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Degenerate Hopf bifurcations, hidden attractors, and control in the extended Sprott E system with only one stable equilibrium

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Abstract: In this paper, we introduce an extended Sprott E system by a general quadratic control scheme with 3 arbitrary parameters for the new system. The resulting system can exhibit codimension-one Hopf bifurcations as parameters vary. The control strategy used can be applied to create degenerate Hopf bifurcations at desired locations with preferred stability. A complex chaotic attractor with only one stable equilibrium is derived in the sense of having a positive largest Lyapunov exponent. The chaotic attractor with only one stable equilibrium can be generated via a period-doubling bifurcation. To further suppress chaos in the extended Sprott E system coexisting with only one stable equilibrium, adaptive control laws are designed to stabilize the extended Sprott E system based on adaptive control theory and Lyapunov stability theory. Numerical simulations are shown to validate and demonstrate the effectiveness of the proposed adaptive control.

Key words: Chaotic attractor, stable equilibrium, Sil'nikov's theorem, degenerate Hopf bifurcations, hidden attractor

1. Introduction

Since chaotic attractors were found by Lorenz in 1963 [10], many chaotic systems have been constructed, such as the Rössler [16], the Chen [4], and the Lü [11] systems. Because of potential applications in engineering, the study of chaotic systems has attracted the interest of more and more researchers.

By exhaustive computer searching, Sprott [21–23] found about 20 simple chaotic systems with no more than 3 equilibria. These systems have either 5 terms and 2 nonlinearities or 6 terms and 1 nonlinearity. Later, many 3-dimensional (3-D) Lorenz-like or Lorenz-based chaotic systems were proposed and investigated [1,3,5,9,12,13,14,24,25,27,29,32]. Methods for generating multiscroll attractors have commonly used analytical criteria for generating and proving chaos in autonomous systems, based on the fundamental work of Sil'nikov [17,18] and its subsequent embellishment and extension [19]. Chaos in the Sil'nikov type of 3-D autonomous quadratic dynamical systems may be classified into 4 subclasses [34]: (1) chaos of homoclinic-orbit type; (2) chaos of heteroclinic-orbit type; (3) chaos of the hybrid type with both homoclinic and heteroclinic orbits; (4) chaos of other types. Therefore, Sil'nikov's criteria are sufficient but certainly not necessary for the emergence of chaos. Creating a chaotic system with a more complicated topological structure such as chaotic attractors with only stable equilibria, therefore, becomes a desirable task and sometimes a key issue for many engineering applications.

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To further the investigation of chaos theory and its applications, it is very important to generate new chaotic systems or to enhance the complex dynamics and topological structure based on the existing chaotic attractors. In this endeavor, Yang et al. [33] studied an unusual 3-D autonomous quadratic Lorenz-like chaotic system with only 2 stable node-foci. Moreover, a new 3-D chaotic system with 6 terms including only 1 nonlinear term in the form of an exponential function was proposed and studied in [30]. This system has double-scroll chaotic attractors in a very wide region of parameter space with only 2 stable equilibria. Wei and Yang [31] analyzed the generalized Sprott C system with only 2 stable equilibria. They computed some basic dynamical properties: Lyapunov exponent spectra, fractal dimensions, bifurcations, and routes to chaos. Wang and Chen [25] obtained chaotic attractors with only one stable node-focus by adding a simple constant control parameter to Sprott's E system. Recently, a chaotic system with no equilibria was proposed by Wei [28], which showed a period-doubling sequence of bifurcations leading to a Feigenbaum-like strange attractor. In 2011, these attractors with no equilibria or only stable equilibria were called it hidden attractors by Leonov et al. [8]. All these findings are indeed surprising from a classical chaos theory point of view, as the systems will be topologically nonequivalent to the original Lorenz and all Lorenz-like systems. Although the fundamental chaos theory for autonomous dynamical systems has reached its maturity today, the aforementioned findings reveal some new features of chaos. On the other hand, the control of chaotic systems is to design state feedback control laws that stabilize the chaotic systems around *the unstable equilibrium points.* Active control technique is used when the system parameters are known and adaptive control technique is used when the system parameters are unknown [15,26]. Therefore, the design of adaptive control of the extended Sprott E system *with only one stable equilibrium* will also be studied.

The current paper further extends the reported result of Wang and Chen [25], utilizing a general quadratic function to create chaotic attractors with one stable equilibrium. We analyze the stability criteria for codimension 1 and 2 Hopf bifurcations by calculating the first and second Lyapunov coefficients, following the approach of Kuznetsov [7]. We verify that the new 3-D system with only one stable equilibrium can also evolve into periodic and chaotic behaviors as parameters vary. We then applied adaptive control theory for the stabilization of extended Sprott E system with unknown system parameters. Numerical simulations are shown to demonstrate the effectiveness of the proposed adaptive stabilization.

2. The extended Sprott E system

2.1. Chaotic attractor

Based on the Sprott E system, we introduce a new chaotic system

$$\begin{cases} \dot{x} = yz + h(x) \\ \dot{y} = x^2 - y \\ \dot{z} = 1 - 4x, \end{cases} \tag{2.1}$$

where $h(x) = ex^2 + fx + g$ and e, f, g are real parameters.

System (2.1) possesses only one equilibrium state, E: $(x, y, z) = (1/4, 1/16, -e - 4f - 16g)$, but has many interesting complex dynamical behaviors. When parameters $(f, g, e) = (0, 0, 0)$, system (2.1) is the Sprott E system. When parameters $(f, g, e) = (-0.1, 0.02, 0.2)$, it displays a chaotic attractor, as shown in Figures 1a and 1b. This chaotic attractor differs from that of the Lorenz system or any existing systems, because the only equilibrium state E is stable for these parameter values; the eigenvalues of the linearised system are $\lambda_1 = -0.9506$, $\lambda_{2,3} = -0.0247 \pm 0.5122i$. Therefore, system (2.1) has no homoclinic orbits joining E. Its

Lyapunov exponents are $L_1 = 0.0450$, $L_2 = 0$, $L_3 = -1.0451$, and the Lyapunov dimension is $L_D = 2.0431$ for initial conditions (–0.6, 0.9, –1.7). Figure 2a shows the Poincaré section on the plane $z = 2$, while Figure 2b shows the time series of $z(t)$ for system (2.1).

