, the 2-form ω extends to a regular 2-form on $\tilde{X}$. An *irreducible symplectic manifold* is a simply connected compact Kähler manifold X admitting an everywhere non-degenerate closed 2-form ω such that $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \omega$.

In this section we are going to prove Theorem 1.1, which should be interpreted as an analogue of the well-known result of Huybrechts [Huy03, Theorem 2.5]. This is the only section where we make use of the complex numbers, which however does not seem to be essential and results as well as proofs should carry over mutatis mutandis to any algebraically closed field of characteristic zero.

The proof relies on Ran's T^1-lifting principle [Ran92, Kaw92, Kaw97]. Essentially it says that a given deformation problem is unobstructed if the tangent space T_X^1 to the deformation space is "deformation invariant" in the sense that for every small deformation $\mathcal{X} \to S$ of X, its relative versions $T^1_{\mathcal{X}/S}$ are free $\mathcal{O}_S$-modules. We refer to [GHJ03, § 14] for a concise account. Recall, for example from [Ser06, 1.2], that the tangent space to the deformation functor $\mathrm{Def}_X^{\mathrm{lt}}$ of locally trivial deformations of an algebraic variety X is $H^1(T_X)$, as opposed to the case of arbitrary deformations, where the tangent space is $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$, and that an obstruction space for $\mathrm{Def}_X^{\mathrm{lt}}$ is given by $H^2(T_X)$.

Let us recall the following well-known result on the local structure of singular symplectic varieties. For convenience we sketch the proof which is due to Kaledin and Namikawa; see [Nam11]. Note that by convention we consider the singular locus as a subscheme (or complex subspace) with the induced reduced structure.

Proposition 2.1. *Let X be a symplectic variety, and let $\Sigma \subset X$ be the singular locus of X^{sing}. Then $\operatorname{codim}_X \Sigma \geqslant 4$ and every $x \in U := X \backslash \Sigma$ has a neighborhood which is locally analytically isomorphic to $(\mathbb{C}^{2n-2}, 0) \times (S, p)$, where $2n = \dim X$ and (S, p) is the germ of a smooth point or a rational double point on a surface. This isomorphism can be chosen to preserve the symplectic structure.*

Proof. Kaledin's result [Kal06, Theorem 2.3] implies that Σ has codimension at least 4 and that every point of U admits the sought for product decomposition in the formal category. By [Art69, Corollary 2.6] the decomposition exists analytically. The last statement is [Nam11, Lemma 1.3]. □

Proposition 2.2. *Let X be a $\mathbb{Q}$-factorial compact symplectic variety, let $\pi\colon \tilde{X} \to X$ be a crepant resolution by a compact Kähler manifold $\tilde{X}$, and let $U \subset X$ be as in Proposition 2.1. Then the restriction $H^1(X, T_X) \to H^1(U, T_U)$ is an isomorphism and $h^1(T_X) = h^1(T_{\tilde{X}}) - m$, where m is the number of irreducible components of the exceptional divisor of π.*

Proof. The exceptional set of π is a divisor by the $\mathbb{Q}$-factoriality hypothesis, and each of its irreducible components meets $\pi^{-1}(U)$ by the semi-smallness property of symplectic resolution; cf. [Kal06, Lemma 2.11]. Let us consider the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(T_X) & \longrightarrow & \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) & \longrightarrow & H^0(T^1_X) \\ & \downarrow & & \downarrow \phi & & \downarrow \\ 0 \longrightarrow & H^1(T_U) & \longrightarrow & \mathrm{Ext}^1(\Omega_U, \mathcal{O}_U) & \longrightarrow & H^0(T^1_U) \end{array}$$

with exact lines, where ϕ is an isomorphism by [Nam01, Proposition 2.1]. As $T^1_X = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$, the space $H^1(T_X)$ consists of extensions that are locally split. Analyzing the construction of the map ϕ described in [KM92, Lemma 12.5.6], we see that an extension on X is locally split if and only if its restriction to U is, and thus $H^1(T_X) \to H^1(T_U)$ is an isomorphism.

Put $\tilde{U} := \pi^{-1}(U) \subset \tilde{X}$ and consider the following sequence:

$$0 \longrightarrow H^1(\pi_* T_{\tilde{U}}) \longrightarrow H^1(T_{\tilde{U}}) \longrightarrow H^0(R^1\pi_* T_{\tilde{U}}) \longrightarrow 0\,.$$

It is exact and we have $h^1(T_U) = h^1(T_{\tilde{U}}) - m$, as was shown in part (ii) of the proof of [Nam01, Theorem 2.2]. As $h^1(T_{\tilde{U}}) = h^1(T_{\tilde{X}})$ by [Nam01, Proposition 2.1], the claim follows. □

In the situation of Theorem 1.1, let $\pi\colon \tilde{X} \to X$ be a crepant resolution, and let $D = \sum_{i=1}^m D_i$ be the exceptional divisor with its decomposition into irreducible components D_i. We put $L_i := \mathcal{O}_{\tilde{X}}(D_i)$ and denote by $\mathscr{X} \to \mathrm{Def}(\tilde{X})$ the universal deformation of $\tilde{X}$. This is the germ of a smooth space of dimension $h^{1,1}(\tilde{X})$ by the Bogomolov–Tian–Todorov theorem. We consider the following subspaces of $\mathrm{Def}(\tilde{X})$:

