Degree formula for connective K-theory

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Abstract

We apply the degree formula for connective K-theory to study incompressibility of algebraic varieties. [1]

1 Introduction

The celebrated *Rost degree formula* says that given a rational map $f\colon Y \dashrightarrow X$ between two smooth projective varieties there is the congruence relation (see [9])

$$\eta_p(Y) \equiv \deg f \cdot \eta_p(X) \mod n_X, \tag{1}$$

where p is a prime, $\eta_p(X)$ is the Rost number of X, $\deg f$ is the degree of f and n_X is the greatest common divisor of degrees of all closed points on X (see [6], [7], [8] and [9]).

It was conjectured by Rost that the degree formula (1) should follow from a *generalized degree formula* for some universal cohomology theory. This conjecture was proven by Levine and Morel in [5], where they constructed the theory of *algebraic cobordism* Ω and provided the respective degree formula (see [5, Theorem 1.2.14]).

Unfortunately, the generalized degree formula has one disadvantage: it deals with elements in the cobordism ring which is too big and usually is hard to compute. On the other hand the classical degree formula (1) is easy to apply but it catches only "pro-p" effects. The reasonable question would be to find a cohomology theory together with a degree formula which doesn't loose much information and is still computable.

The natural candidate for such a theory is the *connective K-theory* denoted by $\mathcal{K}$. It has two important properties: First, $\mathcal{K}$ is the universal oriented cohomology theory for the Chow group CH and Grothendieck's K^0, meaning the following diagram of natural transformations

$$\begin{tikzcd} & \Omega \arrow[d, "pr_{\mathcal{K}}"] & \\ \mathrm{CH} & \mathcal{K} \arrow[l] \arrow[r] & K^0[\beta, \beta^{-1}], \end{tikzcd}$$

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where β denotes the Bott element. Second, it is the *universal birational theory* in the sense that it preserves the fundamental classes for birational maps, i.e. for any proper birational $f\colon Y \to X$ we have $f_*(1_Y) = 1_X$.

The respective degree formula for $\mathcal{K}$ was predicted by Rost and Merkurjev (see [9, Example 11.4]). In the present notes we deduce this formula from the generalized degree formula of Levine and Morel. Namely, we prove the following

Theorem (A). *Let $f\colon Y \dashrightarrow X$ be a rational map between two smooth projective varieties of the same dimension over a field k of characteristic 0. Then there exists a finite family of smooth projective varieties $\{Z_i\}_i$ over k such that each Z_i admits a projective birational map to a proper closed subvariety of X and*

$$\chi(\mathcal{O}_Y) = \deg f \cdot \chi(\mathcal{O}_X) + \sum_i n_i \cdot \chi(\mathcal{O}_{Z_i}), \tag{2}$$

where $n_i \in \mathbb{Z}$ and $\chi(\mathcal{O}_X)$ is the Euler characteristic of the structure sheaf of X.

Recall that an algebraic variety X is called *incompressible* if any rational map $X \dashrightarrow X$ is dominant, i.e. has a dense image. The notion of incompressibility appears to be very important in the study of the splitting properties of G-torsors, where G is a linear algebraic group, in computations of the essential and the canonical dimension of G (see [1], [2] and [4]). For instance, using the Rost degree formula (1) Merkurjev provided a uniform and shortend proof of the incompressibility of certain Severi-Brauer varieties, involution varieties and quadrics (see [9, §5 and §7]). Therefore, it is natural to expect that the formula (2) can provide more examples of incompressible varieties. As a demonstration of this philosophy we prove the following

Theorem (B). *Let X and Y be smooth projective varieties over a field of characteristic 0. Let n_X denote the greatest common divisor of degrees of all closed points on X and let τ_m denote the m-th denominator of the Todd genus.*

Assume there is a rational map $f\colon Y \dashrightarrow X$ whose image is of dimension strictly less than the dimension of Y. Then n_X divides $\chi(\mathcal{O}_Y) \cdot \tau_{\dim Y - 1}$.

In particular, if n_X doesn't divide $\chi(\mathcal{O}_X) \cdot \tau_{\dim X-1}$, then X is incompressible.

For instance, for a geometrically rational variety X we have $\chi(\mathcal{O}_X) = 1$. Therefore, if $n_X \nmid \tau_{\dim X-1}$, then X is incompressible.