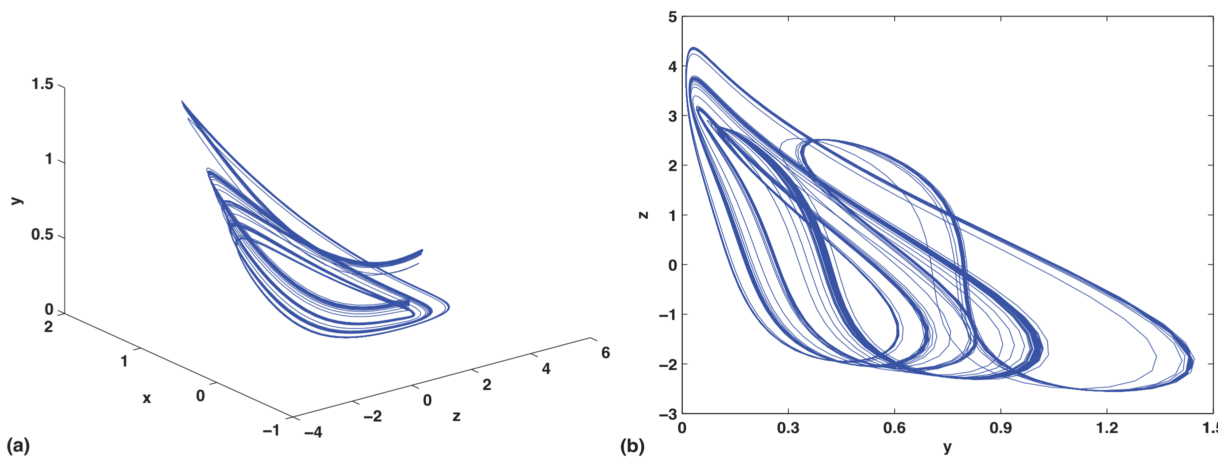

Figure 1. Parameter values when parameter values $(f, g, e) = (-0.1, 0.02, 0.2)$ of system (2.1) with initial value $(-0.6, 0.9, -1.7)$: a) chaotic attractor in 3-D space; b) chaotic attractor projected in y-z plane.

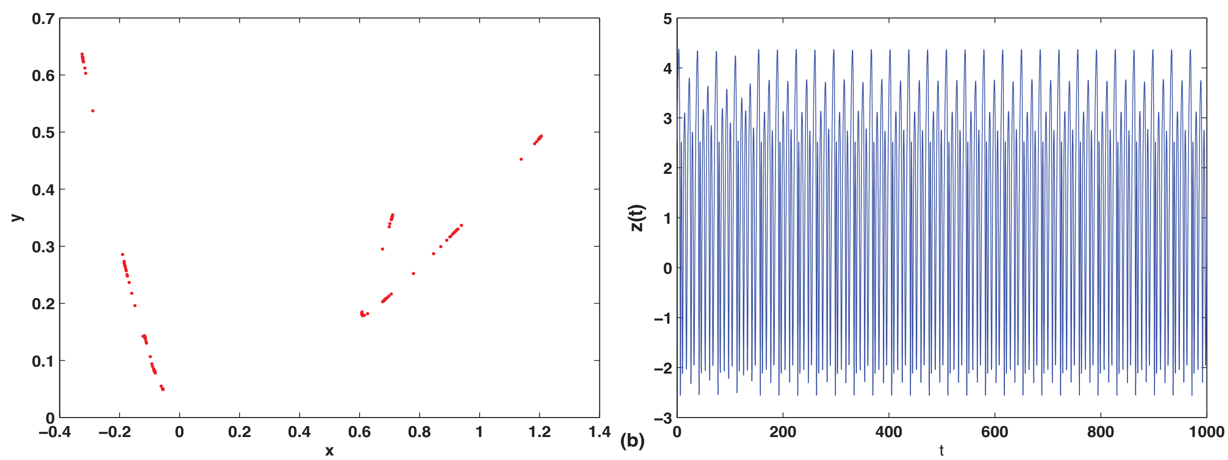

Figure 2. Parameter values $(f, g, e) = (-0.1, 0.02, 0.2)$ of system (2.1) with initial value $(-0.6, 0.9, -1.7)$: a) Poincaré mapping on $z = 2$ section; b) time series of state variable $z(t)$.

2.2. Nonchaotic behaviour

It is straightforward to show that knowledge of fixed points and their properties is insufficient to determine the structure of chaotic attractors. We show here that there are some nonchaotic parameter regions. The following theorem will help to reduce the amount of work spent searching for parameter values for chaos.

Theorem 2.1 *If $e = 0$, $5f + 4g = 0$, and $f > 1$, then system (2.1) is not chaotic.*

Proof From the third equation of (2.1), we obtain

$$z'' = -4x' = -4(yz + h(x)) \tag{2.2}$$

and

$$\begin{aligned} z''' &= -4yz' - 4zy' - 4h'(x) \\ &= -4yz' - 4z(x^2 - y) - 4h'(x). \end{aligned} \tag{2.3}$$

Multiplying both sides of the equation (2.3) by z gives

$$zz''' = -4yzz' - 4z^2(x^2 - y) - 4h'(x)z.$$

Since $yz = x' - h(x)$ and $x = \dfrac{1-z'}{4}$,

$$\begin{aligned} zz''' &= -4z'(x' - h(x)) - 4zh'(x) - 4z^2x^2 \\ &\quad +4z(x' - h(x)) \\ &= -4(2ex + f)z - 4x^2z^2 \\ &\quad -4(\frac{z''}{4} + ex^2 + fx + g)(z - z'). \end{aligned} \tag{2.4}$$

Integrating this equation with respect to t gives

$$\begin{aligned} zz'' - \frac{z'^2}{2} + zz' = \int_0^t [-(5f + 4g + \frac{9}{4}e)z - \frac{z^2}{4} + \frac{e}{4}z'^3 \\ -(f - 1 + \frac{e}{2})z'^2 - \frac{e}{4}zz'^2 - \frac{1}{4}(zz')^2]dt + C, \end{aligned} \tag{2.5}$$

where C is a constant and $t \geq 0$. When $e = 0$, $5f + 4g = 0$, and $f > 1$, the left hand side of (2.5) simplifies to $(1-4x)z - (1-4x)^2/2 - 4yz^2 - 4hz$, a monotonic function of t. It has a limit $L \in \bar{R}$ as t tends to infinity. If L is finite, then any attractor for the equation lies on the surface $(1-4x)z - (1-4x)^2/2 - 4yz^2 - 4hz$ and is not chaotic by virtue of the Poincareé–Bendixson theorem. If $L = \pm\infty$, then at least 1 of the 3 variables is unbounded and cannot be chaotic. □

