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DEFORMATIONS WHICH PRESERVE THE NON-IMMERSIVE LOCUS OF A MAP-GERM

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Introduction.

Singular germs of maps f: $(\mathbf{C}^2, 0) \to (\mathbf{C}^3, 0)$ present three distinct kinds of degeneracy: *non-immersiveness, non-transverse self-intersection*, and *triple self intersection*. In finitely determined germs, these may be measured by the number C of cross-caps into which the singularity breaks up in a stable perturbation, the Milnor number $\mu(D_f^2/Z_2)$ of the quotient, by the natural Z_2-action, of the double-point scheme $D^2 f$, and the number T of ordinary triple points which emerge in a stable perturbation, respectively. In fact, finiteness of all three numbers is equivalent to finite determinacy ([8]), and so is a necessary and sufficient condition for the existence of a versal unfolding ([5], [6]). However, in many contexts, geometrical or other constraints may force one or more of these degeneracies to be infinite. One way of recovering the notion of versal unfolding in such a case is to consider only unfoldings in which the degeneracy in question is left unchanged, and this is our approach here. We introduce the notion of *Σ-trivial unfoldings*: that is, unfoldings in which the non-immersive locus of f is deformed trivially. This generalises a technique used in [9]. Our principal result is an infinitesimal criterion for versality within the category of Σ-trivial unfoldings, for germs of corank 1, Theorem 2.4, and thus in particular an infinitesimal criterion for *Σ-stability*. We apply this in §3 to a variety of examples, and show in particular that the germ parametrising the swallowtail surface of catastrophe theory is Σ-stable; moreover, if $\exp[\gamma]$: $T\mathbf{R} \to \mathbf{R}^3$ is the map parametrising the tangent developable surface of a real analytic space curve, then the germ of $\exp[\gamma]$ at the point $(t, 0) \in T\mathbf{R}$ is Σ-stable if the curvature of γ at t is non-vanishing and the torsion is non-vanishing or vanishes to first order, or if there is a non-degenerate zero of curvature at t. It should be noted that none of these germs has a (finite-dimensional) versal unfolding in the usual sense. Our versality theorem has some unusual technical features; in particular, the space of infinitesimal deformations considered is not a module over the ring of functions on the

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source. In the process of dealing with this difficulty, a canonically defined analytic ring comes to light, leading to a canonical factorisation of the map-germ f. We describe this briefly in §4.

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§1. Preliminary definitions and notation.

Most of our notation is standard in singularity theory, following the usage of Mather in [6] and of Wall in [12], though we prefer Mather's $\theta(f)$ to Wall's $V(f)$.

Let f: $(\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be an analytic map-germ (all maps and germs will be analytic in this paper), let $W \subset J^k(n,p)$ be a subvariety, and suppose that $j^k f(0) \in W$. We say that an unfolding F: $(\mathbf{C}^n \times \mathbf{C}^d, 0) \to (\mathbf{C}^p \times \mathbf{C}^d, 0)$ of f is *W-trivial* if the family

$$\begin{array}{ccc} j^k f^{-1}(W) & \longrightarrow & j^k_x F^{-1}(W) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{C}^d \end{array}$$

is trivial as a deformation of possibly non-reduced spaces. Here $j^k_x F(x,u) = j^k f_u(x)$.

For example, when W is the set of zero 1-jets in $J^1(n, 1)$, our definition is stricter than that of μ-constant deformation – we require that during the unfolding of f, the algebra structure of $\mathcal{O}_n/J_f$ remain constant up to isomorphism.

The W-trivial unfolding F of f is *W-versal* if whenever G is another W-trivial unfolding of f then G is equivalent to an unfolding induced from f. The germ f is *W-stable* if it is its own W-versal unfolding.

We now concentrate on the case where $W = \Sigma^1 \subset J^1(n,p)$, and $n \leqq p$. For brevity we shall write $j^1 f^{-1}(\Sigma^1)$ as $\Sigma^1(f)$. We aim to prove an infinitesimal criterion for Σ^1-versality of Σ^1-trivial unfoldings. This necessitates first a notion of tangent space, and before that, some notation. If F: $(\mathbf{C}^n \times \mathbf{C}^d, 0) \to (\mathbf{C}^p \times \mathbf{C}^d, 0)$ is any unfolding of f: $(\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, with $F(x,u) = (\bar{f}(x,u), u)$, then denote by $\partial_i F$ the partial derivative $\partial \bar{f}/\partial u_i$ evaluated at $u = 0$. Thus, $\partial_i F \in \theta(f)$, the vector space of infinitesimal deformations of f. Then set

$$\theta\Sigma^1(f) = \{\partial_i F \mid F \text{ is a } \Sigma^1\text{-trivial unfolding of } f\}$$

This is the space of infinitesmal Σ^1-trivial deformations of f. A priori it is not even clear that it is a vector space. We will show that it is, and further, that it has a manageable algebraic structure, which will be central in the proof of our versality criterion. One should observe that for a given $f: (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, the set of germs $g: (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ such that $\Sigma^1(f)$ and $\Sigma^1(g)$ are isomorphic, is not a vector or even an affine space, and thus $\theta\Sigma^1(f)$ cannot be identified with this set of germs. Because of this, Damon's general results on versality ([3]) do not apply here.

All map-germs $f: (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ that we consider in this paper will be assumed to be *finite*, and so by the Weierstrass preparation theorem ([4] chapter IV) will induce in $\mathcal{O}_n$ the structure of a finite $\mathcal{O}_p$-module.