- $\mathrm{Def}(\tilde{X}, \underline{L}) \subset \mathrm{Def}(\tilde{X})$ is the base of the universal deformation of $(\tilde{X}, L_1, \ldots, L_m)$; see [Huy99, 1.14]. As the D_i define linearly independent classes in $H^2(\tilde{X}, \mathbb{C})$, the space $\mathrm{Def}(\tilde{X}, \underline{L})$ is smooth and of codimension m in $\mathrm{Def}(\tilde{X})$ by loc. cit.
- $\mathrm{Def}(\tilde{X}, \underline{D}) \subset \mathrm{Def}(\tilde{X})$ is the image of the components containing all D_i of the relative Douady space $\mathscr{D}(\mathscr{X} / \mathrm{Def}(\tilde{X})) \to \mathrm{Def}(\tilde{X})$. This is the space where all components D_i deform along with $\tilde{X}$.

We clearly have $\mathrm{Def}(\tilde{X}, \underline{D}) \subset \mathrm{Def}(\tilde{X}, \underline{L})$. Consequently, $\dim \mathrm{Def}(\tilde{X}, \underline{D}) \leqslant h^{1,1}(\tilde{X}) - m$.

The key step will be to prove the smoothness of the space of locally trivial deformations of the singular variety X. Recall from [FK87] that the universal locally trivial deformation of X exists and that it is just the restriction of the universal deformation to the locally trivial locus $\mathrm{Def}^{\mathrm{lt}}(X) \subset \mathrm{Def}(X)$ in the Kuranishi space, which is a closed subspace.

PROPOSITION 2.3. *Let $\pi\colon \tilde{X} \to X$ be as above. Let $\tilde{\mathscr{X}} \to \mathrm{Def}(\tilde{X}, \underline{D})$ and $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ be the universal deformations. Then there is a diagram*

$$\begin{array}{ccc} \tilde{\mathscr{X}} & \xrightarrow{\ \Pi\ } & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{Def}(\tilde{X}, \underline{D}) & \xrightarrow{\ \pi_*\ } & \mathrm{Def}^{\mathrm{lt}}(X) \end{array} \tag{2.1}$$

with the following properties:

(i) $\mathrm{Def}^{\mathrm{lt}}(X)$ *is smooth of dimension* $h^{1,1}(\tilde{X}) - m$.

(ii) π_* *is the restriction of the natural finite morphism* $\mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ *and is an isomorphism.*

(iii) $\dim \mathrm{Def}(\tilde{X}, \underline{D}) = h^{1,1}(\tilde{X}) - m$; *in particular,* $\mathrm{Def}(\tilde{X}, \underline{D}) = \mathrm{Def}(\tilde{X}, \underline{L})$.

Proof. We will first show that $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth. Let $U \subset X$ be as in Proposition 2.1. The restriction $H^1(T_X) \to H^1(T_U)$ is an isomorphism by Proposition 2.2; in other words, deformations and their local triviality are determined on U. Let $j\colon X^{\mathrm{reg}} \to U$ denote the inclusion. As T_U is reflexive, we have that $T_U \cong j_* T_{X^{\mathrm{reg}}}$. Hence, $H^1(T_X) = H^1(U, j_*\Omega_{X^{\mathrm{reg}}}) = \mathbb{H}^2(U, j_*\Omega^{\geqslant 1}_{X^{\mathrm{reg}}})$, which is deformation invariant, as we will show next. Consider the exact sequence of complexes

$$0 \to j_*\Omega^{\geqslant 1}_{X^{\mathrm{reg}}} \to j_*\Omega^{\bullet}_{X^{\mathrm{reg}}} \to \mathcal{O}_U \to 0\,. \tag{2.2}$$

By Grothendieck's theorem for V-manifolds $\mathbb{H}^k(j_*\Omega^{\bullet}_{X^{\mathrm{reg}}}) = H^k(U, \mathbb{C})$; see, for example, the footnote in the proof of [Nam06, Proposition 1.11]. Moreover, we have

$$H^1(\mathcal{O}_U) = H^1(\mathcal{O}_X) = H^1(\mathcal{O}_{\tilde{X}}) = 0\,,$$

where the first equality holds because X is Cohen–Macaulay and $\mathrm{codim}_X(X\backslash U) \geqslant 4$ and the second because X has rational singularities. In the same way, one finds

$$H^2(\mathcal{O}_U) = H^2(\mathcal{O}_X) = H^2(\mathcal{O}_{\tilde{X}}) \cong \mathbb{C}\,,$$

so that (2.2) gives an exact sequence

$$0 \to H^1(j_*\Omega_{X^{\mathrm{reg}}}) \to H^2(U, \mathbb{C}) \to H^2(\mathcal{O}_U) \to 0\,,$$

where the last map is surjective because the composition $H^2(\tilde{X}, \mathbb{C}) \to H^2(\mathcal{O}_{\tilde{X}}) \to [\cong] H^2(\mathcal{O}_U)$ is. The same line of arguments works in a relative situation and shows that $H^1(T_{\mathcal{X}/S}) = H^1(j_*\Omega_{(\mathcal{X}/S)^{\mathrm{reg}}})$ is a free $\mathcal{O}_S$-module for any small deformation $\mathcal{X} \to S$ over a local artinian scheme S. In other words, the tangent space to the deformation functor $H^1(T_X)$ is deformation invariant, hence by the T^1-lifting argument $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth. In particular, $\dim \mathrm{Def}^{\mathrm{lt}}(X) = \dim H^1(T_X)$, which is equal to $h^{1,1}(\tilde{X}) - m$ by Proposition 2.2.

