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DEFORMING METRICS IN THE DIRECTION OF THEIR RICCI TENSORS

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(An appendix to a paper of R. Hamilton)

1. Introduction. In [2], R. Hamilton has proved that if a compact manifold M of dimension three admits a C^∞ Riemannian metric g_0 with positive Ricci curvature, then it also admits a metric $\bar{g}$ with constant (positive) sectional curvature, and is thus (a quotient of) the sphere S^3. In fact, he shows that the original metric can be deformed into the constant-curvature metric by requiring that, for $t \geqslant 0$, $x \in M$ and $g = g(t, x)$,

$$\frac{\partial g}{\partial t} = \frac{2}{3} r_t g - 2\mathrm{Ric}(g), \qquad g(0, x) = g_0(x), \tag{1}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature of g on M at time t, and r_t is the average scalar curvature of the metric $g_t = g(t, x)$ over M, i.e.,

$$r_t = \frac{1}{\mathrm{Vol}_{g_t}(M)} \int_M \mathrm{Scal}(g_t)\, dV_{g_t}.$$

Hamilton's proof has two parts. In the first part, he proves local-in-time existence for the initial-value problem (IVP) (1), which is equivalent to proving local existence for the IVP

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \qquad g(0, x) = g_0(x) \tag{2}$$

(see [2, §3]). This part of the proof is valid for all dimensions $n \geqslant 3$. In the second part, which is specific to three dimensions, he proves that, as t approaches ∞, $g(t, x)$ approaches $\bar{g}(x)$ and that the Ricci curvature of g remains positive throughout the deformation.

To do the first (local) part of the proof, Hamilton uses a deep and powerful theorem from analysis: the Nash-Moser implicit-function theorem. (Some

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special technique is required because the IVP (2) is almost, but not strictly parabolic.) The purpose of this note is to prove local-in-time existence for (2) without recourse to the Nash-Moser theorem. In fact, our only analytic tool will be the "classical" existence and uniqueness theorem for the initial-value problem for quasilinear parabolic systems.

The idea of our proof is simple: we show that (2) is equivalent to an initial-value problem for a parabolic system. The particular form of the system we study is motivated by the elliptic work in [1], and by the evolution equation for the Ricci (and sectional) curvature tensors implied by (2) (see [2, §7]). Once the correct alternate IVP is written down, it is a tedious but elementary matter to show that it is equivalent to (2) (i.e., that solutions of one are necessarily also solutions of the other). After giving a list of our notation in §2, we will give a rather complete proof for the three-dimensional case in §3. We indicate how to do the n-dimensional case in §4.

2. Notation. We use standard tensor calculus notation, and adapt the notation of [1] and [2] as follows: We will have a flow of metrics $g(t, x)$ on M with $t \in [0, \varepsilon)$ and $x \in M$. At a (possibly arbitrary but) fixed time t, we write g_{ij} for $g(t, x)$; R_{ij} (resp. R_{ijkl}) is the Ricci (resp. sectional) curvature tensor of g_{ij}. For any $S_{ij} \in S^2T^*(M)$, we define:

$$\operatorname{tr}(S) = g^{ij}S_{ij},\ G(S)_{ij} = S_{ij} - \frac{1}{2}(\operatorname{tr} S)g_{ij},$$

$$(\delta S)_i = -g^{jk}S_{ij|k},\ T(S)_i^{st} = \frac{1}{2}g^{sk}g^{tl}\left(S_{ik|l} + S_{ki|l} - S_{kl|i}\right),$$

$$Q(S)_{ij} = g^{pq}g^{rs}\big(S_{ip}S_{qj}g_{rs} + S_{jp}S_{qi}g_{rs} - S_{pq}S_{ij}g_{rs} + S_{pq}S_{rs}g_{ij} - S_{pr}S_{qs}g_{ij} - S_{qr}S_{ps}g_{ij}\big),$$

and for $v_i \in T^*(M)$, we define

$$(v^\#)^i = g^{ij}v_j, \qquad (\delta^* v)_{ij} = \frac{1}{2}(v_{i|j} + v_{j|i}).$$