§2. Construction of the space of infinitesimal Σ-trivial deformations; statement and proof of main theorem.

Suppose that $j^k f(0) \in W$ and that $\rho: (J^k(n,p), j^k f(0)) \to \mathbf{C}^r$ satisfies $\rho^{-1}(0) = W$ as germs at $j^k f(0)$ of analytic spaces. Then the unfolding F of f is W-trivial if and only if the unfolding $d_w F$ of $\rho \circ j^k f$ defined by

$$d_w F(x, u) = (\rho \circ j^k f_u(x), u)$$

is $\mathscr{K}_e$-trivial (see [6] or [12] for the definition of the contact group $\mathscr{K}$). Since it is easy to calculate the tangent space to the set of $\mathscr{K}_e$ trivial unfoldings, this suggests the method we will use for calculating $\theta\Sigma^1(f)$.

Let us now return to the case $W = \Sigma^1$. Given a map-germ f with $j^k f(0) \in \Sigma^1$, choose ρ as above, and set $\tilde{d}f = \rho \circ df$. For an unfolding F of f, write $\tilde{d}F$ instead of $d_{\Sigma^1} F$. From the preceding remarks, it follows that $\partial_i \tilde{d}F \in T\mathscr{K}_e \tilde{d}f$. We now define a map $\tilde{d}_*: \theta(f) \to \theta(\tilde{d}f)$ by $\tilde{d}_*(\partial_i F) = \partial_i \tilde{d}F$. Then $\theta\Sigma^1(f) \subset \tilde{d}_*^{-1}(T\mathscr{K}_e \tilde{d}f)$. We will shortly prove the opposite inclusion. Observe first that we can describe $\tilde{d}_*$ quite explicitly once a choice of ρ is made. So for simplicity suppose that $df_1(0), \ldots, df_{n-1}(0)$ are linearly independent and take for ρ the map $L: (\mathbf{C}^n, \mathbf{C}^p) \to \mathbf{C}^{p-n+1}$ sending the matrix whose rows are $\alpha_1, \ldots, \alpha_p$ to $(\det(\alpha_1, \ldots, \alpha_n), \det(\alpha_1, \ldots, \alpha_{n-1}, \alpha_{n+1}), \ldots, \det(\alpha_1, \ldots, \alpha_{n-1}, \alpha_p))$. If F is an unfolding of f (which for simplicity of notation we take on one parameter), then one calculates easily that

$$\begin{aligned}\partial_1 \tilde{d}F = (&\det(\hat{\alpha}_1, \alpha_2, \ldots, \alpha_n) + \det(\alpha_1, \hat{\alpha}_2, \ldots, \alpha_n) + \ldots + \det(\alpha_1, \alpha_2, \ldots, \hat{\alpha}_n),\\ &\det(\hat{\alpha}_1, \ldots, \alpha_{n-1}, \alpha_{n+1}) + \ldots + \det(\alpha_1, \ldots, \alpha_{n-1}, \hat{\alpha}_{n+1}), \ldots\\ &\det(\hat{\alpha}_1, \ldots, \alpha_{n-1}, \alpha_p) + \ldots + \det(\alpha_1, \ldots, \alpha_{n-1}, \hat{\alpha}_p))\end{aligned}$$

where $\alpha_i = df_i$ ($f_i = i$th component of f), and $\hat{\alpha}_i = d((\partial_1 F)_i)$. Writing elements of $\theta(f)$ and $\theta(\tilde{d}f)$ as column vectors, we can express the linear operator $\tilde{d}_*$ as a matrix; when $n = 2$, $p = 3$, then it is

$$(*)\begin{pmatrix} \dfrac{\partial f_2}{\partial x_2}\dfrac{\partial}{\partial x_1} - \dfrac{\partial f_2}{\partial x_1}\dfrac{\partial}{\partial x_2} & \dfrac{\partial f_1}{\partial x_1}\dfrac{\partial}{\partial x_2} - \dfrac{\partial f_1}{\partial x_2}\dfrac{\partial}{\partial x_1} & 0 \\ \dfrac{\partial f_3}{\partial x_2}\dfrac{\partial}{\partial x_1} - \dfrac{\partial f_3}{\partial x_1}\dfrac{\partial}{\partial x_2} & 0 & \dfrac{\partial f_1}{\partial x_1}\dfrac{\partial}{\partial x_2} - \dfrac{\partial f_1}{\partial x_2}\dfrac{\partial}{\partial x_1} \end{pmatrix}$$

2.1 LEMMA. *Let $G(x,u) = (g(x,u), u)$ be any d-parameter unfolding of $\tilde{d}f$, and let $\hat{f}^{(1)}, \ldots, \hat{f}^{(d)} \in \theta(f)$ satisfy $\tilde{d}_* \hat{f}^{(i)}) = \partial_i G$. Then there exists a d-parameter unfolding $H(x,u) = (h(x,u), u)$ of f, such that*

(i) $\partial_i H = \hat{f}^{(i)} \quad i = 1, \ldots, d$

(ii) $\tilde{d}H = G$.