The paper is organized as follow. First, we provide some preliminaries on algebraic cobordism and connective K-theory. Then we prove Theorem (A) and discuss its applications (Theorem (B)) to the question of incompressibility of algebraic varieties. In the last section we relate the degree formula of Theorem (A) with the classical Rost degree formulas.

Notation and conventions By k we denote a field of characteristic 0. A variety will be a geometrically integral scheme of finite type over k. By pt we denote Spec k. Given a cycle $\alpha \in \mathrm{CH}(X)$ by $\deg \alpha$ we denote the push-forward $p_*(\alpha) \in \mathbb{Z}$, where $p\colon X \to pt$ is the structure map.

2 Algebraic cobordism and connective K-theory

2.1. In [5] M. Levine and F. Morel introduced the theory of *algebraic cobordism* Ω that is a contravariant functor from the category of smooth quasi-projective varieties over a field k of characteristic 0 to the category of graded commutative rings. An element of codimension i in $\Omega(X)$ is the class $[f\colon Y \to X]$ of a projective map of pure codimension i between smooth quasi-projective varieties X and Y. Sometimes we will use the dimension notation meaning the identification $\Omega_{d-i}(X) = \Omega^i(X)$, where $d = \dim X$.

Given f we denote by $f_*\colon \Omega_i(Y) \to \Omega_i(X)$ the induced push-forward and by $f^*\colon \Omega^i(X) \to \Omega^i(Y)$ the induced pull-back.

2.2. Let h be an *oriented cohomology theory* as defined in [5]. Roughly speaking, h is a cohomological functor endowed with characteristic classes c^h. The main result of [5] says that Ω is a universal oriented cohomology theory, i.e. any such h admits a natural transformation of functors $pr_h\colon \Omega \to h$ preserving the characteristic classes.

To any oriented cohomology theory h one assigns a one-dimensional commutative formal group law F_h over the coefficient ring $h(pt)$ via

$$c_1^h(L_1 \otimes L_2) = F_h(c_1^h(L_1), c_2^h(L_2)),$$

where L_1 and L_2 are lines bundles on X and c_1^h is the first Chern class. For Ω the respective formal group law turns to be a *universal formal group law*

$$F(x,y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j.$$

The ring of coefficients $\Omega(pt)$ is generated by the coefficients a_{ij} of F and coincides with the *Lazard ring* $\mathbb{L}$.

2.3. We recall several auxilliary facts about Chow groups CH and graded K_0:

Consider the augmentation map $\varepsilon_a\colon \mathbb{L} \to \mathbb{Z}$ defined by $a_{ij} \mapsto 0$. Define a cohomology theory Ω_a by $\Omega_a(X) = \Omega(X) \otimes_{\varepsilon_a} \mathbb{Z}$. According to [5]

- $\Omega_a(X)$ coincides with the Chow group $\mathrm{CH}(X)$ of algebraic cycles on X modulo rational equivalence;
- the natural transformation $pr_a\colon \Omega(X) \to \Omega_a(X)$ is surjective and its kernel is generated by elements of positive dimensions $\mathbb{L}_{>0}$ of the Lazard ring;
- Ω_a is a universal theory for the additive formal group law $F_a(x,y) = x+y$.

Consider the map $\varepsilon_m\colon \mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$ defined by $a_{11} \mapsto -\beta$ and $a_{ij} \mapsto 0$ for $(i,j) \neq (1,1)$. Define a cohomology theory Ω_m by $\Omega_m(X) = \Omega(X) \otimes_{\varepsilon_m} \mathbb{Z}[\beta, \beta^{-1}]$. According to [5]

- $\Omega_m(X)$ coincides with $K^0(X)[\beta, \beta^{-1}]$, where $K^0(X)$ is Grothendieck's K^0 of X;

- Ω_m is a universal theory for the multiplicative periodic formal group law $F_m(x,y) = x + y - \beta xy$.

2.4. The cohomology theory $\mathcal{K}$ which will be the central object of our discussion is a universal theory for both additive and multiplicative periodic formal group laws. It is called the *connective K-theory* and is defined as $\mathcal{K}(X) = \Omega(X) \otimes_\varepsilon \mathbb{Z}[v]$, where $\varepsilon \colon \mathbb{L} \to \mathbb{Z}[v]$ is given by $a_{11} \mapsto -v$ and $a_{ij} \mapsto 0$ for $(i,j) \neq (1,1)$. It has the following properties (see [5, §4.3.3]):

- The natural transformations $\mathcal{K} \to \Omega_a$ and $\mathcal{K} \to \Omega_m$ are given by the evaluations $v \mapsto 0$ and $v \mapsto \beta$ respectively. Roughly speaking, $\mathcal{K}$ can be viewed as a homotopy deformation between Ω_a and Ω_m.