As explained in [Nam06, § 3], there is a diagram like (2.1) for arbitrary instead of locally trivial deformations. In particular, there is a finite map $\pi_*\colon \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ and for each $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ and every $s \in \mathrm{Def}(\tilde{X})$ mapping to t, the morphism $\tilde{\mathscr{X}}_s \to \mathscr{X}_t$ is a crepant resolution. By abuse of notation we will also denote by $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ the restriction of the universal family to the locally trivial locus $\mathrm{Def}^{\mathrm{lt}}(X)$. Let $\mathscr{S} \subset \mathscr{X}$ be the singular locus of the singular locus of the morphism $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$, that is, the relative version of the subvariety $\Sigma \subset X$ introduced in Proposition 2.1, and denote by $\mathscr{U}$ its complement in $\mathscr{X}$. By the local triviality property, $\mathscr{S}$ is a locally trivial, hence flat, deformation of Σ; in particular, it is fiberwise of codimension at least 4 in $\mathscr{X}$. If we denote $\widetilde{\mathscr{U}} := \Pi^{-1}(\mathscr{U})$, then $\widetilde{\mathscr{U}_s} \to \mathscr{U}_t$ is a crepant resolution of singularities for $t = \pi_*(s)$. Note that by the choice of $\mathscr{U}$ its singular locus is a locally trivial deformation of an ADE-surface singularity, hence $\widetilde{\mathscr{U}} \to \mathscr{U}$ is (fiberwise) the unique minimal relative resolution. In particular, $\widetilde{\mathscr{U}}$ is a locally trivial deformation of its central fiber and the exceptional divisors of $\widetilde{\mathscr{U}} \to \mathscr{U}$ are locally trivial deformations of those of its central fiber $\widetilde{U} \to U$. As $\mathscr{S}$ is fiberwise of codimension at least 4, by taking closures in $\widetilde{\mathscr{X}}$ we obtain deformations of the D_i over $\pi_*^{-1}(\mathrm{Def}^{\mathrm{lt}}(X))$ with irreducible fibers. Therefore, the inclusion $\pi_*^{-1}(\mathrm{Def}^{\mathrm{lt}}(X)) \subset \mathrm{Def}(\tilde{X}, \underline{D})$ holds. As $\dim \mathrm{Def}(\tilde{X}, \underline{D}) \leqslant h^{1,1}(\tilde{X}) - m$, this inclusion is an equality and the restriction of π_* to $\mathrm{Def}(\tilde{X}, \underline{D})$ gives the desired morphism. Statement (iii) also follows from this.

To see that $\pi_*\colon \mathrm{Def}(\tilde{X}, \underline{D}) \to \mathrm{Def}^{\mathrm{lt}}(X)$ is an isomorphism, it suffices to show that its differential $T_{\mathrm{Def}(\tilde{X},\underline{D}),0} \to T_{\mathrm{Def}^{\mathrm{lt}}(X),0} = H^1(T_X)$ is one. We have just seen that $\mathrm{Def}(\tilde{X}, \underline{D}) = \mathrm{Def}(\tilde{X}, \underline{L})$ and by invoking [Huy99, 1.14] once more, we see that

$$T_{\mathrm{Def}(\tilde{X},\underline{D}),0} = \ker\left(\mathrm{c}_1(\underline{L})\colon H^1(T_{\tilde{X}}) \to [\phi] H^2(\mathcal{O}_{\tilde{X}})^m\right),$$

where ϕ is given by cup product with $\mathrm{c}_1(L_i)$ and contraction in the ith component. As explained before, for dimension reasons we may replace X by U and $\tilde{X}$ by $\tilde{U} := \pi^{-1}(U)$ in all cohomology groups involved. The differential of $\pi_*\colon \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ is thus a map $H^1(T_{\tilde{U}}) \to \mathrm{Ext}^1(\Omega_U, \mathcal{O}_U)$ whose restriction to the subspace $H^1(\pi_* T_{\tilde{U}}) \subset H^1(T_{\tilde{U}})$ identifies the latter with $H^1(T_U) \subset \mathrm{Ext}^1(\Omega_U, \mathcal{O}_U)$. Thus, it remains to show that $H^1(\pi_* T_{\tilde{U}}) = \ker \phi|_{\tilde{U}}$. We have already seen that under the symplectic form $H^1(T_{\tilde{U}}) \cong H^1(j_* \Omega_{X^{\mathrm{reg}}}) \subset H^2(U, \mathbb{C})$, and on the other side $\ker \phi$ is identified with the subspace of those $\alpha \in H^{1,1}(\tilde{X})$ which satisfy $q_{\tilde{X}}(\alpha, \mathrm{c}_1(L_i)) = 0$ for $i = 1, \ldots, m$, where $q_{\tilde{X}}$ is the Bogomolov–Beauville–Fujiki form by [Huy99, 1.8]. Certainly, classes which are pullbacks from X are among them. For dimension reasons we have equality, which completes the proof. □

The following lemma is probably well known, we include its proof for convenience.