2.3. Some basic properties of the new system (2.1)

The Jacobian matrix of linearization about the equilibrium E of system (2.1) is given by

$$A = \begin{pmatrix} \frac{e}{2} + f & -e - 4f - 16g & \frac{1}{16} \\ \frac{1}{2} & -1 & 0 \\ -4 & 0 & 0 \end{pmatrix} \tag{2.6}$$

with the characteristic equation

$$\lambda^3 + \left(1 - f - \frac{e}{2}\right)\lambda^2 + \left(\frac{1}{4} + f + 8g\right)\lambda + \frac{1}{4} = 0. \tag{2.7}$$

According to the Routh–Hurwitz stability criterion, the real parts of all the roots λ are negative if and only if

$$\Delta_1 = 1 - f - \frac{e}{2} > 0, \quad \Delta_2 = \frac{1}{4} + f + 8g > 0,$$

$$\Delta_3 = \left(1 - f - \frac{e}{2}\right)\left(\frac{1}{4} + f + 8g\right) - \frac{1}{4} > 0.$$

These inequalities give

$$e < 2(1-f), \quad g > -\frac{e+4ef+2f(-3+4f)}{32(-2+e+2f)}, \tag{2.8}$$

and E is asymptotically stable.

3. Bifurcation analysis in system (2.1)

3.1. Review of the method of Lyapunov coefficients

We first review the projection method described in Chapters 3 and 5 of Kuznetsov [7], but following the analysis of [12,13,20], for the calculation of the first Lyapunov coefficient l_1, associated with the stability of a Hopf bifurcation.

Consider the differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^3$ and $\mu \in \mathbb{R}^3$ are respectively the phase variables and control parameters, and f is a smooth function in $\mathbb{R}^3 \times \mathbb{R}^3$. Suppose that (3.9) has an equilibrium point $\mathbf{x} = \mathbf{x_0}$ at $\mu = \mu_\mathbf{0}$. We write $\mathbf{X} = \mathbf{x} - \mathbf{x_0}$ and

$$F(\mathbf{X}) = f(\mathbf{X}, \mu_0). \tag{3.2}$$

$F(\mathbf{X})$ is also a smooth function and admits a Taylor series expansion in terms of symmetric multilinear vector functions of its variables:

$$\begin{aligned} F(\mathbf{X}) = A\mathbf{X} + \frac{1}{2}B(\mathbf{X},\mathbf{X}) + \frac{1}{6}C(\mathbf{X},\mathbf{X},\mathbf{X}) \\ +\frac{1}{24}D(\mathbf{X},\mathbf{X},\mathbf{X},\mathbf{X}) + \frac{1}{120}E(\mathbf{X},\mathbf{X},\mathbf{X},\mathbf{X},\mathbf{X}) \\ +O(\| X \|^6), \end{aligned} \tag{3.3}$$

where $A = f_x(0, \mu_0)$ is the Jacobian matrix, evaluated at the translated equilibrium state, and, for $i = 1, 2, 3$,

$$B(\mathbf{X},\mathbf{Y}) = \sum_{j,\,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}|_{\xi=0} X_j Y_k,$$

$$C(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \sum_{j,\,k,\,l=1}^{3} \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}|_{\xi=0} X_j Y_k Z_l.$$

Suppose that A has a pair of pure imaginary eigenvalues $\lambda_{2,\,3} = \pm i\omega_0\,(\omega_0 > 0)$ at the equilibrium state $(\mathbf{x_0}, \mu_\mathbf{0})$, with no other eigenvalues on the imaginary axis. Let T^c be the generalized eigenspace of A, the largest invariant subspace spanned by eigenvectors corresponding to $\lambda_{2,\,3}$. In Lemma 3.3 of [7], Kuznetsov introduced eigenvectors $p, q \in \mathbb{C}^3$ such that

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p, \tag{3.4}$$

where we have the normalisation condition

$$\langle p, q\rangle = \sum_{j}^{3} p_j q_j = 1.$$

Here A^T is the transpose of A, p is the complex conjugate of q with $\langle ., .\rangle$ being the standard scalar product over $\mathbb{C}^3$, and the overbar denotes complex comjugation. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle p, y\rangle \in \mathbb{C}$.

The 2-dimensional center manifold associated with the eigenvalues $\lambda_{2,3} = \pm i\omega_0$ can be parameterized by w and $\bar{w}$, by an immersion of the form $X = H(w, \bar{w})$. $H : \mathbb{C}^2 \to \mathbb{R}^3$ is expanded in a Taylor series:

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2\leq j+k\leq 5} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^6), \tag{3.5}$$

where $h_{jk} \in \mathbb{C}^3$ and $h_{jk} = \bar{h}_{kj}$. Differentiating $H(w, \bar{w})$ with respect to t and substituting into (3.2) gives

$$H_w w' + H_w \bar{w}' = F(H(w, \bar{w})), \tag{3.6}$$

where F is given by (3.2) and (3.11). The complex coefficients h_{ij} are obtained by solving the system of linear equations defined by the coefficients of (3.2), so that on the center manifold, w evolves according to

$$\dot{w} = i\omega_0 w + \frac{1}{2} G_{21} w|w|^2 + \frac{1}{12} G_{32} w|w|^4 + O(|w|^6), \tag{3.7}$$

where $G_{21} \in \mathbb{C}$.

Substituting (3.7) into (3.6) and using (3.11), we obtain expressions for the h_{ij}. At quadratic order we have [18]:

$$h_{11} = -A^{-1} B(q, \bar{q}), \quad h_{20} = (2i\omega_0 I_3 - A)^{-1} B(q, q),$$

where I_3 is the 3×3 identity matrix, while at cubic order the coefficient of w^3 is

$$h_{30} = (3i\omega_0 I_3 - A)^{-1}(3B(q, h_{20}) + C(q, q, q)).$$

G_{21} is determined from the condition that the equation for h_{21}, the cubic order $w^2\bar{w}$ coefficient, has a solution. This condition can be written as

$$G_{21} = \langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})\rangle,$$

where we have used the normalisation condition on p and q. The first Lyapunov coefficient is then defined as

$$l_1 = \frac{1}{2}\mathrm{Re}\, \mathrm{G}_{21}, \tag{3.8}$$

and determines the nonlinear stability of a nondegenerate codimension one Hopf bifurcation. If $l_1 = 0$ for some parameter choices, the Hopf bifurcation becomes degenerate and the higher order quintic term G_{32} is required to determine the stability and direction of branching of the bifurcating limit cycles.