- The natural transformations $\Omega \to \mathcal{K}$ and $\mathcal{K} \to \Omega_a = \mathrm{CH}$ are surjective.

- The respective formal group law F is the multiplicative non-periodic formal group law $F_{\mathcal{K}}(x,y) = x + y - vxy$, where v is non-invertible.

2.5. We will extensively use the following fact (see [5, Cor.4.2.5 and 4.2.7]):

The theory $\mathcal{K}$ is universal among all oriented theories for which the *birational invariance* holds, i.e. $f_*(1_Y) = 1_X$ for any birational projective map $f \colon Y \to X$. As a consequence, the kernel of the map $\varepsilon \colon \mathbb{L} \to \mathbb{Z}[v]$ is the ideal generated by elements $[W] - [W']$, where W and W' are birationally equivalent.

3 Degree formula and Todd genus

3.1. Let $f \colon Y \dashrightarrow X$ be a rational morphism between two smooth projective varieties of the same dimension d over a field k. Let $\bar{\Gamma}_f$ be the closure of its graph in $Y \times X$ and $\bar{\Gamma}'_f \to \bar{\Gamma}_f$ be its resolution of singularities. By the generalized degree formula for the composite $\bar{\Gamma}'_f \to Y \times X \xrightarrow{pr_2} X$ (see [5]) there exists a finite family of smooth projective varieties $\{Z_i\}_i$ such that each Z_i admits a projective birational map f_i on the proper closed subvariety of X and

$$[\bar{\Gamma}'_f \to X] = \deg f \cdot [X \xrightarrow{id} X] + \sum_i u_i \cdot [Z_i \xrightarrow{f_i} X], \text{ where } u_i \in \mathbb{L},$$

where $\deg f = [k(Y) : k(X)]$ if f is dominant, and $\deg f = 0$ otherwise. Observe that by definition $\dim Z_i < d$ and $u_i \in \mathbb{L}_{>0}$.

Applying the push-forward $p_* \colon \Omega(X) \to \mathbb{L}$ induced by the structure map $p \colon X \to pt$ we obtain

$$[\bar{\Gamma}'_f] = \deg f \cdot [X] + \sum_i u_i \cdot [Z_i].$$

Then projecting on $\mathcal{K}(pt)$ we obtain the following equality:

$$[Y]_{\mathcal{K}} = \deg f \cdot [X]_{\mathcal{K}} + \sum_i (u_i)_{\mathcal{K}} \cdot [Z_i]_{\mathcal{K}}. \tag{3}$$

Observe that $[Y]_{\mathcal{K}} = [\bar{\Gamma}'_f]_{\mathcal{K}}$, since Y and $\bar{\Gamma}'_f$ are birationally isomorphic and we have property 2.5.

3.2 Lemma. *Given a smooth projective variety X of dimension d we have*

$$[X]_{\mathcal{K}} = \chi(\mathcal{O}_X) \cdot v^d, \tag{4}$$

where $\chi(\mathcal{O}_X)$ is the Euler characteristic of the structure sheaf of X.

Proof. Consider the map $\varepsilon_m \colon \mathbb{L} \to \mathbb{Z}[\beta, \beta^{-1}]$. The image of the class $[X]$ is equal to $[X]_m = p_*([\mathcal{O}_X]) \cdot \beta^d$, where p_* is the push-forward induced by the structure map $p\colon X \to pt$ and the number $p_*([\mathcal{O}_X])$ coincides with the Euler characteristic $\chi(\mathcal{O}_X)$ of the structure sheaf of X (see [9, §10])). Since ε_m factors through $\mathbb{Z}[v]$ and the map $\mathbb{Z}[v] \to \mathbb{Z}[\beta, \beta^{-1}]$, $v \mapsto \beta$, is injective, we obtain the desired formula. □

As an immediate consequence of the lemma we obtain the following classical fact

3.3 Corollary. *Let X and Y be two smooth projective varieties over a field of characteristic 0. If X is birationally isomorphic to Y, then $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$.*

Proof. By property 2.5 of $\mathcal{K}$ we have the equality $[X]_{\mathcal{K}} = [Y]_{\mathcal{K}}$. □

3.4. Observe that by the very definition

$$\chi(\mathcal{O}_X) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{O}_X),$$

where X is a projective variety over k. The number $\dim_k H^i(X, \mathcal{O}_X)$ is known to be a birational invariant if k has characteristic 0.

