Degraded Image Analysis: An Invariant Approach

Jan Flusser and Tomáš Suk

Abstract—Analysis and interpretation of an image which was acquired by a nonideal imaging system is the key problem in many application areas. The observed image is usually corrupted by blurring, spatial degradations, and random noise. Classical methods like blind deconvolution try to estimate the blur parameters and to restore the image. In this paper, we propose an alternative approach. We derive the features for image representation which are invariant with respect to blur regardless of the degradation PSF provided that it is centrally symmetric. As we prove in the paper, there exist two classes of such features: the first one in the spatial domain and the second one in the frequency domain. We also derive so-called combined invariants, which are invariant to composite geometric and blur degradations. Knowing these features, we can recognize objects in the degraded scene without any restoration.

Index Terms—Degraded image, symmetric blur, blur invariants, image moments, combined invariants, object recognition.

1 Introduction

1.1 Motivation

Analysis and interpretation of an image which was acquired by a real (i.e., nonideal) imaging system is the key problem in many application areas such as remote sensing, astronomy and medicine, among others. Since real imaging systems as well as imaging conditions are usually imperfect, the observed image represents only a degraded version of the original scene. Various kinds of degradations (geometric as well as radiometric) are introduced into the image during the acquisition by such factors as imaging geometry, lens aberration, wrong focus, motion of the scene, systematic and random sensor errors, etc.

In the general case, the relation between the ideal image $f(x, y)$ and the observed image $g(x, y)$ is described as $g = \mathcal{D}(f)$, where $\mathcal{D}$ is a degradation operator. In the case of a linear shift-invariant imaging system, $\mathcal{D}$ is realized as

$$g(\tau(x, y)) = (f * h)(x, y) + n(x, y), \tag{1}$$

where $h(x, y)$ is the point-spread function (PSF) of the system, $n(x, y)$ is an additive random noise, τ is a transform of spatial coordinates due to projective imaging geometry and $*$ denotes a 2D convolution. Knowing the image $g(x, y)$, our objective is to analyze the unknown scene $f(x, y)$.

1.2 Present State-of-the-Art

First, let us suppose for simplicity that no geometric distortions and no noise are present, i.e.,

$$g(x, y) = (f * h)(x, y). \tag{2}$$

In some rare cases, the PSF is known explicitly prior to the restoration process or it can be easily estimated, for instance from a point source in the image. This classical image restoration problem has been treated by a lot of techniques, the most popular of which are inverse and Wiener filtering [1] and constrained deconvolution methods [2], [3]. However, this is not the subject of this paper.

In most applications the blur is unknown. Partial information about the PSF or the true image is sometimes available—the symmetry of $h(x, y)$, its bounded support or the positivity of $f(x, y)$, for instance. There are basically two different approaches to degraded image analysis in that case: blind restoration and direct analysis.

Blind image restoration has been discussed extensively in the literature (see [4] or [5] for a basic survey). It is the process of estimating both the original image and the PSF from the degraded image using partial information about the imaging system. However, this is an ill-posed problem, which does not have a unique solution and the computational complexity of which could be extremely high.

There are several major groups of blind restoration methods. Well-known *parametric methods* assume that a parametric model of the PSF is given a priori. Investigating the zero patterns of the Fourier transform or of the cepstrum of $g(x, y)$, the unknown parameters are estimated [6], [7], [8], [9]. This approach is very powerful in motion and out-of-focus deblurring, for instance.

Promising results were achieved by the *zero-sheet separation* method, which was firstly introduced in [10] and further developed in [11]. The method is based on the properties of Z-transform. It was proven that the zeros of the Z-transform of each $f(x, y)$ and $h(x, y)$ lie on distinct continuous surfaces called zero sheets. Separating these two zero sheets from each other, we can restore both $f(x, y)$ and $h(x, y)$ up to a scaling factor. The weakness of the method is in its numerical implementation—it is very sensitive to noise and time-consuming. A more detailed study on the practical

• *The authors are with the Institute of Information Theory and Automation, Department of Image Processing, Academy of Sciences of the Czech Republic, 182 08 Prague 8, Czech Republic. E-mail: {flusser, suk}@utia.cas.cz.*

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applications of the zero-sheet separation method can be found in [12]. An attempt to make the method more robust to noise is described in [13].

Another group of methods is based on modeling of the image by a stochastic process. The original image is modeled as an autoregressive (AR) process and the blur as a moving average (MA) process. The blurred image is then modeled as a mixed autoregressive moving average (ARMA) process and the MA process identified by this model is considered as a description of the PSF. In this way the problem of the PSF estimation is transformed onto the problem of determining the parameters of an ARMA model. The methods of this category differ in how the ARMA parameters are estimated. Basic approaches are *maximum likelihood estimation* [14], [15], [16], and *generalized cross-validation* [17].

Projection-based approach to blind deconvolution proposed in [18] attempts to incorporate prior knowledge about the original image and the PSF through constraint sets. This method was proven to perform well even if the prior information was not perfect. However, the solution may be not unique. Except the projection-based method, there is a number of other nonparametric algorithms which also use prior deterministic constraints, such as image positivity or the size of the PSF support. *Iterative blind deconvolution* [19], [20], [21], and *simulated annealing* [22] fall into this category. A group of deconvolution methods based on *higher-order statistics* (HOS) was designed particularly for restoring images with textures [23], [24].

Direct analysis of the degraded image is based on the different idea: In many cases, one does not need to know the whole original image, one only needs for instance to localize or recognize some objects on it (typical examples are matching of a template against a blurred aerial image or recognition of blurred characters). In such cases, only knowledge of some representation of the objects is sufficient. However, such a representation should be independent of the imaging system and should really describe the original image, not the degraded one. In other words, we are looking for a functional I which is invariant to the degradation operator, i.e., $I(g) = I(f)$. Particularly, if the degradation is described by (2), then the equality

$$I(f) = I(f * h)$$

must hold for any admissible $h(x, y)$.

There have been described only few invariants to blurring in the literature. Most of them are related to very special types of $h(x, y)$ and derived in a heuristic manner. No consistent theory has been published so far. A set of invariants to motion blur was presented in [25]. Recognition of defocused facial photographs by another set of invariants was described in [26]. All those invariants were constructed in spatial domain only. First attempt to find blur invariants in Fourier domain was published in [27], but that paper dealt with 1D signals only.

1.3 The Aim of the Paper

The major attention of this paper is devoted to the following topics:

- To find blur invariants defined in Fourier as well as spatial domains and theoretically prove their property of invariance.
- To find combined blur-geometric invariants (i.e., the features invariant also to some group of spatial transformations of the image plane).
- To approve experimentally the capability of the invariants to recognize objects in a blurred and noisy scene.

The rest of the paper is organized as follows. In Section 2, some basic definitions and propositions are given to build-up necessary mathematical background. Sections 3, 4, and 5 perform the major theoretical part of the paper. Blur invariants in the Fourier domain and in the spatial domain are introduced in Sections 3 and 4, respectively. A close relationship between both classes of invariants is shown in Section 5. Discriminative power of the invariants and their robustness to additive noise are investigated in Sections 6 and 7. Section 8 is devoted to the so-called combined invariants, i.e., invariants to various composite degradations. Finally, Sections 9 and 10 describe numerical experiments.

2 Basic Notations and Mathematical Preliminaries

In this section we introduce some basic terms and relations which will be used later in this paper.

DEFINITION 1. *By image function* (or image) *we understand any real function* $f(x, y) \in L_1$ *which is nonzero on bounded support and*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy > 0 .$$

DEFINITION 2. *Ordinary geometric moment* $m_{pq}^{(f)}$ *of order* $(p + q)$ *of the image* $f(x, y)$ *is defined by the integral*

$$m_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)dxdy . \tag{3}$$

DEFINITION 3. *Central moment* $\mu_{pq}^{(f)}$ *of order* $(p + q)$ *of the image* $f(x, y)$ *is defined as*

$$\mu_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x - x_t^{(f)}\right)^p \left(y - y_t^{(f)}\right)^q f(x,y)dxdy , \tag{4}$$

where the coordinates

$$x_t^{(f)} = \frac{m_{10}^{(f)}}{m_{00}^{(f)}} ,$$

$$y_t^{(f)} = \frac{m_{01}^{(f)}}{m_{00}^{(f)}}$$

denote the centroid of $f(x, y)$.