Defining $\mathcal{H}_{32}$ as

$$\begin{aligned}\mathcal{H}_{32} &= 6B(h_{11},h_{21}) + B(\bar{h}_{20},h_{30}) + 3B(\bar{h}_{21},h_{20}) \\ &+3B(q,h_{22}) + 2B(\bar{q},h_{31}) + 6C(q,h_{11},h_{11}) \\ &+3C(q,\bar{h}_{20},h_{20}) + 3C(q,q,\bar{h}_{21}) + 6C(q,\bar{q},h_{21}) \\ &+6C(\bar{q},h_{20},h_{11}) + C(\bar{q},\bar{q},h_{30}) + D(q,q,q,\bar{h}_{20}) \\ &+6D(q,q,\bar{q},\bar{h}_{11}) + 3 - D(q,\bar{q},\bar{q},h_{20}) + \\ &E(q,q,q,\bar{q},\bar{q}) - 6G_{21}h_{21} - 3\bar{G}_{21}h_{21},\end{aligned}$$

G_{32} is determined from the scalar product $G_{32} = \langle p, \mathcal{H}_{32}\rangle$ and defines the second Lyapunov coefficient l_2 as

$$l_2 = \frac{1}{12}\mathrm{Re}\,\mathrm{G}_{32}. \tag{3.9}$$

If both l_1 and l_2 vanish simultaneously, we require the coefficient G_{43} of the seventh order terms $w^4\bar{w}^3$ to give the third Lyapunov coefficient

$$l_3 = \frac{1}{144}\mathrm{Re}\,\mathrm{G}_{43}, \tag{3.10}$$

where $G_{43} = \langle p, \mathcal{H}_{43}\rangle$. The expression for $\mathcal{H}_{43}$ is too large to be put in print and can be found in [11,12,18].

3.2. Application to system (2.1) for $f = 0$

We now apply the above Hopf bifurcation theory to system (2.1) in the simplified situation where $h(x)$ is an even function so that the parameter $f = 0$.

Substituting $\lambda = i\omega$ into (2.7), system (2.1) undergoes a Hopf bifurcation along the curve, given by equality in the second term of (2.8): $g_h = \frac{e}{32(2-e)}$. The frequency ω satisfies ${\omega_0}^2 = \frac{1}{4-2e} = 8g + 1/4 > 0$, so we require $e < 2$. The third eigenvalue is $\lambda_1 = -(1 - e/2)$. Therefore $\lambda_1 < 0$. The transversality condition, evaluated at g_h

$$\lambda'(g_h) = -\frac{8(2-e)^2}{2 + (2-e)^3} < 0, \tag{3.11}$$

is satisfied and the equilibrium state E undergoes a Hopf bifurcation, whose stability depends upon the first Lyapunov coefficient l_1. This leads to the following Theorem.

Theorem 3.1 *For system (2.1) with $e < 2$, $f = 0$ and $g_h = \frac{e}{32(2-e)}$, the first Lyapunov coefficient at E is given by*

$$l_1 = \frac{G(u)}{2u(1+u)(1024 + 80u^3 + u^6)}, \tag{3.12}$$

where $u = 4 - 2e > 0$ and $G(u) = 512 + 1344u + 960u^2 + 300u^3 - 84u^4 - 19u^5 + 3u^6$. Since $u > 0$ ($e < 2$), the denominator of (3.12) is positive so that the sign and roots of l_1 are determined by $G(u)$. Denoting by

$e_i(i=1,2)$ the only 2 roots of $G(u)$ for which $u>0$, we find that $e_1 \approx -1.65331$ and $e_2 \approx -0.65080$. Moreover, the following results are also obtained:

(i) When $g=g_h$, $e_1<e<e_2$, system (2.1) undergoes a transversal Hopf bifurcation at a stable weak focus E for the flow restricted to the center manifold. Moreover, for each $g<g_h(e_1)$, but close to $g_h(e_1)$, there exists a stable limit cycle near the unstable equilibrium point E.

(ii) When $g=g_h$, $e<e_1$ or $e_2<e<2$, system (2.1) undergoes a transversal Hopf bifurcation at an unstable weak focus E for the flow restricted to the center manifold. Moreover, for each $g>g_h(e_2)$, but close to $g_h(e_2)$, there exists a unstable limit cycle near the stable equilibrium point E.

Proof Since $e<2$ and $f=0$, from (3.11), the transversality condition holds at the Hopf point g_h and we can calculate the first Lyapunov coefficient, determining the stability of the bifurcating limit cycle.

Writing

$$\lambda_1 = -\frac{u}{4},\ \lambda_{2,3} = \pm\frac{1}{\sqrt{u}}\,i,$$

the eigenvectors p,q, satisfying (3.12), are

$$p=\left(\frac{8\sqrt{u}+8iu}{4i-u^{3/2}},-\frac{4i(-4-3u+u^2)}{-4i+u^{3/2}},\frac{(-i+\sqrt{u})u}{2(-4i+u^{3/2})}\right), \tag{3.13}$$

$$q=\left(-\frac{i}{4\sqrt{u}},-\frac{i}{8(i+\sqrt{u})},1\right). \tag{3.14}$$

From (3.10) and (3.11), we have

$$\begin{aligned} B(\mathbf{X},\mathbf{Y}) &= (2eX_1Y_1+X_2Y_3+X_3Y_2,2X_1Y_1,0)\,,\\ C(\mathbf{X},\mathbf{Y},\mathbf{Z}) &= (0,0,0), \end{aligned}$$

so that

$$\begin{aligned} B(q,q) &= \left(-\frac{4-u}{16u}-\frac{1+i\sqrt{u}}{4(1+u)},-\frac{4-u}{16u},0\right),\\ B(q,\bar{q}) &= \left(-\frac{4-u}{16u}-\frac{1}{4(1+u)},\frac{1}{8u},0\right), \end{aligned}$$