3.5 Example. If X is a geometrically rational variety, then $\chi(\mathcal{O}_X) = 1$. Indeed, $\dim_k H^0(X, \mathcal{O}_X) = 1$ and $\dim_k H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.

3.6. Since $\dim_k H^i(X, \mathcal{O}_X)$ doesn't depend on a base change, so is $\chi(\mathcal{O}_X)$. Namely, if $X_l = X \times_k l$ is a base change by means of a field extension l/k, then $\chi(\mathcal{O}_{X_l}) = \chi(\mathcal{O}_X)$.

Combining (3) and (4) we obtain the following

3.7 Theorem. *Let $f\colon Y \dashrightarrow X$ be a rational map between two smooth projective varieties of the same dimension d over a field of characteristic 0. Then there exists a finite family of smooth projective varieties $\{Z_i\}_i$ such that each Z_i admits a projective birational map f_i on a proper closed subvariety of X and*

$$\chi(\mathcal{O}_Y) = \deg f \cdot \chi(\mathcal{O}_X) + \sum_i n_i \cdot \chi(\mathcal{O}_{Z_i}), \text{ where } n_i \in \mathbb{Z}. \tag{5}$$

3.8 Example. If X and Y are curves, then the degree formula (5) turns into

$$\chi(\mathcal{O}_Y) \equiv \deg f \cdot \chi(\mathcal{O}_X) \mod n_X, \tag{6}$$

where n_X denotes the g.c.d. of degrees of all closed points on X. Observe that $\chi(\mathcal{O}_X) = 1 - p_g$, where p_g is the geometric genus of X.

For surfaces X and Y we obtain

$$2\chi(\mathcal{O}_Y) \equiv \deg f \cdot 2\chi(\mathcal{O}_X) \mod n_X, \tag{7}$$

where $\chi(\mathcal{O}_X) = 1 - q + p_g$ with the irregularity q and the geometric genus p_g.

3.9. According to [3, Example 3.2.4] the *Todd class* of a smooth variety X is the following polynomial in Chern classes

$$\mathrm{Td}(X) = 1 + \tfrac{c_1}{2} + \tfrac{c_1^2+c_2}{12} + \tfrac{c_1c_2}{24} + \tfrac{-c_1^4+4c_1^2c_2+3c_2^2+c_1c_3-c_4}{720} + \ldots, \tag{8}$$

where $c_i \in \mathrm{CH}^i(X)$ denotes the i-th Chern class of the tangent bundle of X. The denominators $\tau_0 = 1$, $\tau_1 = 2$, $\tau_2 = 12$, $\tau_3 = 24$, $\tau_4 = 720\ldots$ are called *Todd numbers*. We have the following explicit formula for τ_d (see [9, Example 9.9]):

$$\tau_d = \prod_{p\ prime} p^{[\frac{d}{p-1}]}. \tag{9}$$

In particular, $\tau_{d-1} \mid \tau_d$ for any d.

3.10. To compute the Euler characteristic $\chi(\mathcal{O}_X)$ we may use the following equality (see [3, Corollary 18.3.1]):

$$\chi(\mathcal{O}_X) = \deg \mathrm{Td}(X),$$

where $\deg \mathrm{Td}(X)$ is the degree of the d-th homogeneous component of the Todd class $\mathrm{Td}(X)$ and is called the *Todd genus* of X.

Observe that the Euler characteristic and the Todd genus are multiplicative, i.e. $\chi(\mathcal{O}_{X\times Y}) = \chi(\mathcal{O}_X) \cdot \chi(\mathcal{O}_Y)$.