Of course, all covariant derivatives (denoted by a bar) are taken with respect to g_{ij}. Finally, the Laplacian ΔT of any tensor T is defined as

$$(\Delta T)_{j\cdots}^{i\cdots} = g^{kl}T_{j\cdots|kl}^{i\cdots}.$$

Note that the Bianchi identity for Ricci curvature can be written $\delta G(\operatorname{Ric}(g)) = 0$. Also note that, in three dimensions, (2) implies that

$$\frac{\partial \operatorname{Ric}(g)}{\partial t} = \Delta \operatorname{Ric}(g) - Q(\operatorname{Ric}(g)),$$

and that $Q(g) = 0$.

3. Theorem [2, Theorem 4.2]. *The initial-value problem* (2) *has a unique solution for some small time interval.*

Proof (for $n = 3$). Let c be any constant such that $\mathrm{Ric}(g_0) + cg_0$ is an isomorphism from $T_x(M)$ to $T_x^*(M)$ for all $x \in M$. We will show that (2) is equivalent to the following, strictly parabolic, IVP:

$$(3) \qquad \begin{aligned} &\text{(a)} \quad \frac{\partial g_{ij}}{\partial t} = -2\left(R_{ij} - \left(\delta^*(L^{-1}\delta G(L))\right)_{ij}\right), \\ &\text{(b)} \quad \frac{\partial L_{ij}}{\partial t} = \Delta L_{ij} - Q(L - cg)_{ij} - 2c\left(L_{ij} - cg_{ij}\right), \\ &\text{(c)} \quad g(0, x) = g_0(x), \quad L(0, x) = \mathrm{Ric}(g_0) + cg_0(x). \end{aligned}$$

Clearly, we expect that $L(t, x)$ will always equal $\mathrm{Ric}(g) + cg$, so that (by the Bianchi identity) $\delta G(L) \equiv 0$, and (a) will become $\partial g/\partial t = -2\mathrm{Ric}(g)$. By L^{-1} we mean the tensor which satisfies $(L^{-1})_a^c L_{cb} = g_{ab}$.

To show that the system (3) is strictly parabolic, we note that the right side of 3(a) is elliptic in g, as shown in [1, §3]. If σ_Δ is the symbol of the Laplacian Δ, then the principal symbol of the right side of (3a–b) is

$$\left(\begin{array}{c|c} \sigma_\Delta & * \\ \hline 0 & \sigma_\Delta \end{array}\right),$$

which is clearly elliptic. Thus we can conclude from the standard parabolic existence theorem that the IVP (3) has a unique solution for a short time.

We remark that if (2) has a solution, then the pair $(g, \mathrm{Ric}(g) + cg)$ solves (3). This and the fact that (3) has a unique solution, already proves uniqueness (and regularity) for (2). We must show that a solution of (3) necessarily solves (2), i.e., that the second term on the right side of (3a) vanishes *a posteriori*.

To do this, we consider the quantities

$$u_i = (L^{-1})_i^k (\delta G(L))_k, \quad P_{ij} = L_{ij} - \left(R_{ij} + cg_{ij}\right).$$

We write down the evolution equations for u and P which are implied by (3). They will turn out to form a linear homogeneous parabolic system for the pair (u, P), and since $u = 0$ and $P = 0$ for $t = 0$, we conclude that $u = 0$ and $P = 0$ for all time by uniqueness. This will complete the proof of the theorem.

