Degree Graphs of Simple Linear and Unitary Groups

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ABSTRACT

Let G be a finite group and let $\mathrm{cd}(G)$ be the set of irreducible character degrees of G. The degree graph $\Delta(G)$ is the graph whose set of vertices is the set of primes that divide degrees in $\mathrm{cd}(G)$, with an edge between p and q if pq divides a for some degree $a \in \mathrm{cd}(G)$. We determine the graph $\Delta(G)$ for the finite simple groups of types $A_\ell(q)$ and ${}^2A_\ell(q^2)$, that is, for the simple linear and unitary groups.

1. INTRODUCTION

A problem of current interest is to determine information that can be deduced about the structure of a finite group G from the structure of its set of irreducible character degrees. A tool that has been used to study the relationship between G and its set of character degrees is the character degree graph $\Delta(G)$.

Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of ordinary irreducible characters of G. Denote the set of irreducible character degrees of G by $\mathrm{cd}(G) = \{\chi(1) | \chi \in \mathrm{Irr}(G)\}$ and denote by $\rho(G)$ the set of primes that divide degrees in $\mathrm{cd}(G)$. The character degree graph $\Delta(G)$ of G is the graph whose set of vertices is $\rho(G)$, with primes p, q in $\rho(G)$ joined by an edge if pq divides a for some character degree $a \in \mathrm{cd}(G)$.

The structure of this graph has been studied primarily in the case where G is a solvable group. More recently, however, some results on the structure of $\Delta(G)$ have been obtained for an arbitrary finite group G. In [8], for example, Lewis and the author classified the nonsolvable finite groups for which $\Delta(G)$ is disconnected, and in [9] we proved an upper bound on the diameter of $\Delta(G)$ for a nonsolvable finite group G.

The results on $\Delta(G)$ for nonsolvable G have been obtained essentially by reducing the problem to the structure of $\Delta(G)$ for a finite simple group G.

It is therefore very useful to have as much information about the graphs of the simple groups as possible. In particular, in order to attempt to improve the bound on the diameter of $\Delta(G)$ found in [9], it will probably be necessary to know explicitly the graphs for all finite simple groups.

The character tables of the sporadic simple groups are known (see the Atlas [3]), so it is easy to determine the graphs of these groups. A description of these graphs is given in [9]. The graphs for the alternating groups are easily deduced from the Atlas character tables and the results of [1]. The graphs for the simple groups of exceptional Lie type were found in [12]. By the Classification of Finite Simple Groups, this leaves the graphs for the simple groups of classical Lie type to be determined.

In this paper, we determine the graph $\Delta(G)$ where G is a finite simple group of Lie type either of type A_ℓ, with $\ell \geqslant 1$, or of type ${}^2A_\ell$, with $\ell \geqslant 2$. That is, either G is the linear group $\mathrm{PSL}_{\ell+1}(q)$ for $\ell \geqslant 1$, or G is the unitary group $\mathrm{PSU}_{\ell+1}(q^2)$ for $\ell \geqslant 2$. Work to determine the graphs for the classical groups of types B_ℓ, C_ℓ, D_ℓ, and ${}^2D_\ell$ — the orthogonal and symplectic groups — is in progress.

The character degree graphs of the simple linear and unitary groups tend to be complete graphs, that is, there is an edge joining every pair of vertices. In fact, for $\ell \geqslant 3$, the degree graph of $\mathrm{PSL}_{\ell+1}(q)$ is complete unless $\ell = 3$ and $q = 2$ (Theorem 3.3) and the graph of $\mathrm{PSU}_{\ell+1}(q^2)$ is always complete (Theorem 3.5). The situation for groups of lower rank is more complicated and is described in Theorems 3.1, 3.2, and 3.4.

2. CHARACTER DEGREES

In this section, we construct the character degrees used to determine the character degree graphs for simple groups of type A_ℓ or ${}^2A_\ell$, for $\ell \geqslant 3$.

Throughout this section, q will denote a power of a prime p, $\mathbb{F}_{q^a}$ is the field of q^a elements, and $\mathbb{F}_{q^a}^*$ its multiplicative group. We denote by $\overline{\mathbb{F}}_p$ an algebraic closure of the field $\mathbb{F}_p$ of p elements.

For notation, definitions, and basic properties of groups of Lie type, we refer to [2] or [4]. We will denote by $\boldsymbol{G}$ a simple linear algebraic group of adjoint type defined over $\overline{\mathbb{F}}_p$, and by F a Frobenius endomorphism of $\boldsymbol{G}$ so that the set $\boldsymbol{G}^F$ of fixed points is finite and the derived group of $\boldsymbol{G}^F$ is a simple group. Let $(\boldsymbol{G}^*, F^*)$ denote the dual of $(\boldsymbol{G}, F)$.

Several of the character degrees are determined using the following lemma, which is a direct result of [4, Theorem 13.23, Remark 13.24] or [2, §12.9].

