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Deformed noncommutative tori

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We recall a construction of non-commutative algebras related to a one-parameter family of (deformed) spheres and tori, and show that in the case of tori, the $*$-algebras can be completed into C^*-algebras isomorphic to the standard non-commutative torus. As the former was constructed in the context of matrix (or fuzzy) geometries, it provides an important link to the framework of non-commutative geometry, and opens up for a concrete way to study deformations of non-commutative tori. Furthermore, we show how the well-known fuzzy sphere and fuzzy torus can be obtained as formal scaling limits of finite-dimensional representations of the deformed algebras, and their projective modules are described together with connections of constant curvature.
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I. INTRODUCTION

Over the last decades, there has been an increasing interest in non-commutative geometries and many applications in mathematics, as well as mathematical physics, have emerged. However, it is fair to say that there is no single framework for a non-commutative version of geometry, although there are several interesting ones; one of the more well-known theories was presented by Alain Connes.[12] Another approach, which has been frequently applied in physics, is the concept of "fuzzy spaces", where the algebra of smooth functions on a manifold is replaced by a sequence of matrix algebras (of increasing dimension). This was first introduced in Ref. 16, in the case of S^2, where it was used to regularize Membrane Theory, and it was later extended to the torus.[15,17] Several constructions have been used to establish existence of such matrix analogues for arbitrary surfaces (and manifolds), e.g., Refs. 8, 9, and 18, and, in particular, it was proven that the construction of fuzzy spaces is possible for every (quantizable) compact Kähler manifold.[10] Even though the proof in Ref. 10 is constructive, it provides no practical means, with which explicit realizations can be constructed (apart from the case of the torus). Moreover, it does not provide any insight about differential and metric properties of these fuzzy spaces. Such investigations were later undertaken and culminated in a non-commutative matrix geometry (cf. Refs. 14, 19, and 20).

However, only a few explicit examples were around (mainly the fuzzy sphere and the fuzzy torus), and it was not satisfactorily understood how geometry and topology presented themselves in sequences of matrix algebras. For instance, the constructions giving the fuzzy sphere and the fuzzy torus are very different in nature. In Refs. 1 and 2 explicit sequences of matrix algebras were constructed as fuzzy analogues of surfaces described as inverse images of polynomials in $\mathbb{R}^3$. In particular, a one parameter family of surfaces, interpolating between spheres and tori, was considered and all finite dimensional (Hermitian) representations of the corresponding non-commutative algebras were found and classified. For the first time, one could explicitly study how representation theory reflects the topology of surfaces and how smooth deformations of the geometry induce smooth changes in the representations (see also Refs. 5 and 6).

Although there is (yet) no final definition of non-commutative geometry, there is a very natural generalization of topology to the non-commutative setting. Namely, via the Gelfand representation,

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commutative C^*-algebras can be shown to be in correspondence with function algebras (of continuous functions) on locally compact Hausdorff spaces. Therefore, one may consider C^*-algebras as the analogue of (in general non-commutative) algebras of continuous functions on topological spaces. One of the most well known examples is the non-commutative torus.[11] It is defined as the universal C^*-algebra generated by two unitary elements U, V satisfying the relation $VU = qUV$ for some $q \in \mathbb{C}$ with $|q| = 1$.

In this note, we shall investigate the relation of the construction of the fuzzy tori in Refs. 1 and 2 to the topological non-commutative torus defined above. After all, even though they arise from different tori (in a metric sense; the first is deformed and the latter is round), they should provide the same topology, if these concepts are compatible. This comparison is *a priori* not well defined, since the deformed tori do not have a C^*-algebraic structure; there is no natural norm. However, we shall prove that it is possible to introduce a C^*-norm in which the completion is isomorphic to the non-commutative torus. We believe that this introduces an important link between two distinct frameworks, as well as indicating that they are compatible from a topological point of view.

We start by recalling the notion of a fuzzy manifold and give a short description of the two most well-known examples: the fuzzy sphere and the fuzzy torus. Then we review the construction of the non-commutative algebras related to (deformed) spheres and tori, and a basis for these algebras is constructed, in which one can complete the deformed torus algebras into C^*-algebras isomorphic to the standard non-commutative torus. Finally, we consider projective modules (as well as finite dimensional representations) of the non-commutative torus in this framework, together with connections of constant curvature.