$$\begin{aligned} h_{11} &= \left(0,\frac{1}{8u},\frac{4+3u+3u^2}{u^2+u^3}\right),\\ h_{20} &= (h_{201},h_{202},h_{203})\,, \end{aligned}$$

where $u=4-2e$ and

$$\begin{aligned} h_{201} &= -\frac{-4+12i\sqrt{u}+9u+5iu^{3/2}+2u^2}{6(i+\sqrt{u})u(8i+u^{3/2})},\\ h_{202} &= \frac{-16i-16\sqrt{u}+iu-7u^{3/2}}{24(i+\sqrt{u})\sqrt{u}(8i+u^{3/2})}, \end{aligned}$$

$$h_{203} = \frac{4i + 12\sqrt{u} - 9iu + 5u^{3/2} - 2iu^2}{3(-8\sqrt{u} + 8iu + iu^2 + u^{5/2})}.$$

Since $C(q, q, \bar{q}) = 0$, G_{21} reduces to

$$G_{21} = \langle p, B(\bar{q}, h_{20}) + 2B(q, h_{11})\rangle,$$

so that (3.16) gives

$$l_1 = \frac{G(u)}{2u(1+u)(1024 + 80u^3 + u^6)}, \tag{3.15}$$

where

$$G(u) = 512 + 1344u + 960u^2 + 300u^3 - 84u^4 - 19u^5 + 3u^6.$$

□

Using Mathematica, we find that there are only 2 roots of $l_1 = 0$ for which $u > 0$: $e_1 \approx -1.65331$ (so that $g_h(e_1) \approx -0.01414$) and $e_2 \approx -0.65080$ (so that $g_h(e_2) \approx -0.00767$). It is easy to show that $l_1 > 0$ whenever $e < e_1$ or $e_2 < e < 2$, but $l_1 < 0$ when $e_1 < e < e_2$. Since g_h increases monotonically with e, we obtain the direction of bifurcation. Therefore, Theorem 3.1 is proved.

In order to justify the above theoretical analysis of the first Lyapunov coefficient for the Hopf bifurcation of system (2.1), we chose one set of parameters with $f = 0$, $e = -1.2$, and $g = -0.0142 < g_h(e_1)$. According to Theorem 3.1, a stable periodic solution should be found near the unstable equilibrium point E. This is indeed the case, as shown in Figures 3a and 3b.

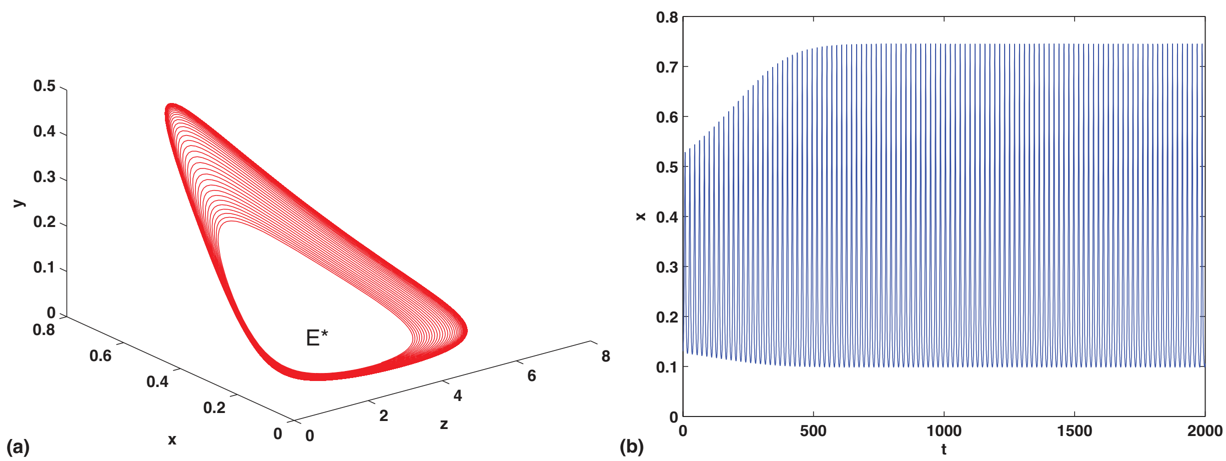

Figure 3. Parameter values $(f, g, e) = (0, -0.016, -2.1)$ of system (2.1) with initial value (0.28, 0.032, 1): a) stable periodic solution in 3-D space; b) times series of state variable $x(t)$.

For $g > g_h(e_2)$, the equilibrium point E is asymptotically stable. Note that for these parameter values, we have the bifurcation value $g = g_h(e_2) \approx -0.00767$. Therefore, system (2.1) undergoes a Hopf bifurcation when the parameter g crosses the critical value $g_h(e_2)$, and an unstable periodic orbit emerges from E with $g > g_h(e_2)$. Choosing $f = 0$, $e = -0.4$, and $g = 0 > g_h(e_2)$, we take initial values $(0.28, 0.032, 0.1)$ near the equilibrium E, the solution of system (2.1) eventually close to 0 (Figure 4a). However, if we take initial values $(-0.6, 0.9, -1.7)$ 'outside' the unstable periodic orbit (it does exist from the Hopf bifurcation), a chaotic attractor exists near the unstable equilibrium E (Figure 4b). Therefore, it seems that when the parameter g moves away from the critical value $g = g_h(e_2)$, a chaotic attractor is generated occurring from the unstable limit cycle that arose in the Hopf bifurcation.