So we must compute $\partial u/\partial t$ and $\partial P/\partial t$. We have

$$\frac{\partial P_{ij}}{\partial t} = \frac{\partial L_{ij}}{\partial t} - \frac{\partial R_{ij}}{\partial t} + c\frac{\partial g_{ij}}{\partial t}.$$

The only term which requires some computation is $\partial R/\partial t$. From the usual formula for the linearization of the Ricci operator ([1, equation (1.1)] or [2, equation (7.3)]), we get

$$\begin{aligned}\frac{\partial R_{ij}}{\partial t} &= \Delta R_{ij} - Q(R)_{ij} + 2(D\,\mathrm{Ric})(\delta^* u)\\ &= \Delta R_{ij} - Q(R)_{ij} + \mathfrak{L}_{u^\#}(R),\end{aligned}$$

where $\mathfrak{L}$ is the Lie derivative, because $2\delta^* u$ is the derivative at 0 of g under the action of the 1-parameter group of diffeomorphisms generated by $u^\#$. Thus

$$\begin{aligned}\frac{\partial P_{ij}}{\partial t} = \Delta P_{ij} + Q(R)_{ij} - Q(L - cg)_{ij} - \mathfrak{L}_{u^\#}(R)\\ -2c(L - cg - R + \delta^* u)_{ij}.\end{aligned}$$

Since Q is quadratic and algebraic, we can easily rewrite this (dropping indices) as

$$\frac{\partial P}{\partial t} = \Delta P + F(P) - \mathfrak{L}_{u^\#}(R) - 2c\delta^* u, \tag{4}$$

where F is a linear operator (whose coefficients depend on g, L, and R, but this does not matter since they have already been determined by (3)).

The computation of $\partial u/\partial t$ is somewhat more complicated. We have, using [1, equation (3.1)],

$$\begin{aligned}\left(\frac{\partial u}{\partial t}\right)_j = (L^{-1})_j^p\left\{-\left(L\delta G\left(\frac{\partial g}{\partial t}\right)\right)_p + T(L)_p^{rs}\left(\frac{\partial g}{\partial t}\right)_{rs} + \left(\delta G\left(\frac{\partial L}{\partial t}\right)\right)_p\right\}\\ + \left(\frac{\partial L^{-1}}{\partial t}\right)_j^p(\delta G(L))_p.\end{aligned}$$

We examine this expression a term at a time. From (3a), the definitions of the operators, the Bianchi identity and the Ricci identities, we get

$$\left(\delta G\left(\frac{\partial g}{\partial t}\right)\right)_j = \Delta u_j + R_j^p u_p.$$

The same kind of considerations give

$$\left(T(L)\frac{\partial g}{\partial t}\right)_j = -2g^{rk}g^{sl}\left(L_{ki|l} - \tfrac{1}{2}L_{kl|i}\right)\left(R_{rs} - (\delta^*(u))_{rs}\right). \tag{5}$$

The $\delta^* u$ term of this is fine, and the rest will be absorbed into the next term. Let $K_{ij} = L_{ij} - cg_{ij}$. Then using the Bianchi and Ricci identities and the expression for R_{ijkl} in terms of R_{ij} in dimension three (see [2, Theorem 8.1]) we have

$$
\begin{aligned}
\left(\delta G\left(\frac{\partial L}{\partial t}\right)\right)_j &= (\delta G(\Delta K))_j - (\delta G(Q(K)))_j - 2c(\delta G(L))_j \\
&= \Delta(\delta G(K))_j - g^{pq}g^{kl}\{2R_{ql}K_{jp|k} + 2R_{qj}K_{pk|l} + R_{kl}K_{pq|j} \\
&\quad -R_{pq}K_{jl|k} - 2R_{jl}K_{pq|k} - 2R_{lq}K_{kp|j} + K_{pj|l}R_{qk} \\
&\quad +\tfrac{1}{2}K_{pq|k}R_{lj} + K_{pj}R_{qk|l} + K_{pk}R_{jl|q} - K_{kq}R_{pl|j} \\
&\quad -6K_{lp|k}K_{qj} - 6K_{lp}K_{qj|k} + 3K_{pq|k}K_{lj} + 3K_{pq}K_{lj|k} \\
&\quad -2K_{kl|j}K_{pq} + 2K_{lp|j}K_{kq} + K_{lp}K_{kq|j}\} - 2cL_j^p u_p.
\end{aligned} \tag{6}
$$