II. THE FUZZY SPHERE AND THE FUZZY TORUS

In general, a fuzzy analogue of a Poisson manifold M can be thought of as a sequence of maps $\{T_\alpha\}_{\alpha=1}^{\infty}$, where each T_α is a linear map from $\mathcal{A} = C^\infty(M)$ (or, perhaps, a subalgebra) to the set of Hermitian $N_\alpha \times N_\alpha$ matrices, such that N_α increases as α tends to infinity. The sequence of maps should (at least) have the following properties:

$$\lim_{\alpha\to\infty} ||T_\alpha(f)|| < \infty, \tag{2.1}$$

$$\lim_{\alpha\to\infty} ||T_\alpha(fh) - T_\alpha(f)T_\alpha(h)|| = 0, \tag{2.2}$$

$$\lim_{\alpha\to\infty} \left|\left| \frac{1}{i\hbar_\alpha}[T_\alpha(f), T_\alpha(h)] - T_\alpha(\{f, h\}) \right|\right| \tag{2.3}$$

for all $f, h \in \mathcal{A}$, where $\hbar_\alpha = \hbar(N_\alpha)$ is a strictly decreasing positive function of N_α (cf. Ref. 3). The third property is the most prominent, telling us that the Poisson bracket of functions is approximated by the commutator of matrices. In Ref. 16, such a sequence was constructed for the round sphere using Hermitian representations of su(2). Namely, define the Hermitian matrices S_1, S_2, S_3 by giving their non-zero matrix elements as

$$\begin{aligned}
\big(S_1\big)_{k,k+1} &= \frac{1}{2}\sqrt{k(N-k)} = \big(S_1\big)_{k+1,k} \qquad k = 1, \ldots, N-1\\
\big(S_2\big)_{k,k+1} &= -\frac{i}{2}\sqrt{k(N-k)} = -\big(S_2\big)_{k+1,k} \qquad k = 1, \ldots, N-1\\
\big(S_3\big)_{k,k} &= \frac{1}{2}(N+1-2k) \qquad k = 1, \ldots, N.
\end{aligned}$$

They satisfy $[S_a, S_b] = i\sum_{c=1}^{3}\epsilon_{abc}S_c$ and $S_1^2 + S_2^2 + S_3^2 = \frac{N^2-1}{4}\mathbb{1}$, which, in particular, means that they provide an N-dimensional representation of su(2). Furthermore, setting $X_a = \frac{2}{\sqrt{N^2-1}}S_a$ implies that

$$[X_a, X_b] = \frac{2i}{\sqrt{N^2-1}}\sum_{c=1}^{3}\epsilon_{abc}X_c$$
$$X_1^2 + X_2^2 + X_3^2 = \mathbb{1}.$$

The round sphere can be considered as embedded in $\mathbb{R}^3$ via the standard spherical coordinates $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$ and $x_3 = r\cos\theta$, together with the Poisson structure

$$\{f, h\} = \frac{1}{\sin\theta}\Big((\partial_\theta f)(\partial_\varphi h) - (\partial_\varphi f)(\partial_\theta h)\Big).$$

The maps T_α (with $N_\alpha = \alpha$) are then defined as $T_\alpha(x_1) = X_1$, $T_\alpha(x_2) = X_2$, $T_\alpha(x_3) = X_3$ and extended to arbitrary polynomials by symmetrization of the corresponding non-commutative polynomial. One can then check that relations (2.1)–(2.3) are fulfilled with $\hbar = 2/\sqrt{N^2-1}$ (see Ref. 16 for details).