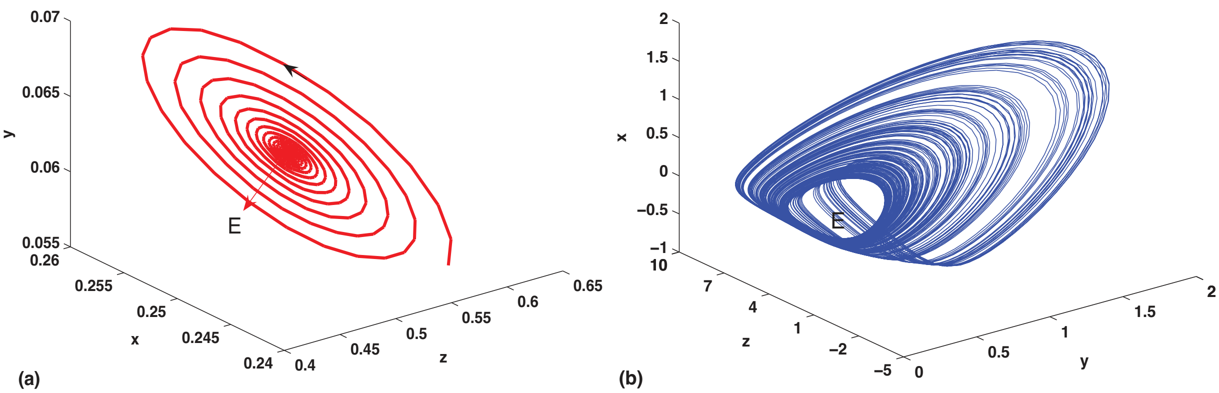

Figure 4. Attractors of system (2.1) with parameter values $f = 0$, $e = -0.5$, and $g = 0 > g_h(e_2)$: a) asymptotically stable equilibrium point E for starting initial values $(0.28, 0.032, 0.1)$; b) chaotic attractor for starting initial values $(-0.6, 0.9, -1.7)$.

Since the sign of the first Lyapunov coefficient, l_1, is determined by the sign of $G(u)$ in (3.23), l_1 vanishes at the roots of $G(u)$, namely for

$$(e_1, u_1, g_h(e_1)) \approx (-1.65331, 7.70662, -0.01414), \tag{3.16}$$

and

$$(e_2, u_2, g_h(e_2)) \approx (-65080, 5.3016, -007667). \tag{3.17}$$

In the next theorem, Theorem 3.2, we determine the sign of the second Lyapunov coefficient when $l_1 = 0$. Because of the complexity of the calculations, we report our results using numerical values for the various quantities evaluated in (3.24) and (3.25).

Theorem 3.2 *Consider system (2.1). If the parameters*

$$(e, f, g) \in Q_i = \left\{ (e, f, g) | e = e_i, f = 0, g_i = \frac{e_i}{32(2 - e_i)} \right\}$$

$(i = 1, 2)$, *then when* $l_1 = 0$, *the second Lyapunov coefficient* l_2 *at the equilibrium state* E *is given by*

$$l_2|_{e=e_1} = -0.00119, \quad l_2|_{e=e_2} = 0.00869. \tag{3.18}$$

Therefore, system (2.1) has a transversal Hopf bifurcation point at the equilibrium state E, *which is a stable weak focus for* $(e, f, g) \in Q_1$ *and an unstable weak focus for* $(e, f, g) \in Q_2$.

Proof The algebraic expressions to calculate the second Lyapunov coefficient are too long to be written out in detail. Instead we present numerical values for the various terms required to determine G_{32} (see above) for $(e, g_h) = (e_1, g_h(e_1)) \approx (-1.65331, -0.01414)$. A similar analysis yields the corresponding G_{32} for $(e, g_h) = (e_2, g_h(e_2)) \approx (-65080, -007667)$. We merely give the final result here.

For (3.23), when the first Lyapunov coefficient $l_1 = 0$, we obtain:

$$
\begin{aligned}
p &= (-0.475974 - 3.05599i, 1.08223 - 5.34361i,\\
&\quad 0.516286 - 0.080412i),\\
q &= (-0.09249i, -0.01505 - 0.04068i, 1),\\
h_{11} &= (0, 0.01711, 0.41961),\\
h_{20} &= (-0.07908 + 0.00616i, -0.03514 + 0.02908i,\\
&\quad -0.03331 - 0.42754i),\\
G_{21} &= -0.01441i,\\
h_{21} &= (0.00362 + 0.00769i, -0.00247 - 0.01009i,\\
&\quad -0.04422 + 0.03922i),\\
h_{30} &= (0.02474 + 0.09458i, 0.05242 + 0.03300i,\\
&\quad -0.34087 + 0.08917i),\\
h_{31} &= (0.02913 + 0.00044i, -0.01364 + 0.00206i,\\
&\quad 0.00616 - 0.00048i),\\
h_{22} &= (0, 0.00689, 0.09363),\\
G_{32} &= -0.01432 - 0.00910i.
\end{aligned}
$$

Therefore, the second Lyapunov coefficient l_2 for (22) when $l_1 = 0$ is

$$l_2 = \frac{1}{12}\mathrm{Re}G_{32} = -0.00119.$$

Moreover, Mathematica gives

$$G_{32} = 0.104257 - 0.049206i,$$

so that

$$l_2 = \frac{1}{12}\mathrm{Re}G_{32} = 0.00869.$$

The proof of Theorem 3.2 is therefore complete. □

4. Dynamical structure of the extended Sprott E system

We now report on our numerical integrations of the extended Sprott E system, summarizing our results in plots of the Lyapunov exponent spectra and bifurcation transition diagrams as e varies. Although the Theorems in Section 3 were applied to the simplified case of $f = 0$, here we also include results for the more general form of the quadratic controller.

4.1. e increasing when $f = 0, g = \frac{e}{32(2-e)}$

We first fix $f = 0, g = \frac{e}{32(2-e)}$ and vary $e \in [-0.1, 0.3]$. According to Theorem 3.1, the equilibrium state E is a nonhyperbolic and unstable weak focus for $f = 0$, $g = \frac{e}{32(2-e)}$, and $e \in [-0.1, 0.3]$. The bifurcation transition diagram for x_{max} as e varies is shown in Figure 5a. Moreover, the corresponding Lyapunov exponent spectrum is shown in Figure 5b. Period-doubling Feigenbaum-type bifurcation is evident in system (2.1), integrated from initial values of $(x_0, y_0, z_0) = (0.28, 0.032, 1)$.