Lemma 2.4. *Let $\tilde{X}$ be an irreducible symplectic manifold, and let $\pi\colon \tilde{X} \to X$ be a proper birational morphism to a Kähler complex space X. If X carries a line bundle L such that $q_{\tilde{X}}(\pi^* L) > 0$, then X is projective.*

Proof. It follows from Huybrechts' projectivity criterion [Huy97, Theorem 3.11] that in such a situation $\tilde{X}$ is projective. Then X is Kähler, Moishezon and has rational singularities, hence is projective by [Nam02, Theorem 1.6]. □

Proposition 2.5. *Let $X \dashrightarrow X'$ be as in Theorem 1.1. Let $\mathscr{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ and $\mathscr{X}' \to \mathrm{Def}^{\mathrm{lt}}(X')$ be the universal locally trivial deformations of X and X', respectively, and let $\pi\colon \tilde{X} \to X$ and $\pi'\colon \tilde{X}' \to X'$ be crepant resolutions of singularities. Then there is an isomorphism $\gamma\colon \mathrm{Def}^{\mathrm{lt}}(X) \to \mathrm{Def}^{\mathrm{lt}}(X')$ fitting into a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Def}(\tilde{X}, \underline{D}) & \xrightarrow{\tilde{\gamma}} & \mathrm{Def}(\tilde{X}', \underline{D}') \\
{\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi'_*} \\
\mathrm{Def}^{\mathrm{lt}}(X) & \xrightarrow{\gamma} & \mathrm{Def}^{\mathrm{lt}}(X')
\end{array}$$

of isomorphisms such that for each $t \in \mathrm{Def}^{\mathrm{lt}}(X)$ we have a birational map $\phi_t\colon \mathscr{X}_t \dashrightarrow \mathscr{X}'_{\gamma(t)}$. For very general t, the map ϕ_t is an isomorphism.

Proof. As $\tilde{X} \dashrightarrow \tilde{X}'$ is an isomorphism in codimension 1, the local Torelli theorem gives an isomorphism $\tilde{\gamma}\colon \mathrm{Def}(\tilde{X}, \underline{D}) \to \mathrm{Def}(\tilde{X}', \underline{D})$. The isomorphism γ is obtained through composition with the isomorphisms $\mathrm{Def}(\tilde{X}, \underline{D}) \to [\pi_*] \mathrm{Def}^{\mathrm{lt}}(X)$ and $\mathrm{Def}(\tilde{X}', \underline{D}) \to [\pi'_*] \mathrm{Def}^{\mathrm{lt}}(X')$ from

Proposition 2.3, as $\gamma = \pi'_* \circ \tilde\gamma \circ (\pi_*)^{-1}$. As $\tilde X$ and $\tilde X'$ are birational by assumption, they are deformation equivalent by Huybrechts' result [Huy03, Theorem 2.5]. So for $s \in \operatorname{Def}(\tilde X, \underline{D})$ the fibers $\tilde{\mathscr{X}}_s$ and $\tilde{\mathscr{X}}'_{s'}$ with $s' = \tilde\gamma(s)$ are deformation equivalent and have the same periods, hence they are birational by Verbitsky's global Torelli theorem [Ver13, Theorem 1.17]. If we denote $t = \pi_*(s) \in \operatorname{Def}^{\mathrm{lt}}(X)$ and $t' = \pi'_*(s') = \gamma(t) \in \operatorname{Def}^{\mathrm{lt}}(X')$, the morphisms $\pi_s\colon \tilde{\mathscr{X}}_s \to \mathscr{X}_t$ and $\pi'_{s'}\colon \tilde{\mathscr{X}}'_{s'} \to \mathscr{X}'_{t'}$ contract the same divisors and we obtain a birational map $\mathscr{X}_t \dashrightarrow \mathscr{X}'_{t'}$, which is an isomorphism in codimension 1. If t is close enough to $0 \in \operatorname{Def}^{\mathrm{lt}}(X)$, then $\mathscr{X}_t$ and $\mathscr{X}'_t$ are Kähler by [Nam02, Proposition 5]. The last statement follows as the general projective deformation of X has Picard number 1 and birational maps between $\mathbb{Q}$-factorial K-trivial varieties of Picard number 1 are isomorphisms. Note that the subspace of $H^2(\tilde X, \mathbb{R})$ spanned by the classes $[D_1], \ldots, [D_m]$ is negative definite with respect to the Bogomolov–Beauville form q_X. This follows, for example, from [Bou04, Theorem 4.5]; see also [Dru11, Theorem 1.3]. We can thus indeed always deform to projective varieties by [GHJ03, Proposition 26.6] and Lemma 2.4. □