3.11 Example. Let X be a complete intersection of m smooth hypersurfaces of degrees $d_1, \ldots, d_m$ in $\mathbb{P}^n$. Then

$$\chi(\mathcal{O}_X) = Res_{z=0} \frac{\prod_{i=1}^m (1 - e^{-d_i z})}{(1 - e^{-z})^{n+1}}. \tag{10}$$

Indeed, by [3, Example 15.2.12.(iii)] we have

$$\chi(\mathcal{O}_X) = \deg\big(\mathrm{Td}(T_{\mathbb{P}^n}) \cdot \prod_{i=1}^m (1 - e^{-d_i z})\big) = \deg\big((\frac{z}{1-e^{-z}})^{n+1} \cdot \prod_{i=1}^m (1 - e^{-d_i z})\big),$$

where $z = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

4 Incompressibility

4.1. The notion of (in-)compressibility of algebraic varieties appeares naturally in the study of the splitting properties of G-torsors and their canonical dimensions. Recall that (see [4, §4] and [2, §1]) a *canonical dimension* $\mathrm{cdim} X$ of a smooth projective algebraic variety X over k is defined to be the minimal dimension of a closed subvariety Z of X such that $Z_{k(X)}$ has a rational point. Obviously $\mathrm{cdim} X \leq \dim X$. If $\mathrm{cdim} X = \dim X$, then X is called incompressible.

To say "$Z_{k(X)}$ has a rational point" is the same as to say that there is a rational dominant map $X \dashrightarrow Z$. Therefore, we obtain the following equivalent definition:

> A variety X is called *incompressible* if any rational map $X \dashrightarrow X$ is dominant.

The next theorem is a direct application of the degree formula (5)

4.2 Theorem. *Let X and Y be smooth projective varieties over a field of characteristic 0. Let n_X denote the greatest common divisor of degrees of all closed points on X and let τ_m denote the m-th Todd number.*

Assume there is a rational map $f \colon Y \dashrightarrow X$ whose image is of dimension strictly less than the dimension of Y. Then n_X divides $\chi(\mathcal{O}_Y) \cdot \tau_{\dim Y - 1}$.

In particular, if n_X doesn't divide $\chi(\mathcal{O}_X) \cdot \tau_{\dim X - 1}$, then X is incompressible.

Proof. Taking the product with a projective space we may assume that X and Y have the same dimension d.

By the degree formula (5) we have

$$\chi(\mathcal{O}_Y) = \sum_i n_i \cdot \chi(\mathcal{O}_{Z_i}) = \sum_{j=0}^{d-1} \Big(\sum_{i, \dim Z_i = j} n_i \cdot \chi(\mathcal{O}_{Z_i}) \Big),$$

where each Z_i admits a projective morphism $Z_i \to X$. The latter implies that all characteristic numbers of Z_i in the numerator of the $(\dim Z_i)$-th homogeneous component of $\mathrm{Td}(Z_i)$ (see (8)) are divisible by n_X. Therefore, we have

$$\chi(\mathcal{O}_Y) = \sum_{j=0}^{d-1} \Big(\sum_{i, \dim Z_i = j} n_i \cdot \frac{n_X \cdot m_i}{\tau_j} \Big) =$$

$$= \sum_{j=0}^{d-1} \frac{n_X}{\tau_j} \Big(\sum_{i, \dim Z_i = j} n_i m_i \Big) = \frac{n_X}{\tau_{d-1}} \sum_{j=0}^{d-1} \frac{\tau_{d-1}}{\tau_j} \Big(\sum_{i, \dim Z_i = j} n_i m_i \Big) = \frac{n_X \cdot m}{\tau_{d-1}},$$

where $m = \sum_{j=0}^{d-1} \frac{\tau_{d-1}}{\tau_j} \big(\sum_{i, \dim Z_i = j} n_i m_i \big) \in \mathbb{Z}$ according to (9). $\square$

4.3 Example. Let X be a smooth projective *geometrically rational* variety of dimension d. Then $\chi(\mathcal{O}_X) = 1$ (see Example 3.5) and, therefore,

$$n_X \nmid \tau_{d-1} \Longrightarrow X \text{ is incompressible.} \qquad (11)$$

Observe that if X is a curve or a surface, the implication (11) can be proven directly using the geometry (see [1, §8] and [2, §2, §3]).

4.4 Example. (cf. [7, Example 8.2] and [6, §7.3]) Let X be a complete intersection of m smooth hypersurfaces of degrees $d_1, d_2, \ldots, d_m$ in $\mathbb{P}^{p+m-1}$, where p is a prime. Observe that $\dim X = p-1$.