DEFINITION 4. *Fourier transform* (or *spectrum*) $F(u, v)$ *of the image* $f(x, y)$ *is defined as*

$$\mathcal{F}(f)(u,v) \equiv F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot e^{-2\pi i(ux+vy)}dxdy\,,$$

where i is the complex unit.

Since $f(x, y) \in L_1$, its Fourier transform always exists.

LEMMA 1. *Let $f(x, y)$ and $h(x, y)$ be two image functions and let $g(x, y) = (f * h)(x, y)$. Then $g(x, y)$ is also an image function and it holds for its moments*

$$m_{pq}^{(g)} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j} m_{kj}^{(h)} m_{p-k,q-j}^{(f)}$$

and

$$\mu_{pq}^{(g)} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j} \mu_{kj}^{(h)} \mu_{p-k,q-j}^{(f)}$$

for any p and q.

LEMMA 2. *Let $h(x, y)$ be a centrally symmetric image function, i.e., $h(x, y) = h(-x, -y)$. Then*

- $\mu_{pq}^{(h)} = m_{pq}^{(h)}$ for every p and q;
- If $p + q$ is odd, then $\mu_{pq}^{(h)} = 0$.

LEMMA 3. *The relationship between the Fourier transform of an image and the geometric moments is expressed by the following equation*:

$$F(u,v) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-2\pi i)^{k+j}}{k!\cdot j!} m_{kj}^{(f)}\cdot u^k v^j\,.$$

The assertions of Lemmas 1, 2, and 3 can be easily proven just using the definition of moments, convolution and Fourier transform.

In the following text, we will assume that the PSF $h(x, y)$ is a centrally symmetric image function and that the imaging system is energy-preserving, i.e.,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)dxdy = 1\,.$$

The invariants with respect to such a system will be called *blur invariants*.

The assumption of centrosymmetry is not a significant limitation of practical utilisation of the method. Most real sensors and imaging systems, both optical and nonoptical ones, have the PSF with certain degree of symmetry. In many cases they have even higher symmetry than central one, such as axial or radial symmetry. Thus, the central symmetry is general enough to describe almost all practical situations. On the other hand, if we restrict ourselves to PSF with radial or axial symmetry, we derive another set of blur invariants (see Appendix A). Generally, the higher degree of symmetry of the PSF is assumed, the more invariants can be obtained.

3 INVARIANTS IN THE SPECTRAL DOMAIN

In this section, blur invariants in the Fourier spectral domain are investigated.

THEOREM 1. *Tangent of the Fourier transform phase is a blur invariant.*

PROOF. Due to the well-known convolution theorem, the corresponding relation to (2) in the spectral domain has the form

$$G(u, v) = F(u, v)\cdot H(u, v), \tag{5}$$

where $G(u, v)$, $F(u, v)$, and $H(u, v)$ are the Fourier transforms of the functions $g(x, y)$, $f(x, y)$, and $h(x, y)$, respectively. Considering the amplitude and phase separately, we get

$$|G(u, v)| = |F(u, v)|\cdot|H(u, v)| \tag{6}$$

and

$$\mathrm{ph}G(u, v) = \mathrm{ph}F(u, v) + \mathrm{ph}H(u, v) \tag{7}$$

(note that the last equation is correct only for those points where $G(u, v) \neq 0$; $\mathrm{ph}G(u, v)$ is not defined otherwise).

Due to the central symmetry of $h(x, y)$, its Fourier transform $H(u, v)$ is real (that means the phase of $H(u, v)$ is only a two-valued function):

$$\mathrm{ph}H(u, v) \in \{0; \pi\}.$$

It follows immediately from the periodicity of tangent that

$$\tan(\mathrm{ph}G(u, v)) =$$

$$\tan(\mathrm{ph}F(u, v) + \mathrm{ph}H(u, v)) = \tan(\mathrm{ph}F(u, v)). \tag{8}$$

Thus $\tan(\mathrm{ph}G(u, v))$ is invariant with respect to convolution of the original image with any centrally symmetric PSF. □

Note, that the phase itself is not invariant with respect to blur and therefore it cannot be directly used for blurred image description and recognition.

4 INVARIANTS IN THE SPACE DOMAIN

In this section, blur invariants based on image moments are introduced.

THEOREM 2. *Let $f(x, y)$ be an image function. Let us define the following function $C^{(f)}: \mathbf{Z}\times\mathbf{Z}\to\mathbf{R}$.*

If $(p + q)$ is even then

$$C(p, q)^{(f)} = 0.$$

If $(p + q)$ is odd then

$$C(p,q)^{(f)} =$$

$$\mu_{pq}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{n=0}^{p}\sum_{\substack{m=0\\0<n+m<p+q}}^{q}\binom{p}{n}\binom{q}{m} C(p-n,q-m)^{(f)}\cdot\mu_{nm}^{(f)}. \tag{9}$$

Then $C(p, q)$ is a blur invariant for any p and q. The number $r = p + q$ is called the order of the invariant.

For proof of Theorem 2, see Appendix B.

Applying (9), we can construct the invariants of any order and express them in the explicit form. The set of invariants of the third, fifth, and seventh orders is listed below:

- Third order:

$$C(3,0) = \mu_{30},$$
$$C(2,1) = \mu_{21},$$
$$C(1,2) = \mu_{12},$$
$$C(0,3) = \mu_{03}.$$

- Fifth order:

$$C(5,0) = \mu_{50} - \frac{10\mu_{30}\mu_{20}}{\mu_{00}},$$
$$C(4,1) = \mu_{41} - \frac{2}{\mu_{00}}(3\mu_{21}\mu_{20} + 2\mu_{30}\mu_{11}),$$
$$C(3,2) = \mu_{32} - \frac{1}{\mu_{00}}(3\mu_{12}\mu_{20} + \mu_{30}\mu_{02} + 6\mu_{21}\mu_{11}),$$
$$C(2,3) = \mu_{23} - \frac{1}{\mu_{00}}(3\mu_{21}\mu_{02} + \mu_{03}\mu_{20} + 6\mu_{12}\mu_{11}),$$
$$C(1,4) = \mu_{14} - \frac{2}{\mu_{00}}(3\mu_{12}\mu_{02} + 2\mu_{03}\mu_{11}),$$
$$C(0,5) = \mu_{05} - \frac{10\mu_{03}\mu_{02}}{\mu_{00}}.$$

- Seventh order:

$$C(7,0) = \mu_{70} - \frac{7}{\mu_{00}}(3\mu_{50}\mu_{20} + 5\mu_{30}\mu_{40}) + \frac{210\mu_{30}\mu_{20}^2}{\mu_{00}^2},$$

$$C(6,1) = \mu_{61} - \frac{1}{\mu_{00}}(6\mu_{50}\mu_{11} + 15\mu_{41}\mu_{20} + 15\mu_{40}\mu_{21} + 20\mu_{31}\mu_{30}) + \frac{30}{\mu_{00}^2}(3\mu_{21}\mu_{20}^2 + 4\mu_{30}\mu_{20}\mu_{11}),$$

$$C(5,2) = \mu_{52} - \frac{1}{\mu_{00}}(\mu_{50}\mu_{02} + 10\mu_{30}\mu_{22} + 10\mu_{32}\mu_{20} + 20\mu_{31}\mu_{21} + 10\mu_{41}\mu_{11} + 5\mu_{40}\mu_{12}) + \frac{10}{\mu_{00}^2}(3\mu_{12}\mu_{20}^2 + 2\mu_{30}\mu_{20}\mu_{02} + 4\mu_{30}\mu_{11}^2 + 12\mu_{21}\mu_{20}\mu_{11}),$$

$$C(4,3) = \mu_{43} - \frac{1}{\mu_{00}}(\mu_{40}\mu_{03} + 18\mu_{21}\mu_{22} + 12\mu_{31}\mu_{12} + 4\mu_{30}\mu_{13} + 3\mu_{41}\mu_{02} + 12\mu_{32}\mu_{11} + 6\mu_{23}\mu_{20}) + \frac{6}{\mu_{00}^2}(\mu_{03}\mu_{20}^2 + 4\mu_{30}\mu_{11}\mu_{02} + 12\mu_{21}\mu_{11}^2 + 12\mu_{12}\mu_{20}\mu_{11} + 6\mu_{21}\mu_{02}\mu_{20}),$$