Proof of Theorem 1.1. The proof of [Huy99, Theorem 4.6] works with minor modifications to give a proof of Theorem 1.1. For convenience we sketch Huybrechts' argument with emphasis on where we have to argue differently. Let L' be an ample line bundle on X' and denote by L the $\mathbb{Q}$-line bundle on X obtained from L' by taking the pullback to a resolution of indeterminacies, the pushforward to X and the double dual. Here we use $\mathbb{Q}$-factoriality. Replacing L' and L by multiples, we may assume that L is a line bundle. Let us denote by $\pi\colon \tilde X \to X$ a resolution of singularities where $\tilde X$ is an irreducible symplectic manifold and by D as before the exceptional divisor of π. Recall from Proposition 2.3 that $\operatorname{Def}^{\mathrm{lt}}(X) \cong \operatorname{Def}(\tilde X, \underline{D})$. Then points in the Kuranishi space $\operatorname{Def}^{\mathrm{lt}}(X, L) \cong \operatorname{Def}(\tilde X, \pi^*L, \underline{D})$ of the pair (X, L) parametrize deformations of X together with a line bundle whose pullback to a resolution of singularities by an irreducible symplectic manifold has positive Beauville–Bogomolov square; in particular, these are projective deformations by Lemma 2.4. We take a 1-dimensional disk $S \subset \operatorname{Def}^{\mathrm{lt}}(X, L)$ which passes through the origin and which is very general in the sense that the fibers of the (restriction to S of the) universal family $\psi\colon \mathscr{X} \to S$ for a very general $s \in S$ have Picard number 1. Denote by $\mathscr{L}$ the universal line bundle restricted to $\mathscr{X}$, which is a deformation of L. Then as in [Huy99, Theorem 4.6] one shows that $\mathscr{L}_s$ is ample for very general $s \in S$ and that $h^0(\mathscr{X}_t, \mathscr{L}_t^m)$ is independent of t in a neighborhood of $0 \in S$, where we have to replace the Kodaira vanishing theorem by the one of Kawamata–Viehweg. Note that $H^i(X, L) = H^i(\tilde X, \pi^*L)$ for all i as X has rational singularities. We may now apply [Huy99, Proposition 4.5], which produces a deformation $\mathscr{X}' \to S$ (maybe after shrinking S) together with a line bundle $\mathscr{L}'$ on $\mathscr{X}'$ such that $(\mathscr{X}'_0, \mathscr{L}'_0) = (X', L')$ and an S-birational map $\mathscr{X} \dashrightarrow \mathscr{X}'$ which is an isomorphism outside the central fiber and the birational map $\phi\colon X \dashrightarrow X'$ we started with. Let us admit this result for a moment and let us see how to complete the proof. It remains to show that the deformation $\mathscr{X}' \to S$ is locally trivial. For this we recall that the universal locally trivial deformation of X' is nothing else than the restriction of the universal deformation of X' to the locally trivial locus $\operatorname{Def}^{\mathrm{lt}}(X') \subset \operatorname{Def}(X')$ in the Kuranishi space. But the locally trivial locus is a Hodge locus by Proposition 2.3 and so we may deduce local triviality from the comparison of the periods of $\mathscr{X}$ and $\mathscr{X}'$.

Let us now comment on the proof of [Huy99, Proposition 4.5]. It is contained in [Huy97, Proposition 4.2] and works roughly like this. By the hypothesis that $h^0(\mathscr{X}_t, \mathscr{L}_t)$ be independent of t we obtain that the coherent sheaf $\psi_*\mathscr{L}$ is locally free and its fiber at $t \in S$ is exactly $H^0(\mathscr{L}_t)$, at least after shrinking S. One considers the rational map $\varphi_{\mathscr{L}}\colon \mathscr{X} \dashrightarrow \mathbb{P}_S((\pi_*\mathscr{L})^*)$ and defines $\psi'\colon \mathscr{X}' \to S$ to be its image. One easily shows that the general fiber of $\mathscr{X}' \to S$ is irreducible and that $X' \subset \mathscr{X}'_0$. This inclusion is then shown to be an equality by comparing the number of sections of the tautological bundle (which is just L' when restricted to X'). Birationality is shown by a similar argument and all these steps carry over without changes for singular varieties. □

3. Minimal log discrepancies

A *log pair* (X, Δ) consists of a normal variety X and an $\mathbb{R}$-Weil divisor $\Delta \geqslant 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A *log resolution* of a log pair (X, Δ) is a projective birational morphism $\pi\colon \tilde{X} \to X$ such that $\tilde{X}$ is smooth and $\pi^*\Delta + Exc(\pi)$ has simple normal crossing support. A birational morphism $f\colon \tilde{X} \to X$ between varieties for which K_X and $K_{\tilde{X}}$ are well defined is called *crepant* if $\pi^* K_X = K_{\tilde{X}}$. A *crepant resolution* is a resolution of singularities which is also a crepant morphism.

3.1 Elementary properties of minimal log discrepancies

If (X, Δ) is a log pair and $\pi\colon \tilde{X} \to X$ is a log resolution of (X, Δ), then we define the log discrepancy $a(E, X, \Delta)$ for a divisor E over X by the formula

$$K_{\tilde{X}} + \tilde{\Delta} = \pi^*(K_X + \Delta) + \sum_{E \subset \tilde{X}} (a(E, X, \Delta) - 1)E\,,$$

where $\tilde{\Delta}$ is the strict transform of Δ.