PROOF. Choose coordinates for f such that

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, f_n(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n)).$$

For $i = 1, \ldots, n-1$, let $h_i(x,u)$ be any function such that $\partial h_i/\partial u_j(x,0) = \hat{f}_1^{(j)}$, $h_i(x,0) = f_i(x)$. In order that H satisfy (ii), it is necessary and sufficient that for $j = n, \ldots, p$,

$$(1_i) \qquad \det(d_x h_1, \ldots, d_x h_{n-1}, d_x h_j) = g_{i-n+1},$$

(where $d_x h_i$ is the differential of h with respect to the x variable). This is a first order linear PDE in the h_j (since we have already fixed $h_1, \ldots, h_{n-1}$), which we may write in the form

$$\sum_{k=1}^{n} \alpha_k(x,u) \partial h_i/\partial x_k(x,u) = g_{i-n+1}.$$

In order that H be an unfolding of f, and satisfy (i), we require also

$$(1') \qquad \partial h_i/\partial u_j(x,0) = \hat{f}_i^{(j)}(x) \quad j = 1, \ldots \mathrm{d}$$

$$(1'') \qquad h_i(x,0) = f_i(x).$$

Observe that in (1_i), $\alpha_n(x,0) = 1$. By the general theory of linear first order PDE's, the solutions of (1_i) satisfy an ODE along the bicharacteristics $(x(\tau), u(\tau))$ defined by

$$(2) \qquad \begin{aligned} x_k(\tau) &= \alpha_k(x(\tau), u(\tau)) \quad & k = 1, \ldots, n \\ u_j(\tau) &= 0 & j = 1, \ldots, d \end{aligned}$$

To get a well-posed Cauchy problem, we choose as initial hypersurface the hyperplane $x_n = 0$, since it is transverse to the bicharacteristics. Along the bicharacteristics, h_i satisfies

$$(3) \qquad dh_i/d\tau(x(\tau), u(\tau)) = g_{i-n+1}(x(\tau), u(\tau))$$

with initial values $h_i(x_1, \ldots, x_{n-1}, 0, u)$ which, if we ignore (1′) and (1″), can be an arbitrary function $b_i(x_1, \ldots, x_{n-1}, u)$. Once b_i is chosen, the standard existence and uniqueness theorems guarantee the existence of a solution. We must show that by choosing b_i appropriately, the solution will satisfy also (1′) and (1″). Clearly, we must set

(4) $$b_i(x_1,\ldots,x_{n-1},0) = f_i(x_1,\ldots,x_{n-1},0)$$

(5) $$\partial b_i/\partial u_j(x_1,\ldots,x_{n-1},0) = f_i^{(j)}(x_1,\ldots,x_{n-1},0).$$

This presents no difficulty. Now restrict to the bicharacteristic through $(x_1,\ldots,x_{n-1},0,0)$, which is simply $(x_1,\ldots,x_{n-1},\tau,0)$; we find that the solution of (3) with initial conditions (4) and (5) automatically satisfies (1″), since G is an unfolding of df, and $g_{i-n+1}(x,0) = \partial f_i/\partial x_n(x)$.

Now differentiate (1_i) with respect to u_j and set $u = 0$. Note that for $k = 1,\ldots,n-1$, $\partial h_k/\partial u_j(x,0) = f^{(j)}(x)$ and $h_k(x,0) = f_k(x)$, by our choice of the h_k, and that we have established that $h_i(x,0) = f_i(x)$. We obtain a differential equation in which the only unknown quantity is $\partial^2 h_i/\partial x_n \partial u_j(x,0)$. Along the bicharacteristics through $(x_1,\ldots,x_{n-1},0,0)$ this can be regarded as an ODE in $\partial h_i/\partial u_j(x,0)$ with initial condition $\partial h_i/\partial u_j(x_1,\ldots,x_{n-1},0,0) = \hat{f}_i^{(j)}(x_1,\ldots,x_{n-1},0)$ (by (4)). However, this is precisely the differential equation satisfied by $\hat{f}_i^{(j)}$, by definition of the operator $\tilde{d}_*$ (this is the first time we invoke the hypothesis that $\tilde{d}_*(f^{(j)}) = \partial_j G$). Thus, the uniqueness of the solutions of ODE's with initial condition, guarantees that (1′) is satisfied, and this completes the proof, relative to the choice of coordinates made at the outset. However, it is clear that the statement of the Lemma is independent of choice of coordinates.

We will refer to the unfolding H constructed above as a *lifting* of G.

2.2 PROPOSITION. $\theta\Sigma^1(f) = \tilde{d}_*^{-1}(T\mathscr{K}_e\tilde{d}f)$.

PROOF. Given $\hat{g} \in T\mathscr{K}_e\tilde{d}f$, there exists a 1-parameter $\mathscr{K}_e$-trivial unfolding G of $\tilde{d}f$, with $\partial_1 G = \hat{g}$. By 2.1, for any element $\hat{f} \in \theta(f)$ such that $\tilde{d}_*(\hat{f}) = \hat{g}$, there exists a 1-parameter unfolding F of f such that $\partial_1 F = \hat{f}$, $\tilde{d}F = G$. As G is $\mathscr{K}_e$-trivial, F is Σ^1-trivial.

Observe that we have now established that $\theta\Sigma^1(f)$ is a vector space. However it is not in general an $\mathcal{O}_n$-submodule of $\theta(f)$, since $\tilde{d}_*$ is not $\mathcal{O}_n$-linear.