After some manipulation, (5) and (6) become

$$
\begin{aligned}
&\left[T(L)\frac{\partial g}{\partial t} + \delta G\left(\frac{\partial L}{\partial t}\right)\right]_j \\
&= \Delta(\delta G(L))_j + g^{kl}\left\{\frac{3}{2}(\operatorname{tr} K)_{|k}(R_{jl} - K_{jl}) - 3K_{jk}(\delta G(L))_l\right\} \\
&\quad +g^{kl}g^{pq}\{5K_{jp|k}(K_{ql} - R_{ql}) + K_{pk}(K_{qj|l} - R_{qj|l}) \\
&\qquad +2K_{lp|k}(K_{qj} - R_{qj}) + K_{qj}(K_{lp|k} - R_{lp|k}) \\
&\qquad +3K_{lp|j}(R_{ql} - K_{ql}) + K_{pk}(R_{ql|j} - K_{ql|j})\} \\
&\quad +\frac{1}{2}(\operatorname{tr} K)_{|j}(\operatorname{tr}(K - R)) - (\operatorname{tr} R)(\delta G(L))_j + \frac{3}{2}(\operatorname{tr} K)(\delta G(L))_j \\
&\quad +2g^{kl}g^{pq}(L_{kj|q} - L_{kq|j})(\delta^* u)_{lp} - 2cL_j^p u_p.
\end{aligned} \tag{7}
$$

Since $P = K - R$ and $\delta G(L) = Lu$, (7) is easily seen to equal $\Delta u + E'(u, P)$, where E' is a homogeneous, linear first-order differential operator whose coefficients involve g, L, and R. But

$$L^{-1}\Delta(Lu) = \Delta u - (\text{first-order linear operator in } u),$$

so we arrive at

$$\frac{\partial u}{\partial t} = 2\Delta u + E(u, P). \tag{8}$$

Since (4) and (8) together form a linear homogeneous parabolic system for u and P, the proof is complete.

4. Higher dimensions. In dimensions greater than three, we must face the fact that the sectional curvature tensor is not completely determined by the Ricci tensor. However, the following strictly parabolic initial-value problem is equivalent to (2):

$$\text{(a)}\quad \frac{\partial g_{ij}}{\partial t} = -2R_{ij} - \left(\delta^*\left(L^{-1}\delta G(L)\right)\right)_{ij},$$

$$\text{(b)}\quad \frac{\partial L_{ij}}{\partial t} = \Delta L_{ij} + 2g^{pr}g^{qs}T_{ipqj}L_{rs} - 2g^{pq}L_{pi}L_{jq} - 2c\left(L_{ij} - cg_{ij}\right),$$

$$\text{(c)}\quad \frac{\partial T_{ijkl}}{\partial t} = \Delta T_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right)$$
$$-g^{pq}\left(T_{pjkl}T_{qi} + T_{ipkl}T_{qj} + T_{ijpl}T_{qk} + T_{ijkp}T_{ql}\right),$$

$$\text{(d)}\quad g(0,x) = g_0(x),\, L(0,x) = \mathrm{Ric}(g_0) + cg_0(x),\, T(0,x) = \mathrm{Sect}(g_0),$$

where we define

$$B_{ijkl} = \tfrac{1}{4}g^{pr}g^{qs}\left(T_{piqj}T_{rksl} + T_{qjpi}T_{rksl} + T_{qjpi}T_{slrk} + T_{piqj}T_{slrk}\right),$$

and $T_{ij} = g^{pq}T_{ipjq}$. As in dimension three, we show that the evolution equations for the following quantities:

$$u_i = (L^{-1})_i^p(\delta G(L))_p,$$
$$P_{ij} = L_{ij} - cg_{ij} - R_{ij},$$
$$S_{ijkl} = T_{ijkl} - R_{ijkl}$$

form a homogeneous linear parabolic system. The calculations involved are similar to those in §3, but longer.

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