Let m_p be the number of degrees d_i which are divisible by p. We claim that

$$p \nmid m_p \text{ and } p \nmid \tfrac{d_1 d_2 \ldots d_m}{n_X} \Longrightarrow X \text{ is incompressible.} \tag{12}$$

Indeed, by the formula (10) the Euler characteristic $\chi(\mathcal{O}_X)$ is equal to the coefficient at z^{p+m-1} in the expansion of

$$d_1 d_2 \ldots d_m z^m \prod_{i=1}^{m} \Big(\sum_{r=0}^{p-1} \tfrac{(-d_i)^r}{(r+1)!} z^r \Big) \Big(\sum_{r=0}^{p-1} \tfrac{B_r}{r!} z^r \Big)^{p+m},$$

where B_r denotes the r-th Bernoulli number. Since the denominator of $\frac{B_r}{r!}$ is not divisible by p for any $r < p-1$ and is divisible by p for $r = p-1$, we obtain

$$\frac{\chi(\mathcal{O}_X)\tau_{p-2}}{n_X} = \frac{\tau_{p-2} \cdot d_1 d_2 \ldots d_m}{n_X} \cdot \Big(\frac{a}{b} - \frac{m_p}{p!} \Big) \notin \mathbb{Z}, \text{ where } p \nmid ab.$$

4.5 Example. Let Y be a smooth hypersurface of degree p^r, $r > 0$, in $\mathbb{P}^p$ with $n_Y = p^r$. Assume that there is a rational map $Y \dashrightarrow X$ with $\dim X < \dim Y$. Then $n_X \mid p^{r-1}$.

Indeed, the set of all closed points of Y of degree p^r is dense in Y. Therefore, $n_X \mid p^r$. By Theorem 4.2 $n_X \mid \chi(\mathcal{O}_Y) \cdot \tau_{p-2}$ and by the previous example the right hand side is not divisible by $n_X = p^r$.

5 Comparison with the classical degree formulas

5.1. We follow the notation of [8]. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ be a partition, i.e. a sequence of integers (possibly empty) $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_r$, and let $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_r$ denote its degree. For any α we define the smallest symmetric polynomial

$$P_\alpha(x_1, x_2, \ldots) = \sum_{(i_1, i_2, \ldots, i_r)} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_r}^{\alpha_r} = Q_\alpha(\sigma_1, \sigma_2, \ldots)$$

containing the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_r^{\alpha_r}$ where the σ_i are the elementary symmetric functions.

5.2. Let X be a smooth projective variety of dimension d. Let $c_i = c_i(-T_X)$ denote the i-th Chern class of the inverse of the tangent bundle. Let α be a partition of d. We define the α-characteristic number of X by

$$c_\alpha = \deg Q_\alpha(c_1, c_2, \ldots).$$

Observe that $c_{(1,1,\ldots,1)} = \deg c_d(-T_X)$ and $c_{(d)}$ defines the so called additive characteristic number of X.

5.3. We fix a prime p. Consider a partition

$$\alpha = (p-1, \ldots, p-1, p^2-1, \ldots, p^2-1, \ldots), \tag{13}$$

where $p^i - 1$ is repeated r_i times (in [6] it was denoted by the sequence $R = (r_1, r_2, \ldots)$). The set of all such partitions α will be denoted by Λ_p. According to [6, §6] for any $\alpha \in \Lambda_p$ the characteristic number c_α is divisible by p. The integer $\frac{1}{p}c_{(p-1,\ldots,p-1)}$ is called the Rost number and is denoted by η_p.

5.4. By [6, Theorem 6.4] for any prime p and any partition $\alpha \in \Lambda_p$ of d we have the degree formula:

$$\frac{c_\alpha(Y)}{p} \equiv \deg f \cdot \frac{c_\alpha(X)}{p} \mod n_X, \tag{14}$$

where $\deg f$ is the degree of a rational map $f\colon Y \dashrightarrow X$ and $d = \dim Y = \dim X$. In particular, if $n_X \nmid \frac{1}{p}c_\alpha(X)$, then X is incompressible.