Lemma 2.1. *There is a bijection between the set of conjugacy classes (s) of semisimple elements s of $\boldsymbol{G}^{*F^*}$ and the set of geometric conjugacy classes $\mathcal{E}(\boldsymbol{G}^F, (s))$ of irreducible characters of $\boldsymbol{G}^F$. For a semisimple element s of $\boldsymbol{G}^{*F^*}$, there is a bijection ψ_s between the set of irreducible characters in $\mathcal{E}(\boldsymbol{G}^F, (s))$ and the set of unipotent characters of $C_{\boldsymbol{G}^*}(s)^{F^*}$.*

Moreover, for $\chi \in \mathcal{E}(\boldsymbol{G}^F, (s))$*, the degree of* χ *is*

$$\chi(1) = \frac{|\boldsymbol{G}^F|_{p'}}{|C_{\boldsymbol{G}^*}(s)^{F^*}|_{p'}} \psi_s(\chi)(1).$$

The elements of $\mathcal{E}(\boldsymbol{G}^F, (1))$ are the unipotent characters of $\boldsymbol{G}^F$. The character χ_s such that $\psi_s(\chi_s)$ is the principal character of $C_{\boldsymbol{G}^*}(s)^{F^*}$ is the semisimple character corresponding to the conjugacy class of s. Thus the lemma says that the irreducible characters of $\boldsymbol{G}^F$ are in bijection with the set of pairs (χ_s, μ_s), where χ_s is the semisimple character corresponding to (s) and μ_s is a unipotent character of $C_{\boldsymbol{G}^*}(s)^{F^*}$. The degree of the character corresponding to (χ_s, μ_s) is $\chi_s(1)\mu_s(1)$.

The degrees of the unipotent characters of the classical groups can be computed using formulas found in [2, §13.8]. Other character degrees will be computed using the formula in Lemma 2.1 for particular semisimple elements s.

2.1. Character Degrees of Linear Groups

We construct here some character degrees for the projective general linear group $\mathrm{PGL}_{\ell+1}(q)$, for $\ell \geqslant 3$. In this case, $\boldsymbol{G}$ is the adjoint group of type A_ℓ, so $\boldsymbol{G} = \mathrm{PGL}_{\ell+1}(\overline{\mathbb{F}}_p)$ and the dual group is $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$. The Frobenius map F^* on $\boldsymbol{G}^*$ is the standard Frobenius map $(a_{ij}) \mapsto (a_{ij}^q)$, so the group of fixed points $\boldsymbol{G}^{*F^*}$ is $\mathrm{SL}_{\ell+1}(q)$, which is dual to $\boldsymbol{G}^F = \mathrm{PGL}_{\ell+1}(q)$.

In order to construct each semisimple character degree of $\mathrm{PGL}_{\ell+1}(q)$, we will exhibit a semisimple element X of $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$ whose eigenvalues are permuted by the Frobenius map F^*. The conjugacy class of X in $\boldsymbol{G}^*$ then intersects the conjugacy class of a semisimple element s of $\mathrm{SL}_{\ell+1}(q)$, and this semisimple class corresponds to a semisimple character χ of $\mathrm{PGL}_{\ell+1}(q)$.

Lemma 2.2. *If* $\ell \geqslant 3$*, then* $\boldsymbol{G}^F = \mathrm{PGL}_{\ell+1}(q)$ *has irreducible characters* $\chi^{(1,1,\ell-1)}$*,* χ_1*,* χ_2*, and* χ_3 *with degrees*

$$\chi^{(1,1,\ell-1)}(1) = q^3 \cdot \frac{(q^{\ell-1}-1)(q^\ell - 1)}{(q-1)(q^2-1)}$$

$$\chi_1(1) = (q-1)(q^2-1)(q^3-1)\cdots(q^{\ell-1}-1)(q^\ell-1)$$

$$\chi_2(1) = (q^2-1)(q^3-1)\cdots(q^{\ell-1}-1)(q^{\ell+1}-1)$$

$$\chi_3(1) = q \cdot \frac{(q^2-1)(q^3-1)\cdots(q^{\ell+1}-1)}{(q^2-1)(q^{\ell-1}-1)}.$$

Proof. By the notation and character degree formula in [2, §13.8], since $\ell \geqslant 3$, $\boldsymbol{G}^F = \mathrm{PGL}_{\ell+1}(q)$ has the unipotent character $\chi^{(1,1,\ell-1)}$ with the degree as claimed.

By [3, §3], we have

$$|\boldsymbol{G}^F|_{p'} = (q^2-1)(q^3-1)\cdots(q^\ell-1)(q^{\ell+1}-1).$$

Let η be a generator of $\mathbb{F}_{q^{\ell+1}}^*$ and $\tau = \eta^{q-1}$, so τ is an element of $\mathbb{F}_{q^{\ell+1}}$ of order

$$\frac{q^{\ell+1}-1}{q-1} = q^\ell + q^{\ell-1} + \cdots + q + 1.$$

Let

$$X_1 = \mathrm{diag}[\tau, \tau^q, \tau^{q^2}, \ldots, \tau^{q^\ell}],$$

so $\det X_1 = 1$ and X_1 is a semisimple element of $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$.

Since $\tau \in \mathbb{F}_{q^{\ell+1}}$, we have $\tau^{q^{\ell+1}} = \tau$, and therefore the Frobenius map cyclically permutes the eigenvalues of X_1. Hence the conjugacy class of X_1 in $\boldsymbol{G}^*$ is F^*-stable and intersects $\boldsymbol{G}^{*F^*} = \mathrm{SL}_{\ell+1}(q)$ in the conjugacy class of a semisimple element s_1. Denote by χ_1 the corresponding semisimple character of $\boldsymbol{G}^F = \mathrm{PGL}_{\ell+1}(q)$.