Let $c_X(E) \in X$ be the center of a divisor over X. This is a not necessarily closed point of X. The *minimal log discrepancy* (mld) at $x \in X$ is

$$\mathrm{mld}(x, X, \Delta) := \inf_{c_X(E) = x} a(E, X, \Delta)$$

and the minimal log discrepancy along a subvariety $Z \subset X$ is

$$\mathrm{mld}(Z, X, \Delta) := \inf_{x \in Z} \mathrm{mld}(x, X, \Delta)\,.$$

Notice that from the definition we have that

$$Z \subset Z' \Rightarrow \mathrm{mld}(Z, X, \Delta) \geqslant \mathrm{mld}(Z', X, \Delta)\,. \tag{3.1}$$

Frequently we will write $\mathrm{mld}(x)$ and $\mathrm{mld}(Z)$ if there is no danger of confusion. We refer to [Amb99, § 1] for more details.

We collect some basic facts about mlds.

Lemma 3.1. *Let $f\colon X \to Y$ be a proper birational morphism with X normal and $\mathbb{Q}$-Gorenstein. Then*

$$\mathrm{mld}(W, Y, D) = \mathrm{mld}(\pi^{-1}(W), X, \pi^*D - K_{X/Y})\,.$$

Proof. This is [EMY03, Proposition 1.3(iv)]. □

For $k \in \mathbb{N}$, let us denote by $X^{(k)} \subset X$ the subset of points of dimension k endowed with the subspace topology. The dimension of a point $x \in X$ is defined to be the dimension of the Zariski closure of x.

Lemma 3.2. *The function* $\mathrm{mld} := \mathrm{mld}_{(X,\Delta)}\colon X^{(k)} \to \mathbb{R} \cup \{-\infty\}$ *takes only finitely many values.*

Proof. This is [Amb99, Theorem 2.3]. □

3.2 Conjectures about minimal log discrepancies

Ambro and Shokurov have made the following two conjectures about mlds in [Amb99, Sho04]. The importance of these conjectures is that if they are fulfilled, then log flips terminate by the main theorem of [Sho04].

Conjecture 3.3 (ACC). Let $\Gamma \subset [0,1]$ be a DCC-set; that is, all decreasing sequences in Γ eventually become constant. For a fixed integer k the set

$$\Omega_k := \left\{ \operatorname{mld}(Z,X,\Delta) \;\middle|\; \begin{matrix} \dim X = k \\ (X,\Delta) \text{ log pair} \\ Z \subset X \text{ closed subvariety} \\ \operatorname{coeff}(\Delta) \in \Gamma \end{matrix} \right\}$$

is an ACC-set; that is, every increasing sequence $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$ in Ω_k eventually becomes stationary.

Conjecture 3.4 (LSC). Let X be a normal $\mathbb{Q}$-Gorenstein variety, and let Δ be an $\mathbb{R}$-Weil divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Then for each d, the function $\operatorname{mld}_{(X,\Delta)}\colon X^{(d)} \to \mathbb{R}\cup\{-\infty\}$ is lower semi-continuous.

Remark 3.5. If LSC holds on X, then for each $a \in \mathbb{R}$ and $d \in \mathbb{N}$, the set

$$X^{(d)}_{\leqslant a} := \left\{ x \in X^{(d)} \mid \operatorname{mld}(x) \leqslant a \right\}$$

is closed; that is, we have $X^{(d)}_{\leqslant a} = X_{\leqslant a} \cap X^{(d)}$, where $X_{\leqslant a}$ is the closure of $X^{(d)}_{\leqslant a}$ in X. Moreover, $X^{(d)}_a := \{x \in X^{(d)} \mid \operatorname{mld}(x) = a\}$ is open in $X^{(d)}_{\leqslant a}$. All this follows directly from Lemma 3.2, which together with the lower semi-continuity implies that for $x \in X^{(d)}$ there is an open neighborhood $U \subset X^{(d)}$ of x such that

$$\forall\, x' \in U\colon\; \operatorname{mld}(x) \leqslant \operatorname{mld}(x')\,.$$

It is well known and an easy consequence of [Amb99, Proposition 2.5] that lower semi-continuity is equivalent to $\operatorname{mld}\colon X^{(0)} \to \mathbb{R}\cup\{-\infty\}$ being lower semi-continuous. Moreover, by loc. cit. one also sees that ACC holds, as soon as

$$\Omega^{(0)}_k := \left\{ \operatorname{mld}(x,X,\Delta) \;\middle|\; \begin{matrix} \dim X = k \\ K_X+\Delta \ \mathbb{Q}\text{-Cartier} \\ x \in X \text{ closed point} \\ \operatorname{coeff}(\Delta) \in \Gamma \end{matrix} \right\}$$

is an ACC-set.

Next we show that LSC descends along crepant morphisms.

Theorem 3.6. *Let Y be a normal projective $\mathbb{Q}$-Gorenstein variety, and let Δ be an effective $\mathbb{R}$-Cartier divisor on Y such that (Y,Δ) is log canonical. If $\pi\colon X \to Y$ is a proper, crepant morphism and LSC holds on X, then*

$$\operatorname{mld}\colon\; Y^{(0)} \to \mathbb{R}\cup\{\infty\}$$

is lower semi-continuous.