In the present section we discuss the relations between the classical degree formulas (14) and the degree formula (5). The following lemma provides an explicit formula for $\chi(\mathcal{O}_X)$ in terms of characteristic numbers $c_\alpha(X)$

5.5 Lemma. *Let X be a smooth projective variety of dimension d over k. Then*

$$\chi(\mathcal{O}_X) = (-1)^d \sum_{\alpha, |\alpha|=d} \frac{c_\alpha(X)}{(\alpha_1+1)!(\alpha_2+1)!\ldots(\alpha_r+1)!}.$$

Proof. By the very definition $\mathrm{Td}(X) = \mathrm{Td}(T_X) = \prod_{i=1}^d Q(x_i)$, where $Q(x_i) = \frac{x_i}{1-e^{-x_i}}$ and x_i are the roots of the tangent bundle T_X. Since $\mathrm{Td}(T_X) = \mathrm{Td}(-T_X)^{-1} = \prod_{i=1}^d Q(-x_i)^{-1}$, its component of degree d is equal to the coefficient at z^d in the expansion of the product

$$\prod_{i=1}^d (1 - \tfrac{x_i}{2!}z + \tfrac{x_i^2}{3!}z^2 - \tfrac{x_i^3}{4!}z^3 + \ldots).$$

Analyzing the product we see that this coefficient is, indeed, given by

$$(-1)^d \sum_{\alpha, |\alpha|=d} \frac{P_\alpha(x_1, x_2, \ldots)}{(\alpha_1+1)!(\alpha_2+1)!\ldots},$$

where P_α is the minimal symmetric polynomial from 5.1. □

5.6 Lemma. *Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a partition of d and let $v_p(m)$ denote the p-adic valuation of an integer m. Then*

$$v_p(\tau_{d-1}) + 1 \geq v_p\Big(\prod_{i=1}^r (\alpha_i+1)!\Big),$$

where the equality holds if and only if $\alpha \in \Lambda_p$.

Proof. Follows from the formulas $v_p(\tau_{d-1}) = [\frac{d-1}{p-1}]$ and $v_p(m!) = \sum_{j=1}^{\infty}[\frac{m}{p^j}]$. $\square$

5.7 Definition. Let p be a prime p and let d be an integer. We define a linear combination u_p of characteristic numbers c_α, $\alpha \in \Lambda_p$, as

$$u_p = \sum_{\alpha \in \Lambda_p, |\alpha| = d} \frac{\tau_{d-1}}{\prod_{i=1}^{r}(\alpha_i + 1)!} c_\alpha.$$

According to Lemma 5.6 we have

$$u_p = \sum_{\alpha \in \Lambda_p, |\alpha| = d} n_\alpha \frac{c_\alpha}{p}, \text{ where } n_\alpha = \frac{p \cdot \tau_{d-1}}{\prod_{i=1}^{r}(\alpha_i + 1)!} \in \mathbb{Z} \text{ and } p \nmid n_\alpha.$$

5.8 Proposition. *Let X be a smooth projective variety of dimension d over a field of characteristic* 0. *Then*

$$n_X \nmid \chi(\mathcal{O}_X)\tau_{d-1} \Longleftrightarrow \exists p \text{ such that } n_X \nmid u_p(X).$$

Proof. Since $n_X \mid c_\alpha$ for any α, $u'_p = \frac{p u_p}{n_X} \in \mathbb{Z}$. We have the following chain of equivalences

$$\begin{aligned} & n_X \mid \chi(\mathcal{O}_X)\tau_{d-1} \Longleftrightarrow n_X \mid \textstyle\sum_p \frac{n_X}{p} u'_p \Longleftrightarrow \sum_p \frac{u'_p}{p} \in \mathbb{Z} \\ & \Longleftrightarrow \forall p \;\; p \mid u'_p \Longleftrightarrow \forall p \;\; \tfrac{u_p}{n_X} \in \mathbb{Z}. \end{aligned}$$

$\square$

5.9 Example. Let X be a smooth projective curve. In this case we have only one non-trivial partition $\alpha = (1) \in \Lambda_2$ and

$$\chi(\mathcal{O}_X) = -\tfrac{1}{2} c_{(1)}(X) = -\eta_2(X) = -u_2(X).$$

Therefore, for curves the degree formula (5) coincides with the classical one.