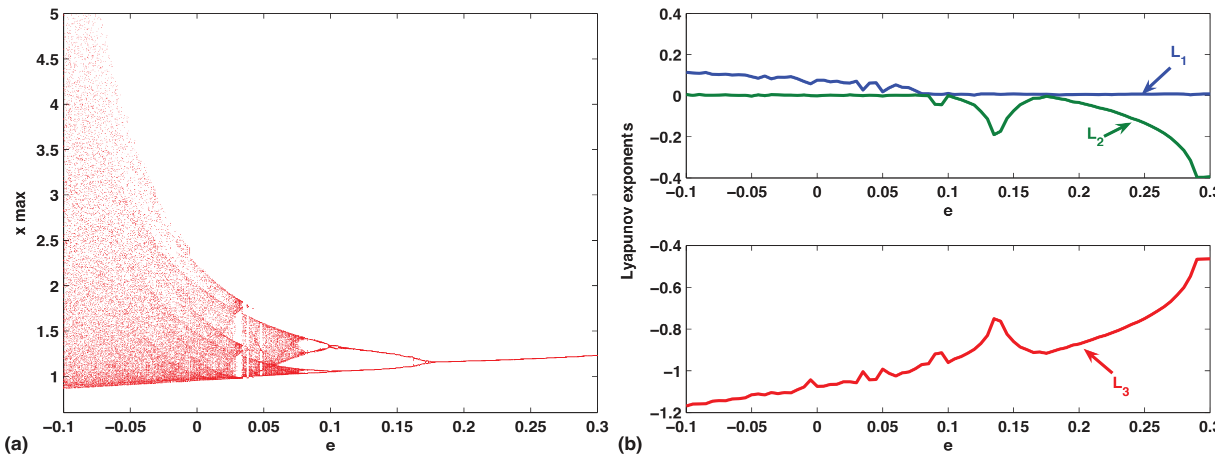

Figure 5. Parameter values $(f, g) = (0, \frac{e}{32(2-e)})$ of system (2.1) with initial value (0.28, 0.032, 1): a) bifurcation diagram of the variable z with $e \in [-0.1, 0.3]$; b) Lyapunov exponent spectrum with $e \in [-0.1, 0.3]$.

As e is decreased, a stable periodic limit cycle undergoes a period-doubling bifurcation when $e = 0.178$. Decreasing e further, a second period-doubling bifurcation to a period-4 attractor occurs for $e = 0.100$. Subsequent period-doubling cascades follow and merge together to produce behavior indicative of the onset of chaos. Also present are windows of odd periodic and corresponding period-doubling cascade, for example the period-5 window at $e \approx 0.05$.

4.2. e increasing when $f = -0.1, g = 0.02$

Figure 6a shows the Lyapunov exponent spectra, starting from the initial value $(x_0, y_0, z_0) = (-0.6, 0.9, -1.7)$ for $f = -0.1$ and $g = 0.02$ as e varies in $e \in [-0.4, 0.5]$. Figure 6b shows the corresponding bifurcation diagram of the state variable $z(t)$. From condition (2.8) in Section 2.3, E is asymptotically stable in this range for e. The maximum Lyapunov exponent is negative for $e \in [-0.4, -0.303)$, implying that (2.1) evolves to a stable sink. For $e > -0.303$, the system undergoes a cascade of period doubling bifurcations, with windows of periodic orbits, interspersing chaotic regimes, before a cascade of period halving bifurcations heralds the reverse bifurcation sequence in the region [0.267, 0.5]. From Figure 6b, it is clear that $-0.015 \leq e < 0.08$ is a periodic window. For $-0.015 \leq e < 0.029$ we have a stable period-2 orbit region, while for $0.029 < e < 0.048$, it is a stable period-4 orbit region. As e increases in $0.048 < e < 0.08$, system (2.1) is chaotic. The periodic windows play an important role in the evolution of dynamical behaviors of system (2.1). It is illustrated in the case of a period-doubling sequence of bifurcations leading to a Feigenbaum-like strange attractor. Although system (2.1) in the parameter region has stable equilibria, the existence of a universal ratio characterizes the transition to chaos via period-doubling bifurcations. Moreover, there is a reestablishing of simple periodic states for $e > 0.3$.

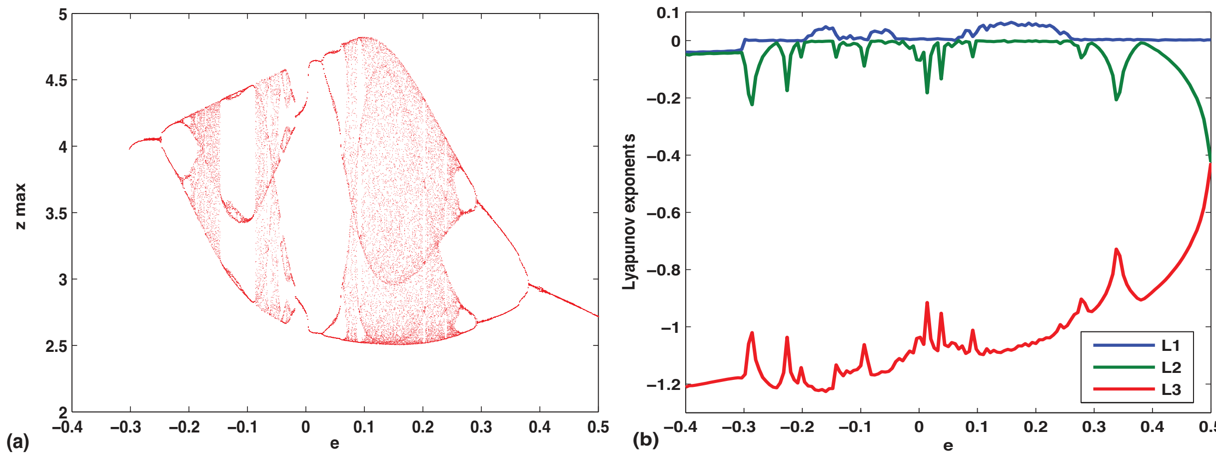

Figure 6. Parameter values $(f, g) = (-0.1, 0.02)$ of system (2.1) with initial value $(-0.6, 0.9, -1.7)$: a) bifurcation diagram of the variable z with $e \in [-0.4, 0.5]$; b) Lyapunov exponent spectrum with $e \in [-0.4, 0.5]$.