Proof. Let us fix a closed point $y \in Y$ and denote $W := \pi^{-1}(y)$. By Lemma 3.1 we have

$$\operatorname{mld}(y,Y,D) = \operatorname{mld}\big(\pi^{-1}(y),X,\pi^*D\big)\,. \tag{3.2}$$

We have to show that there is an open neighborhood $U \subset Y$ of y such that

$$\operatorname{mld}(y') \geqslant \operatorname{mld}(y) \qquad \forall\, y' \in U\,.$$

To this end we spot the "bad" subsets of Y. By Lemma 3.2 the function mld takes only finitely many values. If $a := \operatorname{mld}(y)$ is the smallest mld on $Y^{(0)}$, then there is nothing to prove. Otherwise, let us denote by b the maximal mld on Y with $b < a$. In view of (3.2), the search for mlds smaller than a can be carried out on X, but at the price of having to take into account not only closed points. Consider for each $0 \leqslant d \leqslant n := \dim X$ the set

$$C_d := \left\{x \in X^{(d)} \mid \dim \pi(x) = 0, \operatorname{mld}(x) \leqslant b\right\}.$$

Let $\overline{C}_d$ denote the Zariski closure of C_d in X. By assumption, (LSC) holds on X and hence all $x \in \overline{C}_d$ with $\dim x = d$ satisfy $\operatorname{mld}(x) \leqslant b$. Now we set

$$U := Y \setminus \bigcup_{d=0}^{n} \pi(\overline{C}_d)\,,$$

where $n = \dim(X)$. As π is proper, U is open. We will consecutively prove the following claims:

(i) Every irreducible component of $\overline{C}_d$ has relative dimension at least d over its image.

(ii) $y \in U$.

(iii) We have $\operatorname{mld}(y') \geqslant \operatorname{mld}(y)$ for all $y' \in U$.

Let Σ denote an irreducible component of $\overline{C}_d$ for some d. As $\overline{C}_d$ is the closure of C_d, the set $C_d \cap \Sigma$ is not empty. Thus, the set $\Sigma_{\geqslant d} := \{x \in \Sigma \mid \dim \pi^{-1}\pi(x) \geqslant d\}$ is not empty. By the upper semi-continuity of the fiber dimension [GD66, Corollaire 13.1.5], the set $\Sigma_{\geqslant d}$ is closed and by definition we have $\Sigma_{\geqslant d} \supset \Sigma \cap C_d$. Therefore, as Σ is a component of the closure of C_d, we have $\Sigma_{\geqslant d} = \Sigma$ and the first claim follows.

Suppose $y \notin U$. Then we would have a point $x \in \overline{C}_d$ for some d with $\pi(x) = y$. By the previous statement, we have $\dim(W \cap \overline{C}_d) \geqslant d$, where, we recall, $W = \pi^{-1}(y)$. This implies that there is a point $x' \in W \cap \overline{C}_d$ with $\dim x' = d$ and hence $\operatorname{mld}(x') \leqslant b$ by the definition of B_d. But then, by (3.1)

$$a = \operatorname{mld}(y) \leqslant \operatorname{mld}(x') \leqslant b\,,$$

contradicting the choice of $b < a$. Thus $y \in U$.

Now if there were some $y' \in U$ with $\operatorname{mld}(y') < \operatorname{mld}(y) = a$, the mld at y' would also be at most b by the maximality of b. Let $x \in \pi^{-1}(y')$ be a point with $\operatorname{mld}(x) = \operatorname{mld}(y')$ and denote $d := \dim(x)$. This would imply $x \in C_d \subset \overline{C}_d$, contrarily to the assumption $y' \in U$. This concludes the proof of the theorem. □

By [EMY03, Theorem 0.3], LSC holds on smooth varieties. This immediately yields the following result.

COROLLARY 3.7. *Let Y be a normal projective $\mathbb{Q}$-Gorenstein variety possessing a crepant resolution of singularities. Let Δ be an $\mathbb{R}$-Weil divisor on Y such that $K_Y + \Delta$ is $\mathbb{R}$-Cartier. Then the function $\operatorname{mld}_{(Y,\Delta)}$ is lower semi-continuous.* □

4. Termination

In this section we prove our main application, namely Theorem 1.2. Notice that in its statement we could also drop the log canonical-assumption on (X, Δ), as thanks to $K_X = 0$ we can rescale

Δ at any time. The proof of the theorem will occupy the rest of the section. Let (X, Δ) be a log pair on a projective irreducible symplectic manifold. By [BCHM10, Corollary 1.4.1], we know that $(K_X + \Delta)$-flips exist and so we may run a $(K_X + \Delta)$-MMP. This produces a sequence

$$X = X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} X_2 \dashrightarrow \cdots , \tag{4.1}$$

where the ϕ_i are either divisorial contractions or flips. Let us denote $\Delta_0 := \Delta$ and $\Delta_i = (\phi_i)_* \Delta_{i-1}$. Note that at each step K_{X_i} will be trivial and therefore we can rescale Δ_i such that (X_i, Δ_i) will be Kawamata log terminal (klt), hence the above result applies. We want to show that (4.1) terminates after a finite number of steps. First we notice the following.