$$C(3,4) = \mu_{34} - \frac{1}{\mu_{00}}(\mu_{04}\mu_{30} + 18\mu_{12}\mu_{22} + 12\mu_{13}\mu_{21} + 4\mu_{03}\mu_{31} + 3\mu_{14}\mu_{20} + 12\mu_{23}\mu_{11} + 6\mu_{32}\mu_{02}) + \frac{6}{\mu_{00}^2}(\mu_{30}\mu_{02}^2 + 4\mu_{03}\mu_{11}\mu_{20} + 12\mu_{12}\mu_{11}^2 + 12\mu_{21}\mu_{02}\mu_{11} + 6\mu_{12}\mu_{20}\mu_{02}),$$

$$C(2,5) = \mu_{25} - \frac{1}{\mu_{00}}(\mu_{05}\mu_{20} + 10\mu_{03}\mu_{22} + 10\mu_{23}\mu_{02} + 20\mu_{13}\mu_{12} + 10\mu_{14}\mu_{11} + 5\mu_{04}\mu_{21}) + \frac{10}{\mu_{00}^2}(3\mu_{21}\mu_{02}^2 + 2\mu_{03}\mu_{02}\mu_{20} + 4\mu_{03}\mu_{11}^2 + 12\mu_{12}\mu_{02}\mu_{11}),$$

$$C(1,6) = \mu_{16} - \frac{1}{\mu_{00}}(6\mu_{05}\mu_{11} + 15\mu_{14}\mu_{02} + 15\mu_{04}\mu_{12} + 20\mu_{13}\mu_{03}) + \frac{30}{\mu_{00}^2}(3\mu_{12}\mu_{02}^2 + 4\mu_{03}\mu_{02}\mu_{11}),$$

$$C(0,7) = \mu_{07} - \frac{7}{\mu_{00}}(3\mu_{05}\mu_{02} + 5\mu_{03}\mu_{04}) + \frac{210\mu_{03}\mu_{02}^2}{\mu_{00}^2},$$

THEOREM 3. *Let $f(x, y)$ be an image function. Let us define the following function $M^{(f)}: \mathbf{Z} \times \mathbf{Z} \to \mathbf{R}$.*

If $(p + q)$ is even, then

$$M(p, q)^{(f)} = 0.$$

If $(p + q)$ is odd, then

$$M(p,q)^{(f)} = m_{pq}^{(f)} - \frac{1}{m_{00}^{(f)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} M(p-n, q-m)^{(f)} \cdot m_{nm}^{(f)}. \quad (10)$$

($M(p, q)$ is formally similar to $C(p, q)$ but ordinary geometric moments are used instead of the central ones.)

Then $M(p, q)$ is a blur invariant for any p and q.

PROOF. The proof of Theorem 3 is very similar to that of Theorem 2.

THEOREM 4. *Let $f(x, y)$ be an image function. Let us normalize the above mentioned blur invariants as follows:*

$$M'(p,q)^{(f)} = \frac{M(p,q)^{(f)}}{m_{00}^{(f)}},$$

$$C'(p,q)^{(f)} = \frac{C(p,q)^{(f)}}{\mu_{00}^{(f)}}.$$

Then $M'(p, q)$ and $C'(p, q)$ are blur invariants for any p and q, even if the imaging system is energy nonpreserving (i.e., the system PSF $h(x, y)$ is an arbitrary centrally symmetric image function).

PROOF. Let's define a new PSF

$$h_1(x,y) = \frac{h(x,y)}{m_{00}^{(h)}}.$$

Then

$$g(x,y) = (f * h)(x,y) = m_{00}^{(h)}(f * h_1)(x,y)$$

and

$$M(p,q)^{(g)} = m_{00}^{(h)} M(p,q)^{(f*h_1)} = m_{00}^{(h)} M(p,q)^{(f)}$$

because $h_1(x, y)$ is energy-preserving. It follows from Lemma 1 that

$$m_{00}^{(g)} = m_{00}^{(f)} m_{00}^{(h)}.$$

Thus,

$$M'(p,q)^{(g)} = M'(p,q)^{(f)}. \quad \square$$

The proof of invariance of $C'(p,q)$ is similar.

5 Relationship Between Fourier Domain Invariants and Spatial Domain Invariants

In this section, a close relationship between the Fourier transform phase and the moment-based blur invariants is presented.

THEOREM 5. *Tangent of the Fourier transform phase of any image $f(x, y)$ can be expanded into power series (except the points in which $F(u, v) = 0$ or ph$F(u, v) = \pm\pi/2$)*

$$\tan(\text{ph}F(u,v)) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{kj} u^k v^j, \tag{11}$$

where

$$c_{kj} = \frac{(-1)^{(k+j-1)/2} \cdot (-2\pi)^{k+j}}{k! \cdot j!} M'(k,j) \tag{12}$$

if $k + j$ is odd and

$$c_{kj} = 0$$

if $k + j$ is even.

For proof of Theorem 5 see Appendix B.

Theorem 5 demonstrates that from theoretical point of view both groups of blur invariants are equivalent. However, their suitability for practical usage depends on their numerical behavior, noise robustness, geometric distortions of the input patterns, given object classes, etc., and will be investigated experimentally later in this paper.

6 Discriminative Power of the Invariants

In the previous sections, we dealt with the property of invariance of the functional I and we proved that under certain conditions $I(f) = I(f * h)$ for any function $f(x, y)$ and for an arbitrary centrally symmetric function $h(x, y)$. In this section, an inverse problem will be investigated.

Let's consider the feature vector of the normalized moment invariants

$$I(f) = \{M'(p,q)^{(f)}; \quad p, q = 0, 1, \cdots, \infty\}.$$

The following theorem shows the structure of the set of image functions the feature vectors of which are identical.

THEOREM 6. *Let $f(x, y)$, $g(x, y)$, and $h(x, y)$ be three image functions such that*

$$g(x, y) = (f * h)(x, y)$$

and let $I(f) = I(g)$. Then $h(x, y)$ is centrally symmetric.

PROOF. Since $I(f) = I(g)$, it follows from Theorem 5 that

$$\tan(\text{ph}F(u, v)) = \tan(\text{ph}G(u, v)).$$

Due to the periodicity of tangent,

$$\text{ph}G(u, v) = \text{ph}F(u, v) + \phi(u, v),$$

where $\phi(u, v) \in \{0; \pi\}$. Since $g(x, y) = (f * h)(x, y)$, it holds $G(u, v) = F(u, v) \cdot H(u, v)$ and, particularly,

$$\text{ph}G(u,v) = \text{ph}F(u, v) + \text{ph}H(u, v).$$

Comparison of last two equations implies that $H(u, v)$ is a real function. Thus, $h(x, y)$ is centrally symmetric. $\square$

Theorem 6 says that under certain conditions we are always able to distinguish between two different image functions (modulo convolution with a centrally symmetric PSF). However, it does not say anything about the discriminative power of I in general. In other words, it does not follow from Theorem 6 that I is a unique functional. Anyway, in practice $I(f)$ is a *finite* set of features and, consequently, the discriminative power of I is limited.

The following Lemma deals with a null space of blur invariants. It shows that the null space of the functional I contains centrally symmetric functions only. As a consequence of Lemma 4 we can see that any two different centrosymmetric functions cannot be discriminated from each other. The invariants to radially or axially symmetric PSF have a similar property: The null space is always formed by those functions having the same type of symmetry as the PSF.

LEMMA 4. *The image function $f(x, y)$ is centrally symmetric if and only if $I(f) = 0$.*

PROOF. If $f(x, y)$ is centrally symmetric, then all odd-order moments m_{pq} are zero. Consequently, all $M'(p, q)^{(f)}$ are equal to zero too. On the other hand, if all $M'(p, q)^{(f)}$ are equal to zero, then also all odd-order moments m_{pq} are zero. Lemma 3 implies that $F(u, v)$ is real and, consequently, $f(x, y)$ must be centrally symmetric. $\square$

7 Robustness to Additive Noise

Thus far we have considered a noise-free case only, but in practical applications the noise of various kinds is always present. In this section, we study the influence of additive zero-mean random noise on the values of the blur invariants.