The eigenvalues of s_1 are distinct and $C_{\boldsymbol{G}^*}(s_1)^{F^*} = \langle s_1 \rangle$. Hence

$$|C_{\boldsymbol{G}^*}(s_1)^{F^*}|_{p'} = \frac{q^{\ell+1}-1}{q-1},$$

and by Lemma 2.1 and the value of $|\boldsymbol{G}^F|_{p'}$ given above, $\chi_1(1)$ is as claimed.

Next, let σ be a generator of $\mathbb{F}_{q^\ell}^*$ and

$$\rho = \sigma^{-\frac{q^\ell-1}{q-1}} = \sigma^{-(q^{\ell-1}+q^{\ell-2}+\cdots+q+1)},$$

so ρ is an element of $\mathbb{F}_q$ of order $q-1$. Let

$$X_2 = \mathrm{diag}[\sigma, \sigma^q, \sigma^{q^2}, \ldots, \sigma^{q^{\ell-1}}, \rho],$$

so $\det X_2 = 1$ and X_2 is a semisimple element of $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$.

Since $\rho \in \mathbb{F}_q$, $\rho^q = \rho$. Also, $\sigma \in \mathbb{F}_{q^\ell}$, hence $\sigma^{q^\ell} = \sigma$ and the Frobenius map cyclically permutes the eigenvalues $\sigma, \sigma^q, \ldots, \sigma^{q^{\ell-1}}$ and fixes ρ. Therefore the conjugacy class of X_2 is F^*-stable and its intersection with $\boldsymbol{G}^{*F^*} = \mathrm{SL}_{\ell+1}(q)$ is the conjugacy class of a semisimple element s_2. Denote by χ_2 the corresponding semisimple character of $\boldsymbol{G}^F = \mathrm{PGL}_{\ell+1}(q)$.

The eigenvalues of s_2 are distinct and $C_{\boldsymbol{G}^*}(s_2)^{F^*} = \langle s_2 \rangle$, hence

$$|C_{\boldsymbol{G}^*}(s_2)^{F^*}|_{p'} = q^\ell - 1.$$

Therefore, by Lemma 2.1 and the value of $|\boldsymbol{G}^F|_{p'}$ given above, $\chi_2(1)$ is as claimed.

Let θ be a generator of $\mathbb{F}_{q^{\ell-1}}^*$ and let $\gamma = \theta^{q-1}$, so γ is an element of $\mathbb{F}_{q^{\ell-1}}$ of order

$$\frac{q^{\ell-1}-1}{q-1} = q^{\ell-2} + q^{\ell-3} + \cdots + q + 1.$$

Let

$$X_3 = \mathrm{diag}[1, 1, \gamma, \gamma^q, \gamma^{q^2}, \ldots, \gamma^{q^{\ell-2}}].$$

so $\det X_3 = 1$ and X_3 is a semisimple element of $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$.

Because $\gamma \in \mathbb{F}_{q^{\ell-1}}$, we have $\gamma^{q^{\ell-1}} = \gamma$, so the Frobenius map cyclically permutes the eigenvalues $\gamma, \gamma^q, \ldots, \gamma^{q^{\ell-2}}$, and of course fixes the eigenvalue 1. Therefore the conjugacy class of X_3 is F^*-stable and its intersection with $\boldsymbol{G}^{*F^*} = \mathrm{SL}_{\ell+1}(q)$ is the conjugacy class of a semisimple element s_3. Denote by χ_{s_3} the corresponding semisimple character of $\boldsymbol{G}^F = \mathrm{PGL}_{\ell+1}(q)$.

The eigenvalues of s_3 other than 1 are distinct and $C_{\boldsymbol{G}^*}(s_3)^{F^*}$ is isomorphic to a group of the form $[\mathrm{SL}_2(q) \times \langle\gamma\rangle] \cdot (q-1)$. We have $|\mathrm{SL}_2(q)| = q(q^2-1)$ and $|\langle\gamma\rangle| = (q^{\ell-1}-1)/(q-1)$, hence

$$|C_{\boldsymbol{G}^*}(s_2)^{F^*}|_{p'} = (q^2-1)(q^{\ell-1}-1).$$

Therefore, by Lemma 2.1 and the value of $|\boldsymbol{G}^F|_{p'}$ given above,

$$\chi_{s_3}(1) = \frac{(q^2-1)(q^3-1)\cdots(q^{\ell+1}-1)}{(q^2-1)(q^{\ell-1}-1)}.$$

In this case, $C_{\boldsymbol{G}^*}(s_3)^{F^*}$ has a unipotent character St, the Steinberg character, of degree q. Let χ_3 be the irreducible character of $\boldsymbol{G}^F = \mathrm{PGL}_{\ell+1}(q)$ corresponding to the pair (χ_{s_3}, St). It then follows from Lemma 2.1 that the degree of χ_3 is as claimed. ∎

2.2. Character Degrees of Unitary Groups

We next construct some character degrees for the projective general unitary group $\mathrm{PU}_{\ell+1}(q^2)$, for $\ell \geqslant 3$. Let $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$, and let F^* be the twisted Frobenius map on $\boldsymbol{G}^*$ given by $(a_{ij}) \mapsto ((a_{ij}^q)^t)^{-1}$, so the group of fixed points $\boldsymbol{G}^{*F^*}$ is the special unitary group $\mathrm{SU}_{\ell+1}(q^2)$ defined over the field of q^2 elements. This is dual to $\boldsymbol{G}^F = \mathrm{PU}_{\ell+1}(q^2)$, the adjoint group of type ${}^2A_\ell$.