5. Adaptive control of the extended Sprott E system

5.1. Theoretical results

In this section, we design an adaptive control law for globally stabilizing the extended Sprott E system (2.1) when the parameter value is unknown. Thus, we consider the controlled extended Sprott E system described by

$$\begin{cases} \dot{x_1} = x_2x_3 + h(x) + u_1 \\ \dot{x_2} = x_1^2 - x_2 + u_2 \\ \dot{x_3} = 1 - 4x_1 + u_3, \end{cases} \tag{5.1}$$

where u_1, u_2, and u_3 are feedback controllers to be designed using the states and estimates of the unknown parameter of the system. In order to ensure that the controlled system (5.1) globally converges to the origin asymptotically, we consider the following adaptive control functions:

$$\begin{cases} u_1 = -x_2x_3 - \hat{e}x_1^2 - \hat{f}x_1 - \hat{g} - k_1x_1 \\ u_2 = -x_1^2 + x_2 - k_2x_2 \\ u_3 = -1 + 4x_1 - k_3x_3, \end{cases} \tag{5.2}$$

where $\hat{e}, \hat{f}$, and $\hat{g}$ are the estimate of the parameters e, f and g, respectively, and $k_i (i = 1, 2, 3)$ are positive constants. If we define the parameter estimation error as

$$\begin{cases} e_e = e - \hat{e} \\ e_f = f - \hat{f} \\ e_g = g - \hat{g}, \end{cases} \tag{5.3}$$

for the derivation of the update law for adjusting the parameter estimates, the Lyapunov approach is used. Consider the quadratic Lyapunov function

$$V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + e_e^2 + e_f^2 + e_g^2), \tag{5.4}$$

which is a positive definite function on R^6. Differentiating V along the trajectories of system (5.1), we obtain

$$\begin{aligned}\dot{V} &= -k_1x_1^2 - k_2x_2^2 - k_3x_3^2 + e_e(x_1^3 - \dot{\hat{e}}) \\ &\quad + e_f(x_1^2 - \dot{\hat{f}}) + e_g(x_1 - \dot{\hat{g}}).\end{aligned} \tag{5.5}$$

Therefore, the estimated parameters are updated by the following law:

$$\begin{cases} \dot{\hat{e}} = x_1^3 + k_4e_e \\ \dot{\hat{f}} = x_1^2 + k_5e_f \\ \dot{\hat{g}} = x_1 + k_6e_g, \end{cases} \tag{5.6}$$

where k_4, k_5, k_6 are positive constants. Then

$$\dot{V} = -k_1x_1^2 - k_2x_2^2 - k_3x_3^2 - k_4e_e^2 - k_5e_f^2 - k_6e_g^2. \tag{5.7}$$

which is a negative definite function. Thus, by Lyapunov stability theory [2,6], we obtain the following result.

Theorem 5.1 *The extended Sprott E system with unknown parameters (5.1) is globally and exponentially stabilized for all initial conditions* $(x_1(0), x_2(0), x_3(0)) \in R^3$ *by the adaptive control law (5.2), where the update law for the parameter is given by (5.6) and* $k_i (i = 1, 2, 3, 4, 5, 6)$ *are positive constants.*

5.2. Numerical results

Compared to Figure 1 in Section 2, the parameters of the extended Sprott E system (2.1) are selected as $(e, f, g) = (0.2, -0.1, 0.02)$. For the adaptive and update laws, we take $k_i = 2, (i = 1, 2, 3, 4, 5, 6)$. Suppose that the initial value of the parameter estimates are taken as $\hat{e}(0) = 1, \hat{f}(0) = 4, \hat{g}(0) = -4$. The initial values of the system (5.1) are taken as $x_1(0) = -0.6$, $x_2(0) = 0.9$, $x_3(0) = -1.7$.

When the adaptive control law (5.2) and the parameter update law (5.6) are used, the controlled extended Sprott E system (5.1) converges to the equilibrium (0,0,0) exponentially as shown in Figure 7. Figure 8 shows that the parameter estimates $\hat{e}, \hat{f}, \hat{g}$ converge to the actual values of the system parameters $(e, f, g) = (0.2, -0.1, 0.02)$.

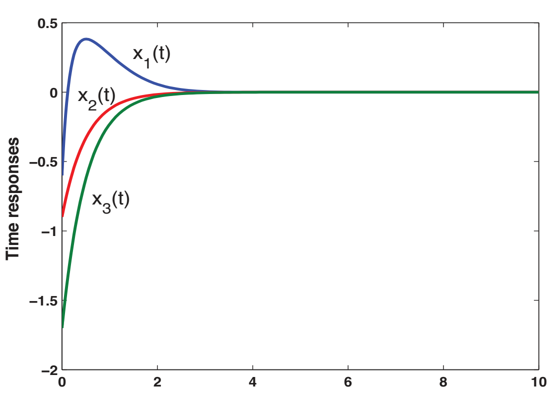

Figure 7. Time responses of the controlled extended Sprott E system (5.27) when parameters values $(e, f, g) = (0.2, -0.1, 0.02)$ and initial value $(-0.6, 0.9, -1.7)$.

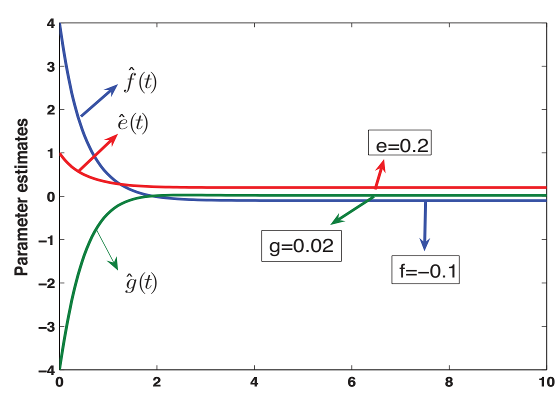

Figure 8. Parameter estimates $\hat{e}(t), \hat{f}(t)$, and $\hat{g}(t)$ when parameters values $(f, g, e) = (-0.1, 0.02, 0.2)$.

6. Conclusion

In this paper, the extended Sprott E system with a nonlinear term $h(x)$ in the form of a quadratic polynomial x has been investigated. Through this analysis we obtained the surfaces for which the system undergoes Hopf bifurcations from the equilibrium state E. Then we extended the analysis to degenerate cases, where the first Lyapunov coefficient vanishes. Calculation of the second Lyapunov coefficient enables the Lyapunov stability to be determined. Basic properties of the system have been analyzed by means of Lyapunov exponent spectrum, bifurcation diagram, and associated Poincaré map. Adaptive control laws are effective to stabilize the extended Sprott E system based on the adaptive control theory and Lyapunov stability theory. Strange chaotic attractors with stable equilibria deserve further investigation and are very desirable for engineering applications such as secure communications in the near future.

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