In the case of the torus,[15,17] one introduces the unitary $N \times N$ matrices g and h, with non-zero components

$$\begin{aligned} g_{kk} &= q^{k-1} \qquad k = 1,\ldots,N \\ h_{k,k+1} &= 1 \qquad k = 1,\ldots,N-1 \\ h_{N,1} &= 1, \end{aligned}$$

where $q = e^{2i\pi/N}$. Note that these matrices satisfy $hg = qgh$. Starting from the Fourier modes

$$Y_{m_1,m_2} = e^{i(m_1\varphi_1 + m_2\varphi_2)},$$

where $m_1, m_2 \in \mathbb{Z}$, together with the Poisson bracket

$$\{f, h\} = \big(\partial_{\varphi_1} f\big)\big(\partial_{\varphi_2} h\big) - \big(\partial_{\varphi_2} f\big)\big(\partial_{\varphi_1} h\big),$$

one defines

$$T_\alpha(Y_{m_1,m_2}) = q^{\frac{1}{2}m_1m_2} g^{m_1} h^{m_2}$$

and extends it through linearity. Again, one may check that these maps fulfill (2.1)–(2.3) with $\hbar = \sin(\pi/N)$.

III. CONSTRUCTION OF NON-COMMUTATIVE ALGEBRAS

Let us recall how to construct non-commutative algebras related to level sets of a polynomial in $\mathbb{R}^3$.[1,2,4–6] Given a polynomial $C \in \mathbb{R}[x, y, z] \equiv \mathbb{R}[x^1, x^2, x^3]$, one can define a Poisson bracket by setting

$$\{f, g\} = \nabla C \cdot \big(\nabla f \times \nabla g\big), \tag{3.1}$$

for $f, g \in C^\infty(\mathbb{R}^3)$. In particular, it follows that $\{x^i, x^j\} = \varepsilon^{ijk}\partial_k C$. By construction, the polynomial $C(x, y, z)$ Poisson commutes with all functions, which implies that the Poisson structure restricts to the inverse image $\Sigma = C^{-1}(0)$. Thus, it defines a Poisson structure on the quotient algebra $\mathbb{R}[x, y, z]/\langle C(x, y, z)\rangle$, which can be identified with polynomial functions on Σ.

To construct a non-commutative version of the above algebra, one starts with the free (non-commutative) associative algebra $\mathbb{C}[X, Y, Z]$ and imposes the relations

$$\begin{aligned} [X, Y] &= i\hbar : \partial_z C : \\ [Y, Z] &= i\hbar : \partial_x C : \\ [Z, X] &= i\hbar : \partial_y C : \end{aligned} \tag{3.2}$$

where $\hbar \in \mathbb{R}$ and $: \partial_i C :$ denotes a choice of ordering of the (commutative) polynomial $\partial_i C$. Let us assume that there exists a sequence of increasing integers $\{N_\alpha\}_{\alpha=1}^\infty$ together with a strictly decreasing sequence $\hbar_\alpha = \hbar(N_\alpha)$ such that one can find a N_α-dimensional representation of the above relations with $\hbar = \hbar_\alpha$ and such that (2.1) holds; we denote the corresponding sequences of matrices by $\{X_\alpha\}$, $\{Y_\alpha\}$, and $\{Z_\alpha\}$. Defining linear maps T_α, from polynomial functions on Σ to $N_\alpha \times N_\alpha$ matrices, by $T_\alpha(x) = X_\alpha$, $T_\alpha(y) = Y_\alpha$ and $T_\alpha(z) = Z_\alpha$, and extending linearly to polynomials in x, y, z by an arbitrary choice of ordering, gives a sequence of maps fulfilling relations (2.2)–(2.3) of a fuzzy space. Namely, since relations (3.3) imply that a change in ordering will only differ by terms proportional to $\hbar$ (hence, tending to 0 as $\alpha \to \infty$), (2.2) is fulfilled. Moreover, by construction, the commutator is proportional to a non-commutative ordering of the Poisson bracket, which, together with the previous fact, implies that (2.3) holds. Thus, the construction as outlined above, will generically provide a fuzzy analogue of the surface Σ.