Lemma 2.3. *If $\ell \geqslant 3$, then $\boldsymbol{G}^F = \mathrm{PU}_{\ell+1}(q^2)$ has irreducible characters $\chi^{(1,1,\ell-1)}$, χ_1, and χ_2 with degrees*

$$\chi^{(1,1,\ell-1)}(1) = q^3 \cdot \frac{(q^{\ell-1}-(-1)^{\ell-1})(q^\ell-(-1)^\ell)}{(q+1)(q^2-1)}$$

$$\chi_1(1) = (q+1)(q^2-1)(q^3+1)\cdots(q^{\ell-1}-(-1)^{\ell-1})(q^\ell-(-1)^\ell)$$

$$\chi_2(1) = (q^2-1)(q^3+1)\cdots(q^{\ell-1}-(-1)^{\ell-1})(q^{\ell+1}-(-1)^{\ell+1}).$$

If (ℓ, q) is not in $\{(3,2),(3,3),(4,2)\}$, then there is an irreducible character χ_3 with degree

$$\chi_3(1) = q \cdot \frac{(q^2-1)(q^3+1)\cdots(q^{\ell+1}-(-1)^{\ell+1})}{(q^2-1)(q^{\ell-1}-(-1)^{\ell-1})}.$$

Proof. By the notation and character degree formula in [2, §13.8], since $\ell \geqslant 3$, $\boldsymbol{G}^F = \mathrm{PU}_{\ell+1}(q^2)$ has the unipotent character $\chi^{(1,1,\ell-1)}$ with the degree as claimed.

By [3, §3], we have

$$|\boldsymbol{G}^F|_{p'} = (q^2-1)(q^3+1)\cdots(q^\ell-(-1)^\ell)(q^{\ell+1}-(-1)^{\ell+1}).$$

Denote by η a generator of $\mathbb{F}^*_{q^{\ell+1}}$ and, if ℓ is even, let ω be a generator of $\mathbb{F}^*_{q^{2(\ell+1)}}$. Let

$$\tau = \begin{cases} \eta^{q+1} & \text{if } \ell \text{ is odd} \\ \omega^{(q^{\ell+1}-1)(q+1)} & \text{if } \ell \text{ is even} \end{cases}$$

so τ is an element of $\overline{\mathbb{F}}_p$ of order

$$\frac{q^{\ell+1}-(-1)^{\ell+1}}{q+1} = (-1)^\ell\left(1-q+q^2-\cdots+(-q)^{\ell-1}+(-q)^\ell\right).$$

Let

$$X_1 = \mathrm{diag}[\tau, \tau^{-q}, \tau^{q^2}, \ldots, \tau^{(-q)^{\ell-1}}, \tau^{(-q)^\ell}],$$

so $\det X_1 = 1$ and X_1 is a semisimple element of $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$.

Since $\tau^{q^{\ell+1}-(-1)^{\ell+1}} = 1$, we have $\tau^{(-q)^{\ell+1}} = \tau$, and therefore the Frobenius map cyclically permutes the eigenvalues of X_1. Hence the conjugacy class of X_1 in $\boldsymbol{G}^*$ is F^*-stable and intersects $\boldsymbol{G}^{*F^*} = \mathrm{SU}_{\ell+1}(q^2)$ in the conjugacy class of a semisimple element s_1. Denote by χ_1 the corresponding semisimple character of $\boldsymbol{G}^F = \mathrm{PU}_{\ell+1}(q^2)$.

The eigenvalues of s_1 are distinct and $C_{\boldsymbol{G}^*}(s_1)^{F^*} = \langle s_1\rangle$, hence

$$|C_{\boldsymbol{G}^*}(s_1)^{F^*}|_{p'} = \frac{q^{\ell+1}-(-1)^{\ell+1}}{q+1}.$$

By Lemma 2.1 and the value of $|\boldsymbol{G}^F|_{p'}$ given above, $\chi_1(1)$ is as claimed.

If ℓ is even, let σ be a generator of $\mathbb{F}^*_{q^\ell}$, and if ℓ is odd, let ζ be a generator of $\mathbb{F}^*_{q^{2\ell}}$ and let $\sigma = \zeta^{q^\ell-1}$. Let

$$\rho = \sigma^{(-1)^\ell\frac{q^\ell-(-1)^\ell}{q+1}} = \sigma^{-(1-q+q^2-\cdots+(-q)^{\ell-2}+(-q)^{\ell-1})},$$

so that σ is an element of $\overline{\mathbb{F}}_p$ of order $q^\ell-(-1)^\ell$ and ρ is an element of $\mathbb{F}_{q^2}$ of order $q+1$. Let

$$X_2 = \mathrm{diag}[\sigma, \sigma^{-q}, \sigma^{q^2}, \ldots, \sigma^{(-q)^{\ell-1}}, \rho],$$

so $\det X_2 = 1$ and X_2 is a semisimple element of $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$.