2.3 PROPOSITION. *$\theta\Sigma^1(f)$ is an $\mathcal{O}_p$-module of $\theta(f)$ via f^*.*

PROOF. Suppose that $\hat{f} \in \theta\Sigma^1(f)$ and $r \in \mathcal{O}_p$. Then we have

$$\tilde{d}_*((r\circ f)\hat{f}) = (r\circ f)\tilde{d}_*(\hat{f}) + f_1\tilde{d}_*((r\circ f)\partial/\partial X_1) + \ldots + f_p\tilde{d}_*((r\circ f)\partial/\partial X_p).$$

As $T\mathscr{K}_e\tilde{d}f$ is an $\mathcal{O}_n$-module, we need only show that for each i, $\tilde{d}_*((r\circ f)\partial/\partial X_i) \in T\mathscr{K}_e\tilde{d}f$. Denote by $\partial/\partial Y_k$, $k = 1,\ldots,p-n+1$, the standard generators of $\theta(\tilde{d}f)$ over $\mathcal{O}_n$. Then $\tilde{d}_*((r\circ f)\partial/\partial X_i)$ is a sum of terms of the form

$$\det(df_{i_1},\ldots,df_{i_{j-1}},d(r\circ f),df_{i_{j+1}},\ldots,df_{i_n})\partial/\partial Y_k.$$

By the chain rule, each of these is contained in $(\tilde{d}f)^* \mathcal{M}_{p-n+1} \cdot \theta(\tilde{d}f)$, and this is itself contained in $T\mathcal{K}_e\tilde{d}f$.

As we are assuming that f is a finite mag-germ, then by the Weierstrass Preparation Theorem, $\theta(f)$ is a finite $\mathcal{O}_p$-module via f^*, and hence since $\mathcal{O}_p$ is Noetherian, by 2.3 $\theta\Sigma^1(f)$ also is a finite $\mathcal{O}_p$-module.

Now we come to the main result:

2.4 THEOREM. *Let f: $(\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a finite map-germ with a singularity of type Σ^1 at 0. Then a Σ^1-trivial d-parameter unfolding F of f is Σ^1-versal if and only if*

$$T\mathcal{A}_e f + \mathbf{C}\langle \partial_1 F, \ldots, \partial_d F\rangle = \theta\Sigma^1(f). \tag{1}$$

(*The second term on the left is the* C*-vector-space generated by the* $\partial_i F$).

PROOF. "Only if" is straightforward; the proof is almost identical to the corresponding part of the versality theorem on page 189 of [5], though note that we use the symbols $T\mathcal{A}_e f$ and $\bar{T}\mathcal{A}_e F$ where Martinet uses Tf and $\bar{T}F$; starting with a 1-parameter Σ^1-trivial unfolding G of f, the supposition that G is equivalent to an unfolding induced from F implies that $\partial_1 G$ belongs to the left hand side of (1). "If" also follows closely the scheme of the proof in [5], but there are important differences. We need

2.5 LEMMA. *Let F: $(\mathbf{C}^n \times \mathbf{C}^d, 0) \to (\mathbf{C}^p \times \mathbf{C}^d, 0)$ and G: $(\mathbf{C}^n \times \mathbf{C}^b, 0) \to (\mathbf{C}^p \times \mathbf{C}^b, 0)$ be Σ^1-trivial unfoldings of f. Then there is a Σ^1-trivial unfolding, L: $(\mathbf{C}^n \times \mathbf{C}^d \times \mathbf{C}^b, 0) \to (\mathbf{C}^p \times \mathbf{C}^d \times \mathbf{C}^b, 0)$, such that $F = i*L$ and $G = j^* L$, where i: $\mathbf{C}^d \to \mathbf{C}^d \times \mathbf{C}^b$ and j: $\mathbf{C}^b \to \mathbf{C}^d \times \mathbf{C}^b$ are the usual embeddings.*

PROOF. $\tilde{d}F$ and $\tilde{d}G$ are both $\mathcal{K}_e$-trivial unfoldings of $\tilde{d}f$, and we may embed them both into a $\mathcal{K}_e$-trivial unfolding R of $\tilde{d}f$, such that $\tilde{d}F = i^* R$ and $\tilde{d}G = j^* R$, as follows: suppose that $\tilde{d}F(x, u) = (M(x, u) \cdot \tilde{d}f(\varphi_u(x)), u)$, $\tilde{d}G(x, v) = (N(x, v) \cdot \tilde{d}f(\psi_v(x)), v)$, where $M(x, u) \in \mathrm{Gl}(p - n + 1, \mathcal{O}_{n+d})$, $N(x, v) \in \mathrm{GL}(p - n + 1, \mathcal{O}_{n+b})$, $M(x, 0) = N(x, 0) =$ Identity matrix, and $\Phi(x, u) = (\varphi_u(x), u)$ and $\Psi(x, v) = (\Psi_v(x), v)$ are diffeomorphisms with φ_0 and ψ_0 both equal to the identity. Then set

$$R(x, u, v) = (M(x, u).\ N(x, v).\ \tilde{d}f(\varphi_u(\psi_v(x))), u, v)$$

The result now follows by 2.1 (observe that the proof of 2.1 allows us to assume that the unfoldings F and G are embedded in the lifting of R).

In order to show that a Σ^1-trivial unfolding F satisfying (1) is Σ^1-versal, following Martinet we take any other Σ^1-trivial unfolding G, form a "direct sum" unfolding H as in 2.5, and then show that H is equivalent to an unfolding induced from F. Martinet's proof consists of two steps, the geometric lemma ([5] XIV.2) and the algebraic lemma ([5] XIV.3). Our proof differs only in the algebraic lemma; in order to apply the geometric lemma we need

2.6 LEMMA. *Let* $H: (\mathbf{C}^n \times \mathbf{C}^r, 0) \to (\mathbf{C}^p \times \mathbf{C}^r, 0)$ *be a* Σ^1*-trivial unfolding of* f: $(\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ *such that*

$$T\mathscr{A}_e f + \mathbf{C}\langle \partial_1 H, \ldots, \partial_{r-1} H\rangle = \theta\Sigma^1(f).$$

(*Note the difference between this and* (1)). *Then there exist vector fields*

$$X = \partial/\partial u_r + \sum_{i=1}^{r-1} \xi_i(u)\partial/\partial u_i + \sum_{j=1}^{n} X_j(x,u)\partial/\partial x_j$$

$$Y = \partial/\partial u_r + \sum_{i=1}^{r-1} \xi_i(u)\partial/\partial u_i + \sum_{j=1}^{p} Y_j(y,u)\partial/\partial y_j$$

such that $DH(X) = Y \circ H$.