In Refs. 1 and 2, the authors considered the polynomial

$$C(x, y, z) = \frac{1}{2}\left(x^2 + y^2 - \mu\right)^2 + \frac{1}{2}z^2 - \frac{1}{2}, \tag{3.3}$$

whose inverse image $C^{-1}(0)$ describes a sphere for $\mu < 1$ and a torus for $\mu > 1$. One computes that

$$\{x, y\} = z \qquad \{y, z\} = 2x(x^2 + y^2 - \mu) \qquad \{z, x\} = 2y(x^2 + y^2 - \mu)$$

and the corresponding non-commutative relations were chosen as

$$[X, Y] = i\hbar Z, \tag{3.4}$$

$$[Y, Z] = i\hbar\Big(2X^3 + XY^2 + Y^2X - 2\mu X\Big), \tag{3.5}$$

$$[Z, X] = i\hbar\Big(2Y^3 + YX^2 + X^2Y - 2\mu Y\Big). \tag{3.6}$$

The algebra is then defined as $\mathcal{C}_{\hbar,\mu} = \mathbb{C}[X, Y, Z]/I$, where I is the two-sided ideal generated by the above relations. Using the "Diamond lemma,"[7] it was proved that $\mathcal{C}_{\hbar,\mu}$ is a non-trivial algebra for which a basis can be computed.[2] We shall also consider $\mathcal{C}_{\hbar,\mu}$ to be a $*$-algebra with $X^* = X$, $Y^* = Y$ and $Z^* = Z$.

In the Poisson algebra, the polynomial C is a Poisson central element of the algebra. It turns out that a non-commutative analogue of C is a central element in $\mathcal{C}_{\hbar,\mu}$. Namely, by setting

$$\hat{C} = \left(X^2 + Y^2 - \mu\mathbb{1}\right)^2 + Z^2, \tag{3.7}$$

one computes that $[X, \hat{C}] = [Y, \hat{C}] = [Z, \hat{C}] = 0$. Thus, in analogy with (3.3) it is natural to also impose $\hat{C} = \mathbb{1}$ in $\mathcal{C}_{\hbar,\mu}$. As we shall see, the presentation of the algebra in terms of X, Y, and Z is appropriate when comparing with spherical geometries (and the "fuzzy sphere"); there is, however, another choice of basis which naturally makes contact with non-commutative tori. By setting $W = X + iY$ and eliminating $Z = \frac{1}{i\hbar}[X, Y]$, the remaining relations may be written as

$$\left(W^2W^* + W^*W^2\right)(1 + \hbar^2) = 4\mu\hbar^2 W + 2(1 - \hbar^2)WW^*W, \tag{3.8}$$

$$\frac{1}{4}\left(WW^* + W^*W - 2\mu\mathbb{1}\right)^2 + \frac{1}{4\hbar^4}\left(WW^* - W^*W\right)^2 = \mathbb{1}. \tag{3.9}$$

Furthermore, by introducing

$$\Lambda = \frac{1}{2\hbar}\left(WW^* - W^*W\right) + \frac{i}{2}\left(WW^* + W^*W - 2\mu\mathbb{1}\right),$$

the above algebra can be presented as

$$W\Lambda = q\Lambda W \quad ; \quad W^*\Lambda = \bar{q}\Lambda W^*, \tag{3.10}$$

$$W^*\Lambda^* = q\Lambda^*W^* \quad ; \quad W\Lambda^* = \bar{q}\Lambda^*W, \tag{3.11}$$

$$\Lambda^*\Lambda = \Lambda\Lambda^* = \mathbb{1}, \tag{3.12}$$

$$WW^* = z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1}, \tag{3.13}$$

$$W^*W = -\bar{z}\Lambda - z\Lambda^* + \mu\mathbb{1}, \tag{3.14}$$

where $q = e^{2\pi i\theta}$, $z = e^{i\pi\theta}/2i\cos\pi\theta$ and θ is related to $\hbar$ via $\hbar = \tan\pi\theta$.

Note that the above relations are quite similar to those of the standard non-commutative torus, generated by two unitary operators U, V satisfying $VU = qUV$. The difference is the deformed unitarity of the operator W. Let us now define the algebra together with the parameter ranges that we shall be interested in.

Definition 3.1: Let $\mu, \theta \in \mathbb{R}$ such that $\mu > 0$ and $|\mu\cos\pi\theta| > 1$. By $A^0_{\mu,\theta}$, we denote the quotient of the (unital) free $*$-algebra $\mathbb{C}\langle W, W^*, \Lambda, \Lambda^*\rangle$ and the two-sided ideal generated by relations (3.10)–(3.14).