Let $g(x, y)$ be a blurred and noisy version of original image $f(x, y)$:

$$g(x, y) = (f * h)(x, y) + n(x, y). \tag{14}$$

Since the image $g(x, y)$ is then a random field, all its moments and all invariants can be viewed as random variables. It holds for any p and q

$$E\left(m_{pq}^{(n)}\right) = 0,$$

$$E\left(m_{pq}^{(g)}\right) = m_{pq}^{(f*h)}$$

and, consequently,

$$E(M(p, q)^{(g)}) = M(p, q)^{(f)}$$

where $E(\cdot)$ denotes the mean value. Thanks to this, if a sufficient number of realizations of $g(x, y)$ is available, we can estimate the invariants of the original image as the mean value of the invariants of all noisy realizations of blurred image.

In practice, however, only a single realization of $g(x, y)$ is available in most cases. The undesired noise impact on the value of the invariant is characterized by its relative error

$$re = \frac{|M(p,q)^{(g)} - M(p,q)^{(f)}|}{M(p,q)^{(f)}}. \qquad (15)$$

Mostafa and Psaltis [28] and Pawlak [29] studied relative errors of moments. They showed that as the order of moments goes up, the relative error becomes higher. The blur invariants behave in the same way.

In practical applications the usability of the invariants depends on how big the relative error is in comparison with a relative distance between two different object classes. As it is demonstrated by experiments in Section 9, the invariants are robust enough to be used for object recognition under "normal" conditions (i.e., SNR higher than 10 dB and relative inter-class distance higher than 0.01).

8 Combined Invariants

In this section, we deal with the invariants to the degradations, which are composed from two degradation factors. Centrally symmetric blur is always supposed to be one of them, the other one can be either a geometric transform of spatial coordinates or an image intensity transform.

More formally, let's suppose that $g \equiv \mathcal{D}(f) = \mathcal{D}_0(f * h)$, where $\mathcal{D}_0$ is some operator acting on the space of image functions. The features invariant to $\mathcal{D}$ will be called *combined invariants*. Clearly, if the functional I is invariant to blurring and simultaneously to $\mathcal{D}_0$, then it is also invariant to $\mathcal{D}$.

In the following text, we will concern with various $\mathcal{D}_0$ operators describing common degradations in imaging. Although the invariants to those degradations are usually well-known from classical image processing literature and the invariants to blurring have been introduced in previous sections, it is not always straightforward to derive combined invariants.

8.1 Change of Contrast

In the case of image contrast global change, $\mathcal{D}_0(g)(x, y) = \alpha \cdot g(x, y)$, where α is a positive constant.

Theorem 7. *$M'(p, q)$ and $C'(p, q)$ defined in Theorem 4 are combined invariants for any p and q.*

Proof. Let $p + q$ be odd (the statement is trivial otherwise). It is sufficient to prove that both $M'(p, q)$ and $C'(p, q)$ are $\mathcal{D}_0$-invariants. Let $g(x, y)$ be an image function and let $g_1(x, y) = \alpha \cdot g(x, y)$; $\alpha > 0$. This means that

$$g_1(x, y) = (h_1 * g)(x, y),$$

where $h_1(x, y) = \alpha \cdot \delta(x, y)$. Since $h_1(x, y)$ is centrally symmetric, $M'(p,q)^{(g_1)} = M'(p,q)^{(g)}$. The proof of $\mathcal{D}_0$-invariance of $C'(p, q)$ is similar. □

It follows from Theorem 5 and Theorem 7 that tangent of the Fourier transform phase is also a combined invariant.

8.2 Change of Brightness

A global change of image brightness is described as $\mathcal{D}_0(g)(x, y) = g(x, y) + \beta \cdot \psi_g(x, y)$, where β is an arbitrary constant and $\psi_g(x, y)$ is a characteristic function of the support of g. According to Definition 1, support of g is bounded. Consequently, it is correct to consider *silhouette moments*

$$\overline{m}_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q \psi_g(x,y)\,dx\,dy.$$

We can construct $\mathcal{D}_0$-invariants easily as

$$B(p,q,s,t) = m_{pq}\overline{m}_{st} - m_{st}\overline{m}_{pq}.$$

However, they do not generate any combined invariants.

If we allow an infinite support of g, then

$$\mathcal{F}(\mathcal{D}_0(g))(u, v) \equiv G'(u, v) = G(u, v) + \beta \cdot \delta(u, v)$$

and, consequently,

$$\mathrm{ph}G'(u, v) = \mathrm{ph}G(u, v).$$

Thus, tangent of the Fourier transform phase is a combined invariant.

8.3 Translation

Translation of the image $g(x, y)$ by vector (x_0, y_0) is described as $\mathcal{D}_0(g)(x, y) = g(x - x_0, y - y_0)$. Clearly, $C(p, q)$ as well as $C'(p, q)$ are combined invariants. Unfortunately, there is no straightforward correspondence with frequency domain.

8.4 Translation and Scaling

In this case, the distortion is expressed as $\mathcal{D}_0(g)(x, y) = g(ax - x_0, ay - y_0)$, where $a > 0$ is a scaling factor.

Theorem 8. *Let $g(x, y)$ be an image function. Let us define the following function $N^{(g)}: \mathbf{Z} \times \mathbf{Z} \to \mathbf{R}$.*

If $(p + q)$ is even then

$$N(p, q)^{(g)} = 0.$$

If $(p + q)$ is odd then

$$N(p,q)^{(g)} = \nu_{pq}^{(g)} - \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} N(p-n, q-m)^{(g)} \cdot \nu_{nm}^{(g)}, \qquad (16)$$

where

$$\nu_{pq} = \frac{\mu_{pq}}{\mu_{00}^{(p+q+2)/2}}.$$

Then $N(p, q)$ is a combined invariant for any p and q.

For proof of Theorem 8 see Appendix B. Note that $N(p, q)$ is formally similar to $C(p, q)$, but the *normalized moments* are used instead of the central ones.

8.5 Translation, Scaling, and Contrast Change

In this case, the distortion operator $\mathcal{D}_0$ has the form $\mathcal{D}_0(g)(x, y) = \alpha \cdot g(ax - x_0, ay - y_0)$. According to the previous cases, the simplest solution seems to be $N(p, q)/\mu_{00}$. Unfortunately, this feature is invariant neither to scaling nor to change of contrast.

THEOREM 9. *Let r be an odd integer and let $p, q, s,$ and t are arbitrary integers such that $r = p + q = s + t$. Then*

$$T(p,q,s,t) = \frac{N(p,q)}{N(s,t)}$$

is a combined invariant.

PROOF. Due to Theorem 8, $T(p, q, s, t)$ is invariant to blur, translation, and scaling. Let's prove its invariance to the change of contrast by an arbitrary positive factor α.

Let $g(x, y)$ be an image function. Then (21) implies

$$N(p,q)^{(\alpha g)} = \frac{C'(p,q)^{(\alpha g)}}{\left(\mu_{00}^{r/2}\right)^{(\alpha g)}}.$$

Theorem 7 says that $C'(p, q)^{(\alpha g)} = C'(p, q)^{(g)}$. Thus,

$$N(p,q)^{(\alpha g)} = \frac{C'(p,q)^{(g)}}{\alpha^{r/2}\left(\mu_{00}^{r/2}\right)^{(g)}} = \alpha^{-r/2} N(p,q)^{(g)}$$

and, consequently,

$$T(p,q,s,t)^{(\alpha g)} = \frac{N(p,q)^{(\alpha g)}}{N(s,t)^{(\alpha g)}} = \frac{N(p,q)^{(g)}}{N(s,t)^{(g)}} = T(p,q,s,t)^{(g)}. \quad \square$$

Using (21) we can derive the useful relation between the functionals T and C:

$$T(p,q,s,t) = \frac{C(p,q)}{C(s,t)}.$$

8.6 Translation, Rotation, and Scaling

This very frequent distortion is described by the operator

$$\mathcal{D}_0(g)(x, y) =$$

$$g(ax \cos\theta + ay \sin\theta - x_0, -ax \sin\theta + ay \cos\theta - y_0),$$

where θ is a rotation angle.