Since $\rho^{q+1} = 1$, $\rho^{-q} = \rho$. Also, $\sigma^{q^\ell-(-1)^\ell} = 1$, hence $\sigma^{(-q)^\ell} = \sigma$ and the Frobenius map cyclically permutes the eigenvalues $\sigma, \sigma^{-q}, \ldots, \sigma^{(-q)^{\ell-1}}$ and fixes ρ. Therefore the conjugacy class of X_2 is F^*-stable and its intersection with $\boldsymbol{G}^{*F^*} = \mathrm{SU}_{\ell+1}(q^2)$ is the conjugacy class of a semisimple element s_2. Denote by χ_2 the corresponding semisimple character of $\boldsymbol{G}^F = \mathrm{PU}_{\ell+1}(q^2)$.

The eigenvalues of s_2 are distinct and $C_{\boldsymbol{G}^*}(s_2)^{F^*} = \langle s_2 \rangle$, hence

$$|C_{\boldsymbol{G}^*}(s_2)^{F^*}|_{p'} = q^\ell - (-1)^\ell.$$

Therefore, by Lemma 2.1 and the value of $|\boldsymbol{G}^F|_{p'}$ given above, $\chi_2(1)$ is as claimed.

Let θ be a generator of $\mathbb{F}^*_{q^{\ell-1}}$ and, if ℓ is even, let ξ be a generator of $\mathbb{F}^*_{q^{2(\ell-1)}}$. Let

$$\gamma = \begin{cases} \theta^{q+1} & \text{if } \ell \text{ is odd} \\ \xi^{(q^{\ell-1}-1)(q+1)} & \text{if } \ell \text{ is even} \end{cases}$$

so γ is an element of $\overline{\mathbb{F}}_p$ of order

$$\frac{q^{\ell-1} - (-1)^{\ell-1}}{q+1} = (-1)^{\ell-2}\left(1 - q + q^2 - \cdots + (-q)^{\ell-3} + (-q)^{\ell-2}\right).$$

Let

$$X_3 = \mathrm{diag}[1, 1, \gamma, \gamma^{-q}, \gamma^{q^2}, \ldots, \gamma^{(-q)^{\ell-3}}, \gamma^{(-q)^{\ell-2}}],$$

so $\det X_3 = 1$ and X_3 is a semisimple element of $\boldsymbol{G}^* = \mathrm{SL}_{\ell+1}(\overline{\mathbb{F}}_p)$.

As $\gamma^{q^{\ell-1}-(-1)^{\ell-1}} = 1$, we have $\gamma^{(-q)^{\ell-1}} = \gamma$, and the Frobenius map cyclically permutes the eigenvalues $\gamma, \gamma^{-q}, \ldots, \gamma^{(-q)^{\ell-2}}$, fixing the eigenvalue 1. Therefore the conjugacy class of X_3 is F^*-stable and its intersection with $\boldsymbol{G}^{*F^*} = \mathrm{SU}_{\ell+1}(q^2)$ is the conjugacy class of a semisimple element s_3. Denote by χ_{s_3} the corresponding semisimple character of $\boldsymbol{G}^F = \mathrm{PU}_{\ell+1}(q^2)$.

Unless $\ell = 3$ and $q = 2$ or $q = 3$, or $\ell = 4$ and $q = 2$, the eigenvalues of s_3 other than 1 are distinct and $C_{\boldsymbol{G}^*}(s_3)^{F^*}$ is isomorphic to a group of the form $\left[\mathrm{SU}_2(q^2) \times \langle\gamma\rangle\right] \cdot (q+1)$. We have $|\mathrm{SU}_2(q^2)| = q(q^2-1)$ and $|\langle\gamma\rangle| = (q^{\ell-1} - (-1)^{\ell-1})/(q+1)$. Hence

$$|C_{\boldsymbol{G}^*}(s_2)^{F^*}|_{p'} = (q^2-1)(q^{\ell-1} - (-1)^{\ell-1}).$$

Therefore, by Lemma 2.1 and the value of $|\boldsymbol{G}^F|_{p'}$ given above,

$$\chi_{s_3}(1) = \frac{(q^2-1)(q^3+1)\cdots(q^{\ell+1} - (-1)^{\ell+1})}{(q^2-1)(q^{\ell-1} - (-1)^{\ell-1})}.$$

In this case, $C_{\boldsymbol{G}^*}(s_3)^{F^*}$ has a unipotent character St, the Steinberg character, of degree q. Let χ_3 be the irreducible character of $\boldsymbol{G}^F = \mathrm{PU}_{\ell+1}(q^2)$ corresponding to the pair (χ_{s_3}, St). It then follows from Lemma 2.1 that the degree of χ_3 is as claimed. ∎

3. THE DEGREE GRAPHS

We are now ready to describe the graphs $\Delta(G)$ for simple groups of types A_ℓ and ${}^2A_\ell$. Let $\pi(n)$ denote the set of prime divisors of a positive integer n. Note that if G is a nonabelian finite simple group, then by the Itô-Michler Theorem (see [10, Remarks 13.13]), $\rho(G) = \pi(|G|)$.