Again, one can make use of the Diamond lemma[7] to explicitly compute a basis of $A^0_{\mu,\theta}$.

Proposition 3.2: A basis for $A^0_{\mu,\theta}$ is given by

$$T_{\vec{m}} = q^{m_1m_2/2}\Lambda^{m_1}W^{m_2}$$
$$S_{\vec{n}} = q^{-n_1n_2/2}\Lambda^{n_1}(W^*)^{n_2},$$

where $\vec{m} = (m_1, m_2) \in \mathbb{Z}\times\mathbb{Z}_{\geq 0}$ and $\vec{n} = (n_1, n_2) \in \mathbb{Z}\times\mathbb{Z}_{\geq 1}$. Moreover, it holds that

$$T_{\vec{m}}T_{\vec{n}} = q^{-\vec{m}\times\vec{n}/2}T_{\vec{m}+\vec{n}}$$
$$S_{\vec{m}}S_{\vec{n}} = q^{\vec{m}\times\vec{n}/2}S_{\vec{m}+\vec{n}},$$

where $\vec{m}\times\vec{n} = m_1n_2 - n_1m_2$.

Proof: To prove that $T_{\vec{m}}$ and $S_{\vec{n}}$ provide a basis for the algebra, we make use of the "Diamond lemma", and refer to Ref. 7 for details. Thus, relations (3.10)–(3.14) are put into the reduction system

$$S_1 = (W\Lambda, q\Lambda W) \quad S_2 = (W\Lambda^*, \bar{q}\Lambda^*W) \quad S_3 = (W^*\Lambda^*, q\Lambda^*W^*)$$
$$S_4 = (W^*\Lambda, \bar{q}\Lambda W^*) \quad S_5 = (\Lambda\Lambda^*, \mathbb{1}) \quad S_6 = (\Lambda^*\Lambda, \mathbb{1})$$
$$S_7 = (WW^*, z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1}) \quad S_8 = (W^*W, -\bar{z}\Lambda - z\Lambda^* + \mu\mathbb{1}),$$

and a compatible ordering is chosen as follows: if two words are of different total order (in $W, W^*, \Lambda, \Lambda^*$), then the one with lower order is smaller than the one with higher order. If two words are of the same order, they are comparable if the orders in $W, W^*, \Lambda, \Lambda^*$ are separately equal. Then the ordering is lexicographic with respect to the alphabet $\Lambda, \Lambda^*, W, W^*$. With this ordering one easily checks that $p_i \geq q_{ij}$, where $S_i = (p_i, \sum_j q_{ij})$. Furthermore, this ordering has the descending chain condition.

There are several ambiguities to be checked in this reduction system. For instance, let us consider $WW^*\Lambda$. One needs to check that it reduces to the same expression if we use S_7 to replace WW^* or S_4 to replace $W^*\Lambda$. One computes

$$\begin{aligned}(z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1})\Lambda - W(\bar{q}\Lambda W^*) &= (z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1})\Lambda - q\bar{q}\Lambda WW^* \\ &= (z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1})\Lambda - \Lambda(z\Lambda + \bar{z}\Lambda^* + \mu\mathbb{1}) = 0.\end{aligned}$$

The other ambiguities can also be checked to be resolvable. Hence, by the Diamond lemma, a basis for the algebra is provided by the irreducible words. Denoting $(\Lambda^*)^n = \Lambda^{-n}$ the irreducible words are given by $T_{\vec{m}}$ and $S_{\vec{n}}$. To prove the product formulas, one simply uses the relations to reorder the expressions. $\square$

IV. RELATION TO THE STANDARD NON-COMMUTATIVE TORUS

Let U, V be the generators of the non-commutative torus C_θ;[11–13] i.e., the universal C^*-algebra generated by the relations

$$VU = e^{i2\pi\theta}UV$$
$$U^*U = UU^* = \mathbb{1}$$
$$V^*V = VV^* = \mathbb{1}.$$

In what follows, we will show that one can map $A^0_{\mu,\theta}$ into C_θ and use the induced norm to complete $A^0_{\mu,\theta}$ to a C^*-algebra isomorphic to C_θ. Let us start by proving a result about the spectrum of a particular element in C_θ, that is used to construct a $*$-homomorphism from $A^0_{\mu,\theta}$ to C_θ.