There have been described lot of $\mathcal{D}_0$-invariants based on various approaches (see [30], [31] for a survey). A large group of them is based on moments. Hu derived seven moment-based $\mathcal{D}_0$-invariants in his fundamental paper [32] which have been employed by many researchers. Recently, Wong [33] has proposed a method how to generate an infinite sequence of rotation moment invariants and has shown Hu's invariants are just particular represantatives of them. It can be proven that some odd-order Wong's invariants (or certain simple functions of them) are also blur invariants. We present here the most simple combined invariants:

$$\Phi_1 = (\nu_{30} - 3\nu_{12})^2 + (3\nu_{21} - \nu_{03})^2,$$

$$\Phi_2 = (\nu_{30} + \nu_{12})^2 + (\nu_{21} + \nu_{03})^2,$$

$$\Phi_3 = (\nu_{30} - 3\nu_{12})(\nu_{30} + \nu_{12})((\nu_{30} + \nu_{12})^2 - 3(\nu_{21} + \nu_{03})^2) + (3\nu_{21} - \nu_{03})(\nu_{21} + \nu_{03})\,(3(\nu_{30} + \nu_{12})^2 - (\nu_{21} + \nu_{03})^2),$$

$$\Phi_4 = (3\nu_{21} - \nu_{03})(\nu_{30} + \nu_{12})((\nu_{30} + \nu_{12})^2 - 3(\nu_{21} + \nu_{03})^2) - (\nu_{30} - 3\nu_{12})(\nu_{21} + \nu_{03})\,(3(\nu_{30} + \nu_{12})^2 - (\nu_{21} + \nu_{03})^2),$$

$$\Phi_5 = [\nu_{50} - 10\nu_{32} + 5\nu_{14} - 10(\nu_{20}\nu_{30} - \nu_{30}\nu_{02} - 3\nu_{12}\nu_{20} + 3\nu_{12}\nu_{02} - 6\nu_{11}\nu_{21} + 2\nu_{11}\nu_{03})]^2 + [\nu_{05} - 10\nu_{23} + 5\nu_{41} - 10(\nu_{02}\nu_{03} - \nu_{03}\nu_{20} - 3\nu_{21}\nu_{02} + 3\nu_{21}\nu_{20} - 6\nu_{11}\nu_{12} + 2\nu_{11}\nu_{30})]^2. \quad (17)$$

Another set of combined invariants can be derived in the frequency domain provided that there occurs no translation. In such a case,

$$\mathcal{F}\big(\mathcal{D}_0(g)\big)(u,v) = \frac{1}{a^2} G\left(\tfrac{u}{a}\cos\theta + \tfrac{v}{a}\sin\theta, -\tfrac{u}{a}\sin\theta + \tfrac{v}{a}\cos\theta\right).$$

Thus, the Fourier transform phase of a rotated and scaled image is the rotated and inverse-scaled phase of the original one. Consequently, any $\mathcal{D}_0$-invariant computed from the tangent of the FT phase is a combined invariant. Particularly, we can apply all Hu's moment invariants. However, in practical implementation they are unstable because of phase discontinuity.

8.7 Affine Transform

Distortion caused by an *affine transform* is the most general one we consider in this paper. It is described by the following operator:

$$\mathcal{D}_0(g)(x, y) = g(a_0 + a_1 x + a_2 y, b_0 + b_1 x + b_2 y)$$

where a_j and b_j are arbitrary real coefficients.

We refer to our previous work [34] in which a set of $\mathcal{D}_0$-invariants called *affine moment invariants* was introduced. It can be proved that some of them are also invariant to blurring. The most simple combined invariant has the form

$$A = \frac{1}{\mu_{00}^{10}}\left(\mu_{30}^2\mu_{03}^2 - 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} + 4\mu_{30}\mu_{12}^3 + 4\mu_{03}\mu_{21}^3 - 3\mu_{21}^2\mu_{12}^2\right).$$

9 EXPERIMENT 1—TEMPLATE MATCHING

To demonstrate the performance of the above described invariants we apply them to the problem of matching a template with a blurred and noisy scene.

9.1 Problem Formulation

Our primary motivation comes from the area of remote sensing. The template matching problem is usually formulated as follows: Having the templates and a digital image of a large scene, one has to find locations of the given templates in the scene. By template we understand a small digital picture usually containing some significant object which was extracted previously from another image of the scene and now is being stored in a database.

There have been proposed numerous image matching techniques in the literature (see the survey paper [35] for instance). A common denominator of all those methods is the assumption that the templates as well as the scene image have been already preprocessed and the degradations like blur, additive noise, etc., have been removed. Those assumptions are not always realistic in practice. For instance, satellite images obtained from Advanced Very High Resolution Radiometer (AVHRR) suffer by blur due to a multiparametric composite PSF of the device [36]. Even if one would know all the parameters (but one usually does not) the restoration would be hard and time-consuming task with an unreliable result.

By means of our blur invariants, we try to perform the matching without any previous deblurring. The current experiment demonstrates that this alternative approach outperforms the classical ones in the case of blurred scene and that it is sufficiently robust with respect to noise.

9.2 Algorithm

For computer implementation, we have to convert the definitions of image moments into the discrete domain first. Among several possibilities how to approximate central moments we employ the most simple one:

$$\mu_{pq}^{(f)} = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(i - x_t^{(f)}\right)^p \left(j - y_t^{(f)}\right)^q f_{ij}\,, \qquad (18)$$

where f_{ij} denotes a gray-level value in the pixel (i, j) and N is the size of the image.

We use the normalized invariants $C'(p, q)$ because of their invariance to image contrast. Moreover, further normalization is necessary to ensure roughly the same range of values regardless of p and q. If we would have enough data for training, we could set up the weighting factors according to interclass and intraclass variances of the features as was proposed by Cash in [37]. However, this is not that case, because each class is represented just by one template. Thus, we normalized the invariants by $(N/2)^{p+q}$ that yields a satisfactory normalization of the range of values.

Let's define for any odd r the following vectors:

$$\overline{C_r} = \left(C'(0,r), C'(1,r-1), \cdots, C'(r,0)\right)$$

and, consequently,

$$\mathbf{C}(r) = \left(\overline{C_3}, \overline{C_5}, \cdots, \overline{C_r}\right).$$

Note that the size K_r of $\mathbf{C}(r)$ is equal to $\frac{1}{4}(r+5)(r-1)$.

Let's define the distance $d_r(f, g)$ between two images as

$$d_r(f, g) = \| \mathbf{C}(r)^{(f)} - \mathbf{C}(r)^{(g)} \|$$

where $\| \cdot \|$ is Euclidean norm in $\ell_2(K_r)$ space. The distance d_r has the properties of a quasimetric: it is nonnegative, symmetric function which satisfies the triangular inequality. However, $d_r(f, g) = 0$ does not imply $f = g$.

Our matching algorithm can be described as follows:

Algorithm *Invar_Match*

1) Inputs:
 g – blurred image of the size $N \times N$,
 f – template of the size $L \times L$, $N \gg L$,
 r – maximal order of the invariants used.
2) Calculate $\mathbf{C}(r)^{(f)}$.
3) for $i = 1$ to $N - L + 1$
  for $j = 1$ to $N - L + 1$
   $t = g(i : i + L - 1, j : j + L - 1)$;
   Calculate $\mathbf{C}(r)^{(t)}$;
   $\mathbf{D}_{ij} = d_r(f, t)$;
  end;
end;
4) Find (i_0, j_0) such that

$$\mathbf{D}_{i_0 j_0} = \min \mathbf{D}_{ij}.$$

5) Output:
 (i_0, j_0)–position of the template in the scene (upper-left corner).

There are several possible modifications of this algorithm. In Step 4, we can interpolate in the distance matrix $\mathbf{D}$ over some neighborhood of the minimal element $\mathbf{D}_{i_0 j_0}$ to find the template location with subpixel accuracy. To speed up the computation, Step 3 can be implemented as a hierarchical procedure. On a coarser level, only a few invariants are used to evaluate the distance matrix. On a finer level, the matching is carried out by means of more invariants, but only in the neighborhood of "hopeful" locations. In some cases we can specify the approximate location of the template and restrict the matching procedure to a search area which is smaller than the entire image.