PROOF. The equation $DH(X) = Y \circ H$ can be solved if and only if

$$\partial\bar{h}/\partial u_d \in \bar{T}\mathscr{A}_e H + \mathcal{O}_d\{\partial\bar{h}/\partial u_1, \ldots, \partial\bar{h}/\partial u_{d-1}\},$$

where $\bar{T}\mathscr{A}_e H = \mathcal{O}_{n+d}\{\partial\bar{h}/\partial x_1, \ldots, \partial\bar{h}/\partial x_n\} + \mathcal{O}_{p+d}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\}$ is just the parametrised version of $T\mathscr{A}_e f$. Now, let $D_*: (\mathcal{O}_{n+d})^p \to (\mathcal{O}_{n+d})^{p-n+1}$ be the parametrised version of $\tilde{d}_*$, obtained from $\tilde{d}_*$ by replacing terms involving the components of f by the corresponding components of $\bar{h}$ (so that $\tilde{d}_*$ is the restriction to $\{u = 0\}$ of D_*). Clearly $D_*(\partial\bar{h}/\partial u_d) \in \bar{T}\mathscr{K}_e\bar{d}H$, where $\bar{T}\mathscr{K}_e\bar{d}H = \mathcal{O}_{n+d}\{\partial\bar{h}/\partial x, \ldots, \partial\bar{h}/\partial x_n\} + H^*\mathscr{M}_{p+d}\cdot\mathcal{O}_{p+d}\{\partial/\partial y_1, \ldots, \partial/\partial y_p\}$, and so we complete the proof with

2.7 LEMMA. *If* H *is an r-parameter* Σ^1*-trivial unfolding of* f, *and if*

(1′) $$T\mathscr{A}_e f + \mathbf{C}\langle \partial_1 H, \ldots, \partial_{r-1} H\rangle = \theta\Sigma^1(f).$$

then

(3) $$\bar{T}\mathscr{A}_e H + \mathcal{O}_d\{\partial\bar{h}/\partial u_1, \ldots, \partial\bar{h}/\partial u_{r-1}\} = D_*^{-1}(\bar{T}K_e\bar{d}H).$$

PROOF. (Compare the proof of XIV.3 in [5]). The proof of 2.3 shows that $D^{*-1}(\bar{T}\mathscr{K}_e\bar{d}H)$ is an $\mathcal{O}_{p+r}$-module via H^*, which is finitely generated since H is a finite map-germ. If $\pi: \mathbf{C}^{p+r} \to \mathbf{C}^d$ is the standard projection, and we define the $\mathcal{O}_{p+d}$-module M by

$$M = D_*^{-1}(\bar{T}\mathscr{K}_e\bar{d}H)/\bar{T}\mathscr{A}_e H$$

then it is a straightforward matter of checking to see that

(2) $$M/\pi^*\mathscr{M}_r \cdot M = \theta\Sigma^1(f)/T\mathscr{A}_e f.$$

Since M is a finite $\mathcal{O}_{p+r}$-module via H^*, (1′) and (2), together with Nakayama's Lemma, imply that (3) holds.

The proof of 2.4 now proceeds exactly as does the proof on page 195 in [5], as described before Lemma 2.6. If F satisfies (1), and G is another unfolding, say on b parameters, then (1′) is satisfied by the direct sum unfolding H (taking $r = d + b$), since $\partial_i H = \partial_i F$, for $i = 1, \ldots, d$. Then by 2.6, 2.7 and Martinet's geometric lemma, H is equivalent to the $(d + b - 1)$-parameter unfolding $t^* H_i$, where $t\colon \mathbf{C}^{d+b} \to \mathbf{C}^{d+b-1}$ forgets the last coordinate and H_1 is obtained from H by restriction of the parameter space to $\mathbf{C}^{d+b-1} \times \{0\}$. Lemmas 2.6 and 2.7 now apply to H_1, and so H_1 is equivalent to the pull-back of the unfolding H_2 obtained from H_1 by restricting the last of its parameters to 0; inductively, one deduces that H is equivalent to the pull-back of F, and since G is a pull-back of H ($G = j^* H$), this proves the versality of F in the category of Σ^1-trivial unfoldings of f.

§3. Examples.

1. If f is finitely $\mathscr{A}$-determined, then df is finitely $\mathscr{K}$-determined, for by the Gaffney-Mather geometric criterion for finite determinacy ([12] page 492), $j^1 f$ must be transverse to Σ^1 off 0; that is, $\tilde{d}f$ must be a submersion at every point of $\tilde{d}f^{-1}(0) \setminus \{0\}$, and this condition is equivalent to the finite $\mathscr{K}$-determinacy of $\tilde{d}f$. It follows that $\theta\Sigma^1(f)$ is of finite codimension in $\theta(f)$. This follows also more directly from the fact that $\theta(f) \supset \theta\Sigma^1(f) \supset T\mathscr{A}_e f$. Since $\dim(\theta(f)/\theta\Sigma^1(f)) = \dim(\theta(df)/T\mathscr{K}_e df)$, by 2.4 a necessary and sufficient condition for Σ^1-stability is that $\operatorname{cod}(\mathscr{A}_e, f) = \operatorname{cod}(\mathscr{K}_e, \tilde{d}f)$. For the germs listed in [7] and [8], this holds only for