Lemma 4.1: If $|\mu\cos\pi\theta| > 1$ and $\mu > 0$ then the element $\mu\mathbb{1} + ze^{i\pi\varphi}U + \bar{z}e^{-i\pi\varphi}U^*$, with $z = e^{i\pi\theta}/2i\cos\pi\theta$, is positive and invertible in C_θ for all $\varphi \in \mathbb{R}$.

Proof: The element is clearly Hermitian and let us write

$$\mu\mathbb{1} + ze^{i\pi\varphi}U + \bar{z}e^{-i\pi\varphi}U^* \equiv \mu\mathbb{1} - B.$$

To study the spectrum, we consider the invertibility of the element $(\mu - \lambda)\mathbb{1} - B$ for different λ. It is a standard fact that this element is invertible if $\frac{1}{|\mu-\lambda|}\,||B|| < 1$. One computes

$$\frac{1}{|\mu-\lambda|}\,||B|| = \frac{1}{2|(\mu-\lambda)\cos\pi\theta|}\left|\left|e^{i\pi\varphi}U + e^{i\pi\varphi}U^*\right|\right| \leq \frac{1}{|(\mu-\lambda)\cos\pi\theta|},$$

which is less than one if $|(\mu - \lambda)\cos\pi\theta| > 1$. Since $|\mu\cos\pi\theta| > 1$ by assumption (and $\mu > 0$), it follows that $\frac{1}{|\mu-\lambda|}\,||B|| < 1$ for all $\lambda \leq 0$. Hence, $\mu\mathbb{1} - B$ is invertible and the spectrum is contained in $(0, \infty)$. □

Thus, it follows from Lemma 4.1 that if $\mu > 0$ is chosen such that $|\mu\cos\pi\theta| > 1$ then both $\sqrt{\mu\mathbb{1} + zU + \bar{z}U^*}$ and its inverse exist in C_θ.

Proposition 4.2: The map ϕ, defined by

$$\phi(W) = \left(\sqrt{\mu\mathbb{1} + zU + \bar{z}U^*}\right)V$$
$$\phi(\Lambda) = U,$$

induces an injective $*$-homomorphism from $A^0_{\mu,\theta}$ to C_θ.

Proof: First of all, one has to check that the map is well defined, i.e., it respects the relations in $A^0_{\mu,\theta}$; for instance, denoting $R = \mu\mathbb{1} + zU + \bar{z}U^*$, one computes

$$\begin{aligned}\phi(WW^* - z\Lambda - \bar{z}\Lambda^* - \mu\mathbb{1}) &= \sqrt{R}VV^*\sqrt{R} - zU - \bar{z}U^* - \mu\mathbb{1}\\ &= R - zU - \bar{z}U^* - \mu\mathbb{1} = 0,\end{aligned}$$

by using the relations in C_θ. The remaining relations are checked in a similar way. Now, let us prove that ϕ is injective. It follows from Proposition 3.2 that an arbitrary element $a \in A^0_{\mu,\theta}$ can be written as

$$a = \sum_{\vec{m}\in\mathbb{Z}\times\mathbb{Z}_{\geq 0}} a_{\vec{m}}\Lambda^{m_1}W^{m_2} + \sum_{\vec{n}\in\mathbb{Z}\times\mathbb{Z}_{\geq 1}} b_{\vec{n}}\Lambda^{n_1}(W^*)^{n_2},$$

(where all but a finite number of coefficients are zero) which implies that

$$\phi(a) = \sum_{\vec{m}} a_{\vec{m}}U^{m_1}(\sqrt{R}V)^{m_2} + \sum_{\vec{n}} b_{\vec{n}}U^{n_1}(V^*\sqrt{R})^{n_2}.$$