LEMMA 4.1. *Each X_i is a symplectic variety and admits a crepant resolution.*

Proof. By induction we may assume that X_{i-1} is a symplectic variety and has a crepant resolution $\tilde{\pi}\colon \tilde{X}_{i-1} \to X_{i-1}$. The symplecticity of X_i is clear, as the exceptional locus of $X_{i-1} \dashrightarrow X_i$ on X_i has codimension at least 2 and thus the symplectic form from X_{i-1} extends. By [BCHM10, Corollary 1.4.3] there exists a proper birational morphism $\pi\colon \tilde{X}_i \to X_i$ such that $\tilde{X}_i$ has only $\mathbb{Q}$-factorial terminal singularities and π is crepant. Let $X_{i-1} \to Z \leftarrow X_i$ be the flipping contraction. Then the compositions $\tilde{X}_i \to X_i \to Z$ and $\tilde{X}_{i-1} \to X_{i-1} \to Z$ are crepant morphisms and $\tilde{X}_{i-1}$ is smooth, hence by [Nam06, Corollary 1, p. 98], the variety $\tilde{X}_i$ is also smooth. □

Proof of Theorem 1.2. In the course of the MMP, only a finite number of divisorial contractions can occur, so that by the preceding lemma we can reduce to the following situation: $X = X_0$ is a symplectic variety having a crepant resolution, $\Delta = \Delta_0$ is an effective $\mathbb{R}$-divisor on X, and we are given a sequence

$$X = X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} X_2 \dashrightarrow \ldots , \tag{4.2}$$

where ϕ_i is a log flip for the pair (X_i, Δ_i). We will show that such a sequence is finite by using Shokurov's criterion; see [Sho04, Theorem]. In fact, by [Sho04, Addendum 2], we only need LSC for the pairs (X_i, Δ_i) and ACC for the set

$$\Omega(X) := \{\operatorname{mld}(E_i, X_i, \Delta_i) \mid i \in \mathbb{N}\} ,$$

where $E_i \subset X_i$ denotes the exceptional locus of $\phi_i\colon X_i \to X_{i+1}$; see also [HM10, § 3]. Recalling that LSC holds by Corollary 3.7 and Lemma 4.1 above, we are left with ACC. By a theorem of Kawakita [Kaw14], the set of all mlds for a fixed finite set of coefficients on a fixed projective variety is finite. More precisely, let Z be a projective variety, let $\Gamma \subset [0, 1]$ be a finite set, and consider

$$M_\Gamma(Z) := \left\{\operatorname{mld}_{(Z,\Delta)}(x) \mid \operatorname{coeff}(\Delta) \in \Gamma,\ (Z, \Delta) \text{ log canonical at } x \in Z^{(0)}\right\} .$$

Then by [Kaw14, Theorem 1.2], the set $M_\Gamma(Z)$ is finite. By loc. cit. the bigger set

$$M_\Gamma^{\mathrm{loc}}(Z) := \left\{\operatorname{mld}_{(U,\Delta)}(x) \mid \operatorname{coeff}(\Delta) \in \Gamma,\ (U, \Delta) \text{ log canonical at } x, U \subset Z \text{ open}\right\}$$

is also finite. Note that $U \subset Z$ is supposed to be open in the Euclidean topology and that Δ is not supposed to be the restriction of a divisor on Z. By Theorem 1.1 all X_i in the sequence (4.2) are locally trivial deformations of one another (notice that the $\mathbb{Q}$-factoriality is ensured by [KM98, Propositions 3.36 and 3.37]). Hence, $M_\Gamma^{\mathrm{loc}}(X_i)$ is independent of i, as mlds are local invariants. Consequently, $\Omega(X_0) \subset \bigcup_{i=0}^{\infty} M_\Gamma(X_i) \subset M_\Gamma^{\mathrm{loc}}(X_0)$. In particular, $\Omega(X_0)$ is finite and thus an ACC-set and we may conclude the proof. □

Remark 4.2. The observation that equivalence by locally trivial deformations implies ACC has already been made by Nakamura in [Nak16, Corollary 1.4], where he considered the case of terminal quotient singularities.

Remark 4.3. It is tempting to try to deduce the termination of the MMP, proved here, from the termination of flips for irreducible symplectic manifolds. The proof of the latter in [MZ13] is much quicker and, notably, does not require any control at all over the local structure of symplectic singularities that appear along a log MMP, as in the framework of [MZ13] every variety is smooth. Using the notation above, if we start from a $\mathbb{Q}$-factorial symplectic variety $X := X_0$ and a boundary divisor $\Delta := \Delta_0$ and if X admits a crepant resolution $f\colon Y \to X$ together with the natural boundary divisor $\Gamma := f^*\Delta$, then the idea would be the following: Given a log flip $X_0 \dashrightarrow X_1$, one could try to run an MMP for (Y, Γ) in such a way that flips and divisorial contractions are interchanged, that is, the MMP is supposed to produce a sequence of flips (Y_i, Γ_i) for $i = 0, \ldots, N$ and a divisorial contraction $Y_N \to X_1$. Then by [MZ13] the sequence of flips for (Y, Γ) would necessarily be finite; however, it seems very plausible that the MMP on (Y, Γ) might produce divisorial contractions right away, which would disallow the use of [MZ13] and make this strategy useless.

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