The only open question is how to choose an appropriate r, i.e., how many invariants should be used for the matching. It is well-known that higher order moments (and, consequently, higher-order invariants too) are more vulnerable to noise. On the other hand, they contain the detail information about the image. It was shown by Pawlak [29] that there exists the optimal number of the moments yielding the best trade-off between those two opposite factors, i.e., giving the best image representation for the given noise variance. The Pawlak's method can be extended to find the optimal number of the moment invariants. However, the features giving optimal representation may not ensure optimal discriminability. Nevertheless, if the signal-to-noise ratio (SNR) is known, we can use the Pawlak's method to find the optimal number of the invariants to represent the given template and employ the same set of the invariants for matching. In most practical cases, such a number of features is too high (often more than 100) to be used for computation. Thus, a user-defined value of r is in common use. In this experiment we used $r = 7$, that means we applied 18 invariants from third to seventh order.

9.3 Data

The experiment was carried out on a simulated AVHRR image. As an input for a simulation, the 512×512 SPOT HRV image covering the north-western part of Prague (Czech capital) was used (see Fig. 1). To simulate AVHRR

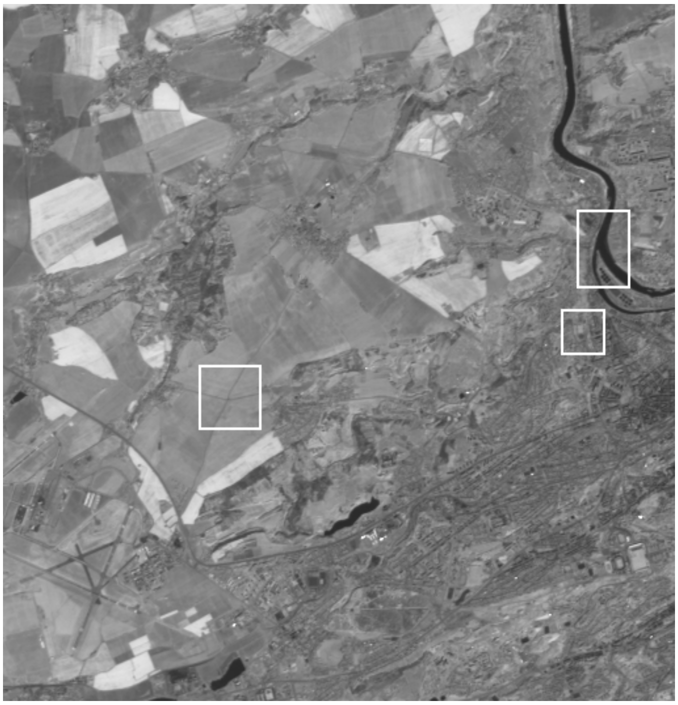


Fig. 1. Original 512 × 512 SPOT scene (band 3) of the north-western part of Prague, Czech Republic, used for simulation. True locations of the templates from Fig. 3 are marked.

Fig. 2. Simulated AVHRR image (9 × 13 blurring mask + additive noise with STD = 10).

acquisition, the image was blurred by a 9×13 mask representing a composite PSF of the AVHRR sensor and corrupted by Gaussian additive noise with standard deviation STD = 10, which yields SNR = 14 in this case (see Fig. 2). We didn't scale the image to simulate lower spatial resolution of AVHRR sensors, because it is not essential in this experiment. The templates were extracted from SPOT image of the same scene and represent significant objects in Prague: the island in Vltava river, the cross of the roads and the soccer stadium (see Fig. 3). The true locations of the templates in the original image are shown in Fig. 1. The task was to localize these templates in the AVHRR image.

9.4 Results

Matching of the templates and the AVHRR image was performed by two different techniques - by our algorithm *Invar_Match* and, for a comparison, by the *Sequential Similarity Detection Algorithm (SSDA)* which is probably the most popular representative of the correlation-like matching methods [38].

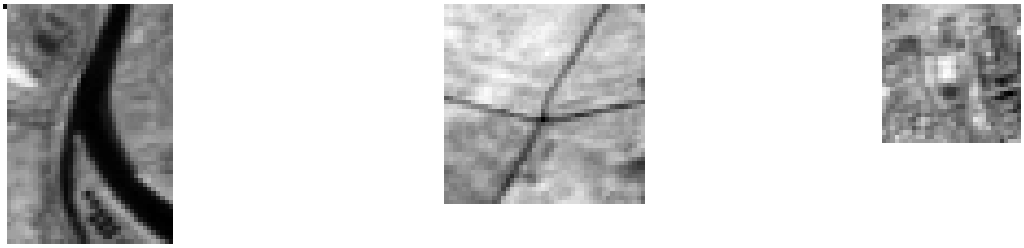
The results are summarized in Table 1. It can be seen that by *Invar_Match* all templates were placed correctly or almost correctly with a reasonable error 1 pixel. On the other hand, SSDA did not yield satisfactory results. Only the "Island" template was placed correctly (because of its distinct structure), whereas the other templates were misplaced.

We have performed a lot of experiments like this one with various templates, template sizes, blurring masks and noise variance. The "average" results are summarized in Fig. 4. The noise standard deviation is on the horizontal axis, the ratio w between the size of the blurring mask and the size of the template is on the vertical axis. The value of w is an important factor. Due to blurring, the pixels laying near the boundary of the template in the image are affected by those pixels laying outside the template. The higher w is, the larger part of the template is involved in this boundary effect. In principle, the invariants cannot be invariant to boundary effect and this might lead to mismatch, especially when w is higher than 0.15.

The area below each graph corresponds to the domain in which the algorithms *Invar_Match* and SSDA, respectively, work successfully. *Invar_Match* algorithm clearly outperforms SSDA, especially when the noise corruption becomes significant. It proves the fact that our features are not only invariant to blurring but also robust to additive noise. However, if the SNR becomes lower than 10 dB, *Invar_Match* failed in most cases.

TABLE 1
TEMPLATE MATCHING:
SUMMARY OF THE RESULTS OF EXPDRIMENT 1

Template	Ground truth	Invar_Match	SSDA	Invar_Match error	SSDA error
Island	438,152	438,152	438,152	0	0
Cross	150,265	149,265	59,31	1	251
Stadium	426,225	426,224	401,13	1	213



(a) (b) (c)

Fig. 3. The templates: “Island” (a), “Cross” (b), and “Stadium” (c).

10 Experiment 2—Registration of Rotated Images

In this section, we demonstrate the capability of the combined invariants for registering rotated and blurred images.

Image-to-image registration usually consists of four steps. First, specific objects or features are identified either automatically or manually in the images, and the correspondence between them is established by matching of object descriptors. After that, distinctive points, such as window centers or object centroids, are considered as *control points* (CP). The coordinates of corresponding control points are then used to calculate the coefficients of the parametric transformation model. In satellite imagery, low-order polynomials are usually employed and their coefficients are computed by least-square approximation. In the last step, the sensed image is resampled and overlaid over the reference one.

Possible application of combined invariants to image registration is in window matching.

In this experiment, the simulated AVHRR image from Section 9 served as the reference one. The original SPOT scene from Fig. 1 was rotated by 30° around its center and shifted horizontally and vertically by 94 pixels (see Fig. 5) to get the image to be registered. Two circular windows containing an island in the river and a runway cross on the airport, respectively, were selected manually (their locations are depicted in Fig. 5). The positions of the window centers were (561, 388) and (100, 375), respectively. Now the task was to localize these templates in the reference image.

An algorithm similar to that presented in Section 9.2 was used for the window matching. The only difference is that five combined translation-rotation-blur invariants (17) were employed instead of $C'(p, q)$. The windows were localized in the AVHRR image according to minimum distance in the invariant space on the following positions: the island at (459, 183) and the runway cross at (53, 401) (center positions).

Since the geometric difference between the images consists of translation and rotation only, two pairs of control points are sufficient to estimate the transformation parameters. The estimated value of the rotation angle is 29.86° and the estimated translation parameters were 93 and 94.5, respectively. These results demonstrate sufficient registration accuracy.