3.1. Linear Groups

In this section, G is the projective special linear group $\mathrm{PSL}_{\ell+1}(q)$ of type $A_\ell(q)$, where q is a power of a prime p and $\ell \geqslant 1$. If $\ell = 1$, then we take $q \geqslant 4$ as $\mathrm{PSL}_2(2)$ and $\mathrm{PSL}_2(3)$ are not simple. Note also that $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$.

THEOREM 3.1. *Let $G \cong \mathrm{PSL}_2(q)$, where $q \geqslant 4$ is a power of a prime p.*

1. *If q is even, then $\Delta(G)$ has three connected components, $\{2\}$, $\pi(q-1)$, and $\pi(q+1)$, and each component is a complete graph.*
2. *If $q > 5$ is odd, then $\Delta(G)$ has two connected components, $\{p\}$ and $\pi\left((q-1)(q+1)\right)$.*
   - *(a) The connected component $\pi\left((q-1)(q+1)\right)$ is a complete graph if and only if $q-1$ or $q+1$ is a power of 2.*
   - *(b) If neither of $q-1$ or $q+1$ is a power of 2, then $\pi\left((q-1)(q+1)\right)$ can be partitioned as $\{2\} \cup M \cup P$, where $M = \pi(q-1) - \{2\}$ and $P = \pi(q+1) - \{2\}$ are both nonempty sets. The subgraph of $\Delta(G)$ corresponding to each of the subsets M, P is complete, all primes are adjacent to 2, and no prime in M is adjacent to any prime in P.*

Proof. The character table of $\mathrm{PSL}_2(q)$ is well-known. See [5, §38], for example.

If $q = 2^n$, $n \geqslant 2$, then

$$\mathrm{cd}(G) = \{1, 2^n - 1, 2^n, 2^n + 1\}.$$

As $2^n - 1$ and $2^n + 1$ are odd and relatively prime, $\Delta(G)$ has three connected components $\{2\}$, $\pi(2^n - 1)$, and $\pi(2^n + 1)$. For each connected component, there is a degree divisible by all primes in that component, hence each component is a complete graph.

If $q = p^n > 5$ is odd, then

$$\mathrm{cd}(G) = \{1, q-1, q, q+1, (q+\epsilon)/2\},$$

where $\epsilon = (-1)^{(q-1)/2}$. Now 2 divides both $q-1$ and $q+1$, and $(q+\epsilon)/2$ divides $q+\epsilon$, hence the two connected components of $\Delta(G)$ are $\{p\}$ and $\pi((q-1)(q+1))$.

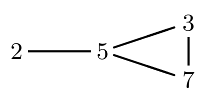
We have $(q-1, q+1) = 2$, so no prime in M is adjacent to any prime in P, but every prime in $\pi((q-1)(q+1))$ is adjacent to 2 in $\Delta(G)$. If $q-1$ or $q+1$ is a power of 2, then either M or P is empty, and the graph of the component $\pi((q-1)(q+1))$ is complete. If neither $q-1$ nor $q+1$ is a power of 2, then both M and P are nonempty, hence the graph is not complete. ∎

We next describe the degree graph for $\mathrm{PSL}_3(q)$. Note that $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, so if $q = 2$, then the graph is described in Theorem 3.1. We will therefore take $q > 2$.

THEOREM 3.2. *Let $G \cong \mathrm{PSL}_3(q)$, where $q > 2$ is a power of a prime p.*

1. *The graph $\Delta(G)$ is complete if and only if q is odd and $q - 1 = 2^i 3^j$ for some $i \geqslant 1$, $j \geqslant 0$.*

2. *(a) If $q = 4$ then $G \cong \mathrm{PSL}_3(4)$ and $\Delta(G)$ is*



   (b) If $q \neq 4$, then $\rho(G) = \{p\} \cup \pi\left((q-1)(q+1)(q^2+q+1)\right)$. The subgraph of $\Delta(G)$ corresponding to $\pi\left((q-1)(q+1)(q^2+q+1)\right)$ is complete and p is adjacent to precisely those primes dividing $q+1$ or q^2+q+1.

Proof. If $q = 4$, then $G \cong \mathrm{PSL}_3(4)$ and by the character table in the Atlas [3], we have

$$\mathrm{cd}(\mathrm{PSL}_3(4)) = \{1, 2^2 \cdot 5, 5 \cdot 7, 3^2 \cdot 5, 3^2 \cdot 7, 2^6\}.$$

Hence $\Delta(\mathrm{PSL}_3(4))$ is as claimed in 2a. Note that the graph is not complete, so 1 holds in case $q = 4$.

If $q = 3$, then $G \cong \mathrm{PSL}_3(3)$ and again by the Atlas character table, we have

$$\mathrm{cd}(\mathrm{PSL}_3(3)) = \{1, 2^2 \cdot 3, 13, 2^4, 2 \cdot 13, 3^3, 3 \cdot 13\}.$$

Hence $\Delta(\mathrm{PSL}_3(3))$ is complete as claimed in 1. Moreover, in this case $p = 3$, $q-1 = 2$, $q+1 = 4$, and $q^2+q+1 = 13$, hence 2b holds as well.