$$S_k\colon (x, y) \to (x, y^2, y^3 + x^{k+1} y) \quad k \geqq 0$$

$$X_4\colon (x, y) \to (x, y^3 - x^2 y, xy^2 + y^4),$$

and for those listed in [10], it holds only for

$$(x, y) \to (x, y^3 + x^{k+1} y), \quad k \geqq 0$$

2. Let $f(x, y) = (x, xy + y^{3k-1}, y^3)$ (H_k in the notation of [7]). Then $\tilde{d}f(x, y) = (x + (3k - 1)y^{3k-2}, 3y^2)$, so $\operatorname{cod}(\mathscr{K}_e, \tilde{d}f) = 1$. The unfolding

$$F(x, y, u_1, \ldots, u_{k-1}) =$$
$$= (x, xy + y^{3k-1} + u_1 y^2 + u_2 y^5 + \ldots + u_{k-1} y^{3k-4}, y^3, u_1, \ldots, u_{k-1})$$

is Σ^1-trivial, and since an $\mathscr{A}_e$-versal unfolding of f requires one extra parameter (add $u_k y$ to the third component), it follows that F is Σ^1-versal. In this unfolding, the $k - 1$ virtual triple points of f (see [8], page 374) are realised, in the sense that for generic parameter values u, f_u has $k - 1$ distinct triple points. This may be shown by calculating the equations of the triple-point variety of F in the target $\mathbf{C}^3 \times \mathbf{C}^{k-1}$, (using for example the second Fitting ideal of a presentation of

$fF_*(\mathcal{O}_{\mathbf{C}^{n+k-1}})$, which turn out to be

$$X = Y = u_1 + u_2 Z + \ldots + u_{k-1} Z^{k-1} = 0,$$

with respect to coordinates X, Y, Z on $\mathbf{C}^3$.

3. The germs $f_1(x, y) = (x, y^2, y^3)$, $f_2(x, y) = (x, y^2, xy^3)$, $f_3(x, y) = (x, y^3 + xy, y^4 + \frac{2}{3}xy^2)$, $f_4(x, y) = (x, y^3 - x^2 y, y^4 - \frac{2}{3}x^2 y^2)$ are all Σ^1-stable. We leave the proof for f_1 and f_2 as an exercise in the definitions: using the matrix (*) one shows first that $\theta\Sigma^1(f) = \mathcal{O}_2 \partial/\partial X + \mathcal{O}_2 \partial/\partial Y + \{\mathbf{C}\{x, y^2\} + y^3 \mathbf{C}\{x, y^2\}\partial/\partial Z$, and then calculates $T\mathscr{A}_e f_i$ using 4.1:16(ii) of [7]. For f_4, we now sketch the proof: if $g(x, y) = (x, y^3 - x^2 y)$ is obtained from f by forgetting the last component, then $T\mathscr{A}_e g = \mathcal{O}_2 \partial/\partial X + \{\mathcal{O}_2 - \{y\}\}\partial/\partial Y$, so every element $\hat{f} \in \theta(f)$ satisfies an equation

$$\hat{f} \equiv \alpha y \partial/\partial Y + c\partial/\partial Z \ (\mathrm{mod}\ T\mathscr{A}_e f)$$

where $\alpha \in C$ and $c \in \mathcal{O}_2$. Now a calculation shows easily that $\hat{f} \in \theta\Sigma^1(f)$ if and only if $\alpha = 0$ and $c \in R_0$, where $R_0 = \{r \in \mathcal{O}_2 \mid \partial r/\partial y \in (3y^2 - x^2)\}$. Now R_0 is an $\mathcal{O}_3$-submodule of $\mathcal{O}_2$ via f^*, and finite, as f is clearly finite. So by Nakayama's Lemma, we need only check that

$$R_0 \partial/\partial Z \subset T\mathscr{A}_e f + \mathscr{M}_3 \cdot R_0 \partial/\partial Z.$$

By using the Division Theorem ([4] chapter IV), one sees easily that $R_0 = I^2 + f^{-1}(\mathcal{O}_3)$ (where $I = (3y^2 - x^2)$), and since $f^{-1}\mathcal{O}_3 \partial/\partial Z \subset T\mathscr{A}_e f$, we need only check that

$$I^2 \partial/\partial Z \subset T\mathscr{A}_e f + \mathscr{M}_3 \cdot I^2 \partial/\partial Z.$$

Now $\mathscr{M}_3 \cdot I^2 \supset (x, y^3) I^2$, and therefore we need only find $(3y^2 - x^2)^2 \partial/\partial Z$, $y(3y^2 - x^2)^2 \partial/\partial Z$ and $y^2(3y^2 - x^2)^2 \partial/\partial Z$ in $T\mathscr{A}_e f$. In fact these are, respectively, equal to $\omega f(\{9Y + X^4\}\partial/\partial Z)$, $tf(\{-9/4(y^3 - x^2 y)\}\partial/\partial x + 3/2\, y^2 \partial/\partial y) + \omega f((9/4) Y\partial/\partial X - Z\partial/\partial Y - X^2 Y \partial/\partial Z)$ and $\omega f(\{9Y^2 + 12X^2 Z)\partial/\partial Z)$.

The proof of the Σ^1-stability of f_3 closely resembles this last proof, except that it is slightly simpler, since in this case the map g obtained by forgetting the last component of f_3 is stable.