Note that Lemma 4.1 implies that $R(q^k) = \mu\mathbb{1} + zq^kU + \bar{z}q^kU^*$ is positive and invertible for all $k \in \mathbb{Z}$, which in particular implies that $(\sqrt{R})V = V\sqrt{R(\bar{q})}$ and $V^*\sqrt{R} = \left(\sqrt{R(\bar{q})}\right)V^*$. Thus, one concludes that

$$\phi(a)=\sum_{\vec m}a_{\vec m}\Big(\prod_{k=0}^{m_2-1}\sqrt{R(\bar q^k)}\Big)U^{m_1}V^{m_2}+\sum_{\vec n}b_{\vec n}\Big(\prod_{k=1}^{n_2}\sqrt{R(\bar q^k)}\Big)U^{n_1}(V^*)^{n_2}.$$

Since elements of the form

$$\sum_{\vec m\in\mathbb{Z}\times\mathbb{Z}}c_{\vec m}U^{m_1}V^{m_2}$$

form a basis of a dense subset of C_θ, it follows that if $\phi(a)=0$ then one must have $a_{\vec m}=b_{\vec n}=0$ for all $\vec m$ and $\vec n$. Hence, $a=0$, which proves that ϕ is injective. □

Since ϕ is injective, one can define a C^*-norm on $A^0_{\mu,\theta}$ by setting $||a||=||\phi(a)||$ for all $a\in A^0_{\mu,\theta}$, and by $A_{\mu,\theta}$ we denote the completion of $A^0_{\mu,\theta}$ in this norm. Moreover, ϕ can be extended to $A_{\mu,\theta}$ by continuity, and (by a slight abuse of notation) we shall also denote the extended map by ϕ.

Proposition 4.3: The map $\phi:A_{\mu,\theta}\to C_\theta$ is an isomorphism of C^*-algebras.

Proof: As in Lemma 4.1, one can show that $\mu\mathbb{1}+z\Lambda+\bar z\Lambda^*$ is positive and invertible in $A_{\mu,\theta}$. Hence, one constructs the inverse of ϕ by setting

$$\phi^{-1}(V)=\frac{1}{\sqrt{\mu\mathbb{1}+z\Lambda+\bar z\Lambda^*}}W$$
$$\phi^{-1}(U)=\Lambda,$$

(which is easily shown to be a well defined map) and extending it as a $*$-homomorphism through continuity. □

V. PROJECTIVE MODULES

In Ref. 2 all finite-dimensional Hermitian $*$-representations of $\mathcal{C}_{\hbar,\mu}$ were constructed and classified. It was found that, in the case of algebras related to tori, the parameter θ has to be a rational number for finite dimensional representations to exist; which is in the same spirit as for C_θ. Firstly, we compare these finite dimensional representations and show that one can be obtained as a formal scaling limit of the other. Secondly, we display how the standard projective modules of the non-commutative torus can be presented for $A_{\mu,\theta}$, as well as their connections of constant curvature.

A. Finite dimensional representations

For rational θ, where there exists an integer N such that $q^N=e^{i2\pi N\theta}=1$, there are finite-dimensional representations of C_θ, given by setting $U=g$ and $V=h$, where g and h are the matrices introduced in Sect. II. If one considers relations (3.10)–(3.14), and introduces $\tilde W=\varepsilon W$ and sets $\mu=1/\varepsilon^2$, relations (3.13) and (3.14) become

$$\tilde W\tilde W^*=\varepsilon^2\big(z\Lambda+\bar z\Lambda^*\big)+\mathbb{1},$$
$$\tilde W^*\tilde W=\varepsilon^2\big(-z\Lambda-\bar z\Lambda^*\big)+\mathbb{1},$$

which reduces to the fact that $\tilde W$ is unitary as $\varepsilon\to0$. Clearly, Eqs. (3.10)–(3.12) are invariant under this rescaling, and the resulting set of relations coincides with those defining C_θ as $\varepsilon\to0$. In Ref. 2, the non-zero matrix elements of $W=X+iY$, in an N-dimensional irreducible representation (for $\mu>1$), were found to be