Assume now that $q > 4$. By the character table of G in either [11] or the CHEVIE system [6], every character degree of G divides one of the degrees in the subset

$$\{q^3, q(q+1), (q-1)(q^2+q+1), q(q^2+q+1), (q+1)(q^2+q+1), (q-1)^2(q+1)\}$$

of $\mathrm{cd}(G)$. Hence all primes dividing $q-1$, $q+1$, or q^2+q+1 are adjacent in $\Delta(G)$ and p is adjacent to precisely those primes dividing $q+1$ or q^2+q+1. Statement 2b of the theorem follows.

It also follows from the list of degrees above that for $q > 4$, $\Delta(G)$ is complete if and only if p is adjacent to all primes dividing $q-1$. This holds if and only if every prime dividing $q-1$ also divides either $q+1$ or q^2+q+1.

We have

$$(q-1, q+1) = \begin{cases} 1 & \text{if } q \text{ is even} \\ 2 & \text{if } q \text{ is odd} \end{cases}$$

and

$$(q-1, q^2+q+1) = \begin{cases} 1 & \text{if } q \not\equiv 1 (\mathrm{mod}\ 3) \\ 3 & \text{if } q \equiv 1 (\mathrm{mod}\ 3). \end{cases}$$

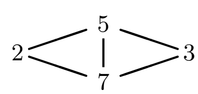
Thus for $q > 4$, $\Delta(G)$ is complete if and only if no primes other than 2 or 3 divide $q-1$, that is, if and only if $q-1 = 2^i 3^j$ for some $i \geqslant 0$, $j \geqslant 0$.

If $q-1 = 2^i 3^j$ and q is even, then $q = 2^m$ for some positive integer m and $i = 0$, so $q-1 = 2^m - 1 = 3^j$. Hence either $j = 0$ and $q = 2$, contradicting $q > 4$, or $2^m - 1$ is divisible by 3. Since $2 \equiv -1 (\mathrm{mod}\ 3)$, this implies $m = 2k$ is even, and $3^j = 2^{2k} - 1 = (2^k - 1)(2^k + 1)$. But then $2^k - 1$ and $2^k + 1$ are powers of 3 and are relatively prime, and this implies $2^k - 1 = 1$. Hence $k = 1$, $m = 2$, and $q = 2^m = 4$, again contradicting $q > 4$.

Therefore, 1 holds for $q > 4$ and we saw above that 1 holds for $q \leqslant 4$, completing the proof of the theorem. ∎

We now consider the simple groups of type A_ℓ for all $\ell \geqslant 3$. The one exceptional case is $\mathrm{PSL}_4(2)$, which is isomorphic to the alternating group $\mathrm{Alt}(8)$. This is one of only three simple alternating groups whose graph is not complete, along with $\mathrm{Alt}(5) \cong \mathrm{PSL}_2(4)$ and $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$.

Theorem 3.3. *Let $G \cong \mathrm{PSL}_{\ell+1}(q)$, where $\ell \geqslant 3$ and q is a power of a prime p. The graph $\Delta(G)$ is complete unless $\ell = 3$ and $q = 2$. If $\ell = 3$ and $q = 2$, then $G \cong \mathrm{PSL}_4(2) \cong \mathrm{Alt}(8)$ and $\Delta(G)$ is*



Proof. The order of $G \cong \mathrm{PSL}_{\ell+1}(q)$ is given by

$$|G| = \frac{1}{d}\, q^{\ell(\ell+1)/2}(q^2-1)(q^3-1)\cdots(q^\ell-1)(q^{\ell+1}-1),$$

where $d = (\ell+1, q-1) = [\mathrm{PGL}_{\ell+1}(q) : \mathrm{PSL}_{\ell+1}(q)]$. Therefore, denoting by $\Phi_k = \Phi_k(q)$ the value of the kth cyclotomic polynomial evaluated at q, we have that $\rho(G)$ is precisely the set of primes dividing q or some Φ_k for $1 \leqslant k \leqslant \ell+1$.

The character degrees found in Lemma 2.2 are degrees of $\mathrm{PGL}_{\ell+1}(q)$. By [7, Corollary 11.29], if χ is an irreducible character of $\mathrm{PGL}_{\ell+1}(q)$ and μ is an irreducible constituent of the restriction of χ to $\mathrm{PSL}_{\ell+1}(q)$, then

$\chi(1)/\mu(1)$ divides d and hence divides $q-1$ as well. In particular, if $q-1$ divides $\chi(1)$, then $\chi(1)/(q-1) = \chi(1)/\Phi_1$ divides $\mu(1)$.