The map-germs f_1, f_2 f_3 and f_4 are all defined over $\mathbf{R}$, and f_1, f_2 and f_4 are in fact equivalent to the germ at points $(t_0, 0) \in T\mathbf{R}$ of the map $\exp[\gamma]$ parametrising the tangent developable surface of a space curve, where, respectively, the curvature and torsion of γ at t_0 are both non-zero, curvature is non-zero and torsion vanishes to first order at t_0, and there is a non-degenerate zero of curvature at t_0, see [2], [9], [11]. The germ f_3 has as image the *swallowtail surface* (the discriminant of a versal unfolding of an A_4 singularity) familiar from catastrophe theory.

Our theorems on Σ-stability and versality apply without change in the real analytic category, but in order to extend them to the C^∞ category, it is necessary to prove that $\theta\Sigma^1(f)$ is a finite module over the ring $\mathscr{E}_p$ of smooth function germs on $(\mathbf{R}^p, 0)$. The corresponding result in the analytic case follows from the fact that $\mathcal{O}_p$ is Noetherian, and the Noetherian property fails for rings of C^∞ function germs. However, Edward Bierstone has informed me that finiteness of $\theta\Sigma^1(f)$ over $\mathscr{E}_p$ can be proved using the results of [1].

§4. The ring $R_0(f)$ and the canonical factorisation.

In the proof of 2.3 (that $\theta\Sigma^1(f)$ is an $\mathcal{O}_p$-module via f^*), it was in fact shown that $\theta\Sigma^1(f)$ is closed under multiplication by elements of the set

$$R_0(f) = \{g \in \mathcal{O}_n \mid \det(df_{i_1}, \ldots, df_{i_{n-1}}, dg) \in (df)^*(I_\Sigma) \cdot \mathcal{O}_n \text{ for all } 1 \leqq i_1 < \ldots < i_{n-1} \leqq p\}.$$

Here I_Σ is the ideal generated by the $n \times n$ minors of the generic $n \times p$ matrix. The smooth part of the variety it defines is of course the submanifold Σ^1 of $L(n, p)$. In fact $R_0(f)$ is a local subring of $\mathcal{O}_n$, with maximal ideal $\mathcal{M}_n \cap R_0(f)$, over which, as we have seen, $\theta\Sigma^1(f)$ is a module. We list some of its further properties, including a more succint characterisation, in

4.1 Proposition. *Let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be an analytic map-germ (not necessarily of corank 1), and suppose $n \leqq p$. Then*

(i) $R_0(f) = \{g \in \mathcal{O}_n \mid dg \wedge f^{-1}\Omega^{n-1}(\mathbf{C}^p) \subset f^{-1}\Omega^n(\mathbf{C}^p)\mathcal{O}_n\}$.

(ii) *If f is of corank 1, then* $R_0(f) = \{g \in \mathcal{O}_n \mid \partial g \in \partial(f^{-1}\mathcal{O}_p)\mathcal{O}_n$ for all $\mathbf{C}$-linear derivations $\partial: \mathcal{O}_n \to \mathcal{O}_n\}$.

(iii) *If $f_1 = \psi \circ f_2 \circ \varphi$ (with ψ, φ diffeomorphisms), then $R_0(f_1) = \varphi^{-1}(R_0(f_2))$.*

(iv) *$R_0(f)$ is an analytic ring: if $g_1, \ldots, g_s \in R_0(f)$ and $g = (g_1, \ldots, g_s): (\mathbf{C}^n, 0) \to (\mathbf{C}^s, 0)$, then $g^{-1}\mathcal{O}_s \subset R_0(f)$. Moreover, $R_0(f)$ is finitely generated as an analytic ring.*

Proof. Parts (i), (ii) and (iii) are straightforward, as is the fact that $R_0(f)$ is an analytic ring. Its finite generation follows from the fact that since f is finite, $\mathcal{O}_n$ is a finite $f^{-1}\mathcal{O}_p$ module, and therefore its $\mathcal{O}_p$-submodule $R_0(f)$ is also finite. A fortiori, $R_0(f)$ itself is also Noetherian. If $g_1, \ldots, g_s$ generate the maximal ideal in $R_0(f)$ (as an ideal), then since $R_0(f) \supset f^{-1}(\mathcal{O}_p)$, we have $(g_1, \ldots, g_s)\mathcal{O}_n \supset f^*\mathcal{M}_p\mathcal{O}_n$, and so from the finiteness of f we deduce the finiteness of g. It follows that $R_0(f)$ is a finite $\mathcal{O}_s$-module via g^*, and hence, by Nakayama's Lemma, that $R_0(f) = g^{-1}\mathcal{O}_s$, since clearly $R_0(f) = g^{-1}\mathcal{O}_s + g^*\mathcal{M}_s R_0(f)$.

The last property leads to the canonical factorisation of the title of this section: if as in the proof of 4.1 (iv), $R_0(f) = g^{-1}(\mathcal{O}_s)$, then in particular $f^{-1}(\mathcal{O}_p) \subset g^{-1}(\mathcal{O}_s)$, and so there exist analytic functions $h_1, \ldots, h_p \in \mathcal{O}_s$ such that, setting

$h = (h_1, \ldots, h_p)$, we have $f = h \circ g$. Maps g constructed in this way have a special property:

4.2 PROPOSITION. *Let f and g be as described in the preceding paragraph. Then $R_0(g) = g^{-1}(\mathcal{O}_s)$.*

PROOF. This follows from 4.1 (ii); for evidently, by 4.2(ii), for any C-derivation $\partial: \mathcal{O}_n \to \mathcal{O}_n$, $\partial(R_0(f))\mathcal{O}_n = \partial(f^{-1}\mathcal{O}_p)\mathcal{O}_n$.