$$W_{N,1}=\sqrt{\mu+\frac{1}{\cos\pi\theta}}$$
$$W_{l,l+1}=\sqrt{\mu+\frac{\cos2\pi l\theta}{\cos\pi\theta}}$$

for $l = 1, \ldots, N - 1$, which implies that

$$\tilde{W}_{N,1} = \varepsilon\sqrt{\frac{1}{\varepsilon^2} + \frac{1}{\cos\pi\theta}} \quad \to \quad 1$$
$$\tilde{W}_{l,l+1} = \varepsilon\sqrt{\frac{1}{\varepsilon^2} + \frac{\cos 2\pi l\theta}{\cos\pi\theta}} \quad \to \quad 1$$

as $\varepsilon \to 0$. Even without any rescaling, it holds that Λ is a matrix with the N roots of unity on the diagonal. As a complementary remark, let us note that the Lie algebra su(2), which defines the fuzzy sphere (as in Sec. II), can be obtained from (3.4)–(3.6) as the limit $\varepsilon \to 0$ when setting $\tilde{X} = X/\varepsilon$, $\tilde{Y} = Y\varepsilon$, $\tilde{Z} = Z/\varepsilon$ as well as $\hbar = k\varepsilon$.

B. Projective modules

In Ref. 11, finitely generated projective modules of the non-commutative torus C_θ were introduced. For the sake of comparison, let us see how these modules can be presented for $A^0_{\mu,\theta}$. Let $\xi_{m,n}$ be the vector space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}_n)$, i.e., the space of Schwartz functions in one real variable x and one discrete variable $k \in \mathbb{Z}_n$. By defining

$$\big(\phi W\big)(x,k) = W(x,k)\phi(x-\varepsilon, k-1), \tag{5.1}$$

$$\big(\phi W^*\big)(x,k) = W(x+\varepsilon, k+1)\phi(x+\varepsilon, k+1), \tag{5.2}$$

$$\big(\phi\Lambda\big)(x,k) = e^{2\pi i(x-mk/n)}\phi(x,k), \tag{5.3}$$

$$\big(\phi\Lambda^*\big)(x,k) = e^{-2\pi i(x-mk/n)}\phi(x,k), \tag{5.4}$$

where $\varepsilon = (m + n\theta)/n$ and

$$W(k,x) = \sqrt{\mu + \frac{\sin\big(2\pi(x-mk/n) - \pi\theta\big)}{\cos\pi\theta}}, \tag{5.5}$$

one can check that $\xi_{m,n}$ becomes a right $A^0_{\mu,\theta}$ module.

The standard derivations on C_θ, defined by

$$\partial_1 U = iU \qquad \partial_2 U = 0$$
$$\partial_1 V = 0 \qquad \partial_2 V = iV,$$

and extended to the smooth part of C_θ, can be pulled back to the smooth part of $A_{\mu,\theta}$ (defined as the inverse image of the smooth part of C_θ) giving

$$\partial_1\Lambda = i\Lambda \qquad \partial_2\Lambda = 0$$
$$\partial_1 W = i(z\Lambda - \bar{z}\Lambda^*)\big(\mu\mathbb{1} + z\Lambda + \bar{z}\Lambda^*\big)^{-1}W$$
$$\partial_2 W = iW.$$

Furthermore, a connection may be defined on the above modules in a standard manner. Namely, the linear operators $\nabla_1, \nabla_2 : \xi_{m,n} \to \xi_{m,n}$, given as

$$\big(\nabla_1\phi\big)(x,k) = \frac{1}{2\pi}\frac{d\phi}{dx}(x,k)$$
$$\big(\nabla_2\phi\big)(x,k) = \frac{i}{\varepsilon}x\phi(x,k),$$

define a connection on $\xi_{m,n}$, i.e., they fulfill

$$\nabla_i(\phi \cdot a) = \big(\nabla_i \phi\big) \cdot a + \phi \cdot \big(\partial_i a\big)$$

for $i = 1, 2$ and a in the smooth part of $A_{\mu,\theta}$. One easily computes that

$$[\nabla_1, \nabla_2] = \frac{i}{2\pi\varepsilon}\mathbb{1},$$

i.e., the connection has constant curvature.

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