5.10 Example. For a smooth projective surface X we have two partitions $\alpha = (1, 1)$ and (2), where the first one belongs to Λ_2 and the second one to Λ_3. We have

$$\chi(\mathcal{O}_X) = \tfrac{1}{4} c_{(1,1)}(X) + \tfrac{1}{6} c_{(2)}(X) \text{ and } u_2 = \tfrac{1}{2} c_{(1,1)} = \eta_2,\ u_3 = \tfrac{1}{3} c_{(2)} = \eta_3.$$

The degree formula (5) turns into a sum of the classical degree formulas

$$(\eta_2 + \eta_3)(Y) \equiv \deg f \cdot (\eta_2 + \eta_3)(X) \mod n_X$$

and

$$n_X \nmid \tau_{d-1}\chi(\mathcal{O}_X) \Longleftrightarrow n_X \nmid \eta_2(X) \text{ or } n_X \nmid \eta_3(X).$$

So from the point of view of incompressibility the degree formula (5) provides the same answer as the classical degree formulas.

5.11 Example. Let X be a smooth projective 3-fold. In this case we have three partitions $(1, 1, 1)$, $(1, 2)$ and (3), where the first and the last one belong to Λ_2. We have

$$\chi(\mathcal{O}_X) = -\tfrac{1}{8} c_{(1,1,1)} - \tfrac{1}{12} c_{(1,2)} - \tfrac{1}{24} c_{(3)} \text{ and } u_2 = \tfrac{3}{2} c_{(1,1,1)} + \tfrac{1}{2} c_{(3)}.$$

Therefore,

$$n_X \nmid \chi(\mathcal{O}_X)\tau_2 \Longleftrightarrow n_X \nmid u_2 \stackrel{(*)}{\Longrightarrow} n_X \nmid \tfrac{1}{2}c_{(1,1,1)} \text{ or } n_X \nmid \tfrac{1}{2}c_{(3)}$$

which means that for 3-folds the classical degree formulas (14) detect the incompressibility better than (5).

5.12. Since each characteristic number c_α is divisible by n_X, to say that $n_X \mid q \cdot c_\alpha$, $q \in \mathbb{Q}$, is equivalent to say that $q \cdot C_\alpha \in \mathbb{Z}$, where $C_\alpha := c_\alpha/n_X \in \mathbb{Z}$. Hence, the implication $(*)$ can be rewritten as

$$\tfrac{1}{2}C_{(1,1,1)} + \tfrac{1}{2}C_{(3)} \notin \mathbb{Z} \Longrightarrow \tfrac{1}{2}C_{(1,1,1)} \notin \mathbb{Z} \text{ or } \tfrac{1}{2}C_{(3)} \notin \mathbb{Z}$$

In particular, the implication $(*)$ becomes an equivalence if and only if a 3-folds X satisfies the following condition

$$C_{(1,1,1)} = \tfrac{c_{(1,1,1)}(X)}{n_X} \text{ is even or } C_{(3)} = \tfrac{c_{(3)}(X)}{n_X} \text{ is even.} \tag{15}$$

5.13 Example. Let X be a complete intersection of m hypersurfaces of degrees $d_1, \ldots, d_m$ in $\mathbb{P}^{m+3}$. Using the formula from [7, §8] we obtain:

$$C_{(3)} = \frac{\partial}{\partial z}\Big(\frac{\prod_{i=1}^m (1+d_i^3 z)}{(1+z)^{m+4}}\Big)_{z=0} = (\sum_{i=1}^m d_i^3) - m - 4 \equiv \sigma_1 + m \mod 2,$$

where σ_1 denotes the sum of all degrees. And

$$C_{(1,1,1)} = \frac{1}{6}\frac{\partial^3}{\partial z^3}\Big(\frac{\prod_{i=1}^m (1+d_i z)}{(1+z)^{m+4}}\Big)_{z=0} \equiv \binom{m+2}{3} + \binom{m+1}{2}\sigma_1 + m\sigma_2 \mod 2,$$

where σ_2 is the second elementary symmetric function in d_i-s.

Hence, a complete intersection X satisfies (15) if and only if it satisfies one of the following conditions

- $m = 4k$;
- $m = 4k+2$, σ_1 is even;
- $m = 4k+1$, σ_1 or σ_2 is odd;
- $m = 4k-1$, σ_1 is odd or σ_2 is even.

5.14 Example. Let X be a complete intersection of m smooth hypersurfaces of degrees $d_1, d_2, \ldots, d_m$ in $\mathbb{P}^{m+4}$. As in the previous example one can show that the degree formula (5) for X is equivalent to the classical degree formulas if and only if X satisfies one of the following conditions

- m is odd;
- $m = 4k$, $-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4$ is even;
- $m = 4k+2$, $-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4$ is odd.

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