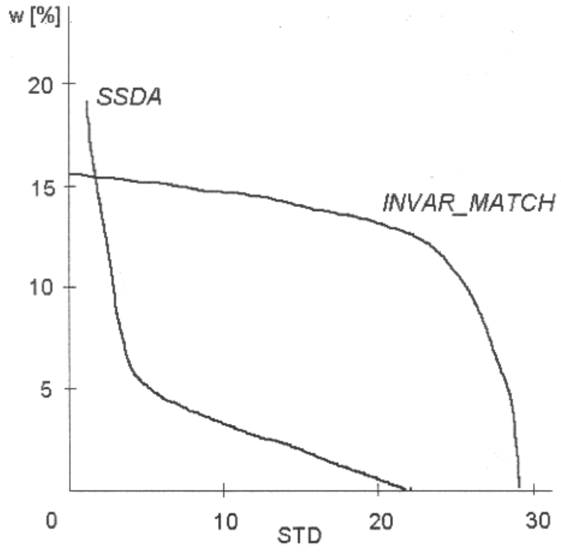


Fig. 4. Experimental evaluation and comparison of Invar_Match and SSDA algorithms. Horizontal axis: standard deviation of the noise; vertical axis: the ratio between the size of the blurring mask and that of the template. The area below the graph corresponds to the domain of good performance of the method.

11 Conclusion

The paper was devoted to the image features which are invariant to blurring by a filter with centrally symmetric PSF. The invariants in the spectral domain as well as in the spatial domain were derived and the relationship between them was investigated. It was proven that both kinds of features are equivalent from the theoretical point of view. Combined invariants, i.e., invariants with respect to degradations composed from symmetric blur and another geometric or gray-level transform were investigated too. Particular attention was paid to translation, rotation, scaling and contrast changes. Finally, original theoretical results of the paper were approved by numerical experiments on satellite images.

Practical applications of the new invariants may be found in object recognition in blurred and noisy environment, in template matching, image registration, etc.

Basically, there are two major directions of the further research on this field. The first one should be focused on special types of degradations such as motion blur, vibration blur or out-of-focus blur. Knowing the parametric expressions of their PSF, we should be able to derive more invariants than for general centrally symmetric PSF. Oppo-

Fig. 5. The original SPOT image from Fig. 1 rotated by 30 degrees with two circular templates extracted for the registration purposes.

site direction will be focused on combined invariants where a general method of their derivation should be developed, particularly in the case of image rotation and affine transform.

Appendix A

We present here an analogon of Theorem 2 showing blur invariants in the case of radially symmetric PSF.

If $h(x,y) = h\left(\sqrt{x^2+y^2}\right)$ then the functional

$$R(p,q) =$$

$$\mu_{pq} - a\mu_{qp} - \frac{1}{\mu_{00}} \sum_{\substack{n=0 \\ (m \wedge n) even}}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} R(p-n, q-m) \cdot \mu_{nm}$$

where $a = 1$ if both p and q are even and $a = 0$ otherwise is a blur invariant for any p and q. In this case we get more invariants than for centrosymmetric PSF, because invariants of even order are also defined. On the other hand, $R(p, q)$ consists of less number of terms than $C(p, q)$ because the summation goes only over those μ_{nm} both indices of which are even.

Blur invariants for axisymmetric PSF (i.e., $h(x, y) = h(-x, y) = h(x, -y)$) are exactly the same as those for radially symmetric PSF if p or q are odd and they are indefined if both p and q are even.

The proof of these assertions is similar to that of Theorem 2.

Appendix B

Proof of Theorem 2. The statement of the Theorem is trivial for any even r. Let us prove the statement for odd r by an induction.

- $r = 1$

$$C(0,1)^{(g)} = C(0,1)^{(f)} = 0,$$

$$C(1,0)^{(g)} = C(1,0)^{(f)} = 0$$

regardless of f and h.

- $r = 3$

There are four invariants of the third order: $C(1, 2)$, $C(2, 1)$, $C(0, 3)$, and $C(3, 0)$. Evaluating their recursive definition (9) we get them in the explicit forms:

$$C(1,2) = \mu_{12},$$

$$C(2,1) = \mu_{21},$$

$$C(0,3) = \mu_{03},$$

$$C(3,0) = \mu_{30}.$$

Let us prove the Theorem for $C(1,2)$; the proofs for the other invariants are similar.

$$C(1,2)^{(g)} = \mu_{12}^{(g)} = \sum_{k=0}^{1}\sum_{j=0}^{2}\binom{1}{k}\binom{2}{j}\mu_{kj}^{(h)}\mu_{1-k,2-j}^{(f)} = \mu_{12}^{(f)} = C(1,2)^{(f)}.$$

- Provided the theorem is valid for all invariants of order $1, 3, \cdots, r-2$. Then using Lemma 1 we get

$$C(p,q)^{(g)} = \mu_{pq}^{(g)} - \frac{1}{\mu_{00}^{(g)}}\sum_{n=0}^{p}\sum_{\substack{m=0\\0<n+m<p+q}}^{q}\binom{p}{n}\binom{q}{m}C(p-n,q-m)^{(g)}\cdot\mu_{nm}^{(g)}$$

$$= \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(h)}\mu_{p-k,q-j}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{n=0}^{p}\sum_{\substack{m=0\\0<n+m<p+q}}^{q}\binom{p}{n}\binom{q}{m} C(p-n,q-m)^{(f)}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\mu_{kj}^{(h)}\mu_{n-k,m-j}^{(f)}$$

Using the identity

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c},$$

changing the order of the summation and shifting the indices we get

$$C(p,q)^{(g)} = C(p,q)^{(f)} + \sum_{k=0}^{p}\sum_{\substack{j=0\\0<k+j}}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(h)}\mu_{p-k,q-j}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{n=0}^{p}\sum_{\substack{m=0\\0<n+m<p+q}}^{q}\sum_{k=0}^{n}\sum_{\substack{j=0\\0<k+j}}^{m}\binom{p}{n}\binom{q}{m}\binom{n}{k}\binom{m}{j} C(p-n,q-m)^{(f)}\mu_{kj}^{(h)}\mu_{n-k,m-j}^{(f)}$$

$$= C(p,q)^{(f)} + \sum_{k=0}^{p}\sum_{\substack{j=0\\0<k+j}}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(h)}\mu_{p-k,q-j}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{n=0}^{p}\sum_{\substack{m=0\\0<n+m<p+q}}^{q}\sum_{k=0}^{n}\sum_{\substack{j=0\\0<k+j}}^{m}\binom{p}{k}\binom{q}{j}\binom{p-k}{n-k}\binom{q-j}{m-j} C(p-n,q-m)^{(f)}\mu_{kj}^{(h)}\mu_{n-k,m-j}^{(f)}$$

$$= C(p,q)^{(f)} + \sum_{k=0}^{p}\sum_{\substack{j=0\\0<k+j}}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(h)}\mu_{p-k,q-j}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{k=0}^{p}\sum_{\substack{j=0\\0<k+j}}^{q}\sum_{n=k}^{p}\sum_{\substack{m=j\\n+m<p+q}}^{q}\binom{p}{k}\binom{q}{j}\binom{p-k}{n-k}\binom{q-j}{m-j} C(p-n,q-m)^{(f)}\mu_{kj}^{(h)}\mu_{n-k,m-j}^{(f)}$$

$$= C(p,q)^{(f)} + \sum_{k=0}^{p}\sum_{\substack{j=0\\0<k+j}}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(h)} \left(\mu_{p-k,q-j}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{n=k}^{p}\sum_{\substack{m=j\\n+m<p+q}}^{q}\binom{p-k}{n-k}\binom{q-j}{m-j} C(p-n,q-m)^{(f)}\mu_{n-k,m-j}^{(f)}\right)$$

$$= C(p,q)^{(f)} + \sum_{k=0}^{p}\sum_{\substack{j=0\\0<k+j}}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(h)} \left(\mu_{p-k,q-j}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{n=0}^{p-k}\sum_{\substack{m=0\\n+m<p+q-k-j}}^{q-j}\binom{p-k}{n}\binom{q-j}{m} C(p-n-k,q-m-j)^{(f)}\mu_{nm}^{(f)}\right).$$

Using a shorter notation we can rewrite the last equation in the form

$$C(p,q)^{(g)} = C(p,q)^{(f)} + \sum_{k=0}^{p}\sum_{\substack{j=0\\0<k+j}}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(h)}\cdot D_{kj}, \qquad (19)$$

where

$$D_{kj} = \mu_{p-k,q-j}^{(f)} - \frac{1}{\mu_{00}^{(f)}}\sum_{n=0}^{p-k}\sum_{\substack{m=0\\n+m<p+q-k-j}}^{q-j}\binom{p-k}{n}\binom{q-j}{m} C(p-n-k,q-m-j)^{(f)}\mu_{nm}^{(f)}.$$