It would be interesting to characterise the maps h arising in the paragraph preceding 4.2.
Our final observation on this topic is

4.3 PROPOSITION. *If f is a finite map-germ of corank 1, then $R_0(f)$ separates points, and thus the first map in the canonical factorisation is injective.*

PROOF. We show that if $I \subset \mathcal{O}_n$ is any proper ideal not contained in $(x_1, \ldots, x_{n-1})$ and $R(I) = \{g \in \mathcal{O}_n \mid \partial g/\partial x_n \in I\}$, then R separates points. The conclusion then follows, for once a map-germ f of corank 1 is written in linearly adapted coordinates $f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, f_n((x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n))$, $R_0(f) = R(I)$ with $I = (\partial f_n/\partial x_n, \partial f_{n+1}/\partial x_n, \ldots, \partial f_p/\partial x_n)$. The finiteness of f is then equivalent to $I \not\subset (x_1, \ldots, x_{n-1})$.

It will be convenient to write y instead of x_n, and to denote points in $\mathbf{C}^n$ in the form (x, y).

If $I \subset J$ then $R(I) \subset R(J)$, so if $R(I)$ separates points, so also does $R(J)$. Therefore we may assume that I is principal, generated by $p(x, y)$. We may also assume that $p \notin (x_1, \ldots, x_{n-1})$, and then by the division theorem that p is a polynomial $y^k + c_{k-1}(x)y^{k-1} + \ldots + c_0(x)$. Evidently all functions of $x_1, \ldots, x_{n-1}$ alone lie in $R(I)$, so it is necessary to show only that $R(I)$ separates points with equal x coordinates. Let $U \subset \mathbf{C}^{n-1}$ be a neighbourhood of 0 in which all of the coefficients c_i are holomorphic, let $x_0 \in U$, and let y_1, y_2 and z_1, z_2 be two pairs of distinct points in C. Then there exists $g \in \mathbf{C}[y]$ such that $g'(y)$ is divisible in $\mathbf{C}[y]$ by $p(x_0, y)$, and $g(y_i) = z_i$. The existence of such a polynomial g is an easy exercise in polynomial interpolation: the condition on g' amounts to no more than a condition on its divisor, that $v(g') \geqq v(p(x_0, -)$ at each of the zeros of $p(x_0, -)$, and to this finite condition is appended the condition on the values of g at y_1 and y_2. Now, by the division algorithm, we may write any polynomial $g(y)$ in the form

$$(1) \qquad g(y) = q(y)p(x_0, y)^2 + \alpha_0 + \alpha_1 y + \ldots + \alpha_k y^k +$$

$$+ \alpha_{k+1} \int p(x_0, y)\, dy + \alpha_{k+2} \int yp(xp(x_0, y)dy + \ldots + \alpha_{2k-1} \int y^{k-2} p(x_0, y) dy,$$

where we integrate formally, i.e. $\int y^j dy = y^{j+1}/j + 1$. Upon differentiation, it is clear that in order that g' be divisible by $p(x_0, y)$, we must have $\alpha_1 = \ldots = \alpha_k = 0$. We conclude that the polynomial $g(y)$ whose existence was invoked in the previous paragraph, must have this special form (with $\alpha_1 = \ldots = \alpha_k = 0$). Then g can be extended to a function $\tilde{g}$ on $U \times \mathbf{C}$ by allowing x to vary in the defining equation (1). Evidently, $\tilde{g} \in R(I)$, and $\tilde{g}(x_0, y_1) \neq \tilde{g}(x_0, y_2)$. This completes the proof.

REFERENCES

1. E. Bierstone and P. Milman, *Relations among analytic functions*, I, Ann. Inst. Fourier 37 (1987) No. 1, 187–239, II, Ann. Inst. Fourier (1987) No. 2, 49–75.
2. J. Cleave, *The form of the tangent developable at points of zero curvature on space curves*, Math. Proc. Cambridge Philos. Soc. 88 (1980), 403–407.
3. J. Damon, *Unfolding and determinacy theorems for subgroups of A and K*, Mem. Amer. Math. Soc. 50, No. 306 (1984).
4. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer, Berlin, 1973.
5. J. Martinet, *Singularities of smooth functions and mappings*, London Math. Soc. Lecture Notes 58, Cambridge University Press, Cambridge, 1982.
6. J. Mather, *Stability of C^∞-mappings* III: Finitely determined map-germs, Publ. Math. I.H.E.S. 35, 127–156.
7. D. Mond, *On the classification of germs of maps from R^2 to R^3*, Proc. London Math. Soc. 50 (1985), 333–369.
8. D. Mond, *Some remarks on the geometry and classification of germs of maps from surfaces to 3-space*, Topology 26 (1987), 361–383.
9. D. Mond, *Singularities of the tangent developable of a space-curve*, Quart. J. Math. Oxford ser. (2) 40 (1989), 79–91.
10. J. Rieger, *Families of maps from the plane to the plane*, Proc. London Math. Soc. 36 (1987), 351–369.
11. O. P. Scherbak, *Projectively dual space curves and Legendre singularities*, Trudy Tbilisskogo Universiteta (= Proceedings of Tbilisi University), Vol. 232–233, series Mathematics, Mechanics, Astronomy, 1982, 280–336 (in Russian).
12. C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. 13 (1981), 481–539.

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