If $k + j$ is odd then Lemma 2 implies that $\mu_{kj}^{(h)} = 0$. If $k + j$ is even then it follows from the definition (9) that

$$C(p-k,q-j) = \mu_{p-k,q-j} - \frac{1}{\mu_{00}}\sum_{n=0}^{p-k}\sum_{\substack{m=0\\0<n+m<p+q-k-j}}^{q-j}\binom{p-k}{n}\binom{q-j}{m}C(p-k-n,q-j-m)\cdot\mu_{nm}.$$

Consequently,

$$D_{kj} = C(p-k,q-j)^{(f)} - \frac{1}{\mu_{00}^{(f)}}C(p-k,q-j)^{(f)}\mu_{00}^{(f)} = 0.$$

Thus, (19) implies that $C(p,q)^{(g)} = C(p,q)^{(f)}$ for every p and q. □

PROOF OF THEOREM 5. Lemma 3 implies that

$$\operatorname{Re} F(u,v) = \sum_{k=0}^{\infty}\sum_{\substack{j=0\\(k+j)even}}^{\infty}\frac{(-2\pi i)^{k+j}}{k!\cdot j!}m_{kj}\cdot u^k v^j = \sum_{k=0}^{\infty}\sum_{\substack{j=0\\(k+j)even}}^{\infty}\frac{(-1)^{(k+j)/2}(-2\pi)^{k+j}}{k!\cdot j!}m_{kj}\cdot u^k v^j$$

and

$$\mathrm{Im}\, F(u,v) = \sum_{k=0}^{\infty} \sum_{\substack{j=0 \\ (k+j) odd}}^{\infty} \frac{(-2\pi i)^{k+j}}{k! \cdot j!} m_{kj} \cdot u^k v^j =$$

$$\sum_{k=0}^{\infty} \sum_{\substack{j=0 \\ (k+j) odd}}^{\infty} \frac{(-1)^{(k+j-1)/2}(-2\pi)^{k+j}}{k! \cdot j!} m_{kj} \cdot u^k v^j .$$

Thus, $\tan(\mathrm{ph}F(u, v))$ is a ratio of two absolutely convergent power series and, therefore, it can be also expressed as a power series

$$\tan\left(\mathrm{ph}F(u,v)\right) = \frac{\mathrm{Im}\, F(u,v)}{\mathrm{Re}\, F(u,v)} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{kj} u^k v^j ,$$

which must accomplish the relation

$$\sum_{k=0}^{\infty} \sum_{\substack{j=0 \\ (k+j) odd}}^{\infty} \frac{(-1)^{(k+j-1)/2}(-2\pi)^{k+j}}{k! \cdot j!} m_{kj} \cdot u^k v^j =$$

$$\sum_{k=0}^{\infty} \sum_{\substack{j=0 \\ (k+j) even}}^{\infty} \frac{(-1)^{(k+j)/2}(-2\pi)^{k+j}}{k! \cdot j!} m_{kj} \cdot u^k v^j \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{kj} u^k v^j \quad (20)$$

It follows immediately from (20) that if $k + j$ is even then $c_{kj} = 0$. Let's prove by an induction that c_{kj} has the form (12) if $k + j$ is odd.

- $k + j = 1$

It follows from (20) that

$$c_{10} = \frac{-2\pi m_{10}}{m_{00}}.$$

On the other hand,

$$\frac{(-2\pi)(-1)^0}{1! \cdot 0!} M'(1,0) = \frac{-2\pi m_{10}}{m_{00}} = c_{10}.$$

- Let's suppose the assertion has been proven for all k and j, $k + j \le r$, where r is an odd integer and let $p + q = r + 2$. It follows from (20) that

$$\frac{(-1)^{(p+q-1)/2}(-2\pi)^{p+q}}{p! \cdot q!} m_{pq} =$$

$$m_{00} c_{pq} + \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m}}^{q} \frac{(-1)^{(n+m)/2}(-2\pi)^{n+m}}{n! \cdot m!} m_{nm} c_{p-n,q-m} .$$

Introducing (12) into the right side we get

$$\frac{(-1)^{(p+q-1)/2}(-2\pi)^{p+q}}{p! \cdot q!} m_{pq} = m_{00} c_{pq} +$$

$$\sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \frac{(-1)^{(p+q-1)/2}(-2\pi)^{p+q}}{n! \cdot m! (p-n)! \cdot (q-m)!} M'(p-n, q-m) \cdot m_{nm}$$

and, consequently,

$$c_{pq} = \frac{(-1)^{(p+q-1)/2}(-2\pi)^{p+q}}{p! \cdot q! \cdot m_{00}} \cdot \left(m_{pq} - \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} M'(p-n, q-m) \cdot m_{nm} \right).$$

Finally, it follows from (10) that

$$c_{pq} = \frac{(-1)^{(p+q-1)/2}(-2\pi)^{p+q}}{p! \cdot q!} M'(p,q).$$

□

PROOF OF THEOREM 8. Since the normalized moments do not depend on the scale and the translation, $N(p, q)$ is $\mathcal{D}_0$-invariant for any p and q. Let's prove $N(p, q)$ is also invariant to blurring.

To do that, let's prove by an induction that

$$N(p,q) = \frac{C(p,q)}{\mu_{00}^{(p+q+2)/2}} \quad (21)$$

for any odd $r = p + q$.

- The assertion is trivial for $r = 1$ because

$$N(0, 1) = C(0, 1) = 0.$$

- $r = 3$

$$N(3,0) = \nu_{30} - \sum_{n=1}^{2} \binom{3}{n} N(3-n, 0) \cdot \nu_{n0} =$$

$$\nu_{30} - 3N(1,0) \cdot \nu_{20} = \nu_{30} = \frac{\mu_{30}}{\mu_{00}^{5/2}} = \frac{C(3,0)}{\mu_{00}^{5/2}}.$$

The proofs for $N(2, 1)$, $N(1, 2)$, and $N(0, 3)$ are quite similar.

- Provided the assertion has been proven for all $N(p, q)$ of order $1, 3, \cdots, r - 2$. Then for $r = p + q$ we get the following:

$$N(p,q) =$$

$$\frac{\mu_{pq}}{\mu_{00}^{(p+q+2)/2}} - \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} \frac{C(p-n, q-m)}{\mu_{00}^{(p+q-n-m+2)/2}} \cdot \frac{\mu_{nm}}{\mu_{00}^{(n+m+2)/2}}$$

$$= \frac{1}{\mu_{00}^{(p+q+2)/2}} \left(\mu_{pq} - \frac{1}{\mu_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} C(p-n, q-m) \cdot \mu_{nm} \right)$$

$$= \frac{C(p,q)}{\mu_{00}^{(p+q+2)/2}}.$$

Since $C(p, q)$ as well as μ_{00} are blur invariants, $N(p, q)$ is a combined invariant. □

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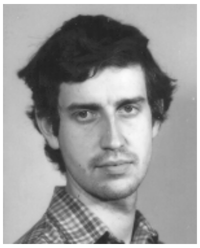
Jan Flusser received the MSc degree in mathematical engineering from the Czech Technical University, Prague, Czech Republic in 1985 and the PhD degree in computer science from the Czechoslovak Academy of Sciences in 1990. Since 1985, he has been with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague. Currently, he is the head of the Department of Image Processing. Since 1991, he has been an assistant professor in the Department of Mathematics and Physics, Charles University, Prague. His current research interests include digital image processing, pattern recognition, and remote sensing. He has authored or coauthored more than 30 scientific publications in these areas. Dr. Flusser is a member of the Pattern Recognition Society, the IEEE Signal Processing Society, and the IEEE Geoscience and Remote Sensing Society.



Tomáš Suk received the MSc degree in technical cybernetics from the Czech Technical University, Prague, Czech Republic in 1987 and the PhD degree in computer science from the Czechoslovak Academy of Sciences in 1992. Since 1987, he has been with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague. His current research interests include image preprocessing, geometric invariants, and remote sensing. He is an author or coauthor of more than 20 scientific publications in these areas.