First, let $\ell \geqslant 4$. By Lemma 2.2, $\mathrm{PGL}_{\ell+1}(q)$ has an irreducible character χ_3 of degree

$$\chi_3(1) = q \cdot \frac{(q^2-1)(q^3-1)\cdots(q^{\ell-1}-1)(q^\ell-1)(q^{\ell+1}-1)}{(q^2-1)(q^{\ell-1}-1)}.$$

Since $\ell \geqslant 4$, $\chi_3(1)$ is divisible by at least $q^\ell - 1$ and $q^{\ell+1}-1$, one of which is divisible by q^2-1 and the other by $q-1$. Hence Φ_1^2 and Φ_2 both divide $\chi_3(1)$, in spite of the division by q^2-1 in the degree formula. It follows that $\mathrm{PSL}_{\ell+1}(q)$ has an irreducible character whose degree is divisible by p and all Φ_k with $1 \leqslant k \leqslant \ell+1$ except $\Phi_{\ell-1}$. Thus all primes in $\rho(G)$, except those dividing only $\Phi_{\ell-1}$, are adjacent in $\Delta(G)$.

Again by Lemma 2.2, $\mathrm{PGL}_{\ell+1}(q)$ has irreducible characters of degrees

$$\chi_1(1) = (q-1)(q^2-1)(q^3-1)\cdots(q^{\ell-1}-1)(q^\ell-1)$$

and

$$\chi_2(1) = (q^2-1)(q^3-1)\cdots(q^{\ell-1}-1)(q^{\ell+1}-1).$$

As $\ell \geqslant 4$, both degrees are divisible by Φ_1^2, hence $\mathrm{PSL}_{\ell+1}(q)$ has irreducible characters μ_1 and μ_2 whose degrees are divisible by the same primes as $\chi_1(1)$ and $\chi_2(1)$, respectively. These degrees show that the primes dividing $\Phi_{\ell-1}$ are adjacent in $\Delta(G)$ to all primes in $\rho(G)$ except possibly to p.

It remains to show that p is adjacent to all primes dividing $\Phi_{\ell-1}$. The character $\chi^{(1,1,\ell-1)}$ in Lemma 2.2 is a unipotent character of degree

$$\chi^{(1,1,\ell-1)}(1) = q^3 \cdot \frac{(q^{\ell-1}-1)(q^\ell-1)}{(q-1)(q^2-1)}.$$

By the results of [2, §12.1], $\mathrm{PSL}_{\ell+1}(q)$ has a unipotent character of the same degree. Because $\ell \geqslant 4$, both p and $\Phi_{\ell-1}$ divide this degree. Hence $\Delta(G)$ is a complete graph for $\ell \geqslant 4$.

Now let $\ell = 3$, so that $G = \mathrm{PSL}_4(q)$. In this case, the primes in $\rho(G)$ are those primes dividing one of q, Φ_1, Φ_2, Φ_3, or Φ_4. The degrees listed in Lemma 2.2 are

$$\chi^{(1,1,\ell-1)}(1) = q^3 \cdot \frac{(q^2-1)(q^3-1)}{(q-1)(q^2-1)} = q^3\Phi_3,$$

$$\chi_1(1) = (q-1)(q^2-1)(q^3-1) = \Phi_1^3\Phi_2\Phi_3,$$
$$\chi_2(1) = (q^2-1)(q^4-1) = \Phi_1^2\Phi_2^2\Phi_4,$$

and

$$\chi_3(1) = q \cdot \frac{(q^2-1)(q^3-1)(q^4-1)}{(q^2-1)(q^2-1)} = q\Phi_1\Phi_3\Phi_4.$$

Note that in this case, $d = (4, q-1) = (4, \Phi_1)$. If q is even, then $d = 1$ and $\mathrm{PSL}_4(q) = \mathrm{PGL}_4(q)$. If q is odd, then $d = 2$ or $d = 4$, but d divides Φ_1 and $\Phi_4 = q^2 + 1$ is even. Hence, in any case, the primes dividing the degree of an irreducible constituent of the restriction of χ_3 to $\mathrm{PSL}_4(q)$ are the same as the primes dividing $\chi_3(1)$. Therefore, all primes in $\rho(G)$ are adjacent in $\Delta(G)$ except possibly those dividing Φ_2.

Irreducible constituents of the restrictions of χ_1 and χ_2 to $\mathrm{PSL}_4(q)$ will have degrees divisible by $\Phi_1^2\Phi_2\Phi_3$ and $\Phi_1\Phi_2^2\Phi_4$, respectively. Hence the primes dividing Φ_2 are adjacent in $\Delta(G)$ to all primes dividing Φ_1, Φ_3, or Φ_4.

If q is even then $p = 2$ and if q is odd then p is adjacent to all primes dividing $\Phi_1 = q - 1$, so in particular p is adjacent to 2. Hence, it remains to determine whether p is adjacent to the odd primes dividing Φ_2.

If $q = p = 2$, then $G = \mathrm{PSL}_4(2) \cong \mathrm{Alt}(8)$ and $\Phi_2 = 3$. By the character table of G in the Atlas [3], we have

$$\mathrm{cd}(G) = \{1, 7, 2 \cdot 7, 2^2 \cdot 5, 3 \cdot 7, 2^2 \cdot 7, 5 \cdot 7, 3^2 \cdot 5, 2^3 \cdot 7, 2^6, 2 \cdot 5 \cdot 7\}.$$

Hence 2 and 3 are not adjacent in $\Delta(G)$, and the graph is as claimed.

If $q = 3$, then $\Phi_2 = 4$ and there are no odd primes dividing Φ_2, so p is adjacent to all primes dividing Φ_2. Hence $\Delta(G)$ is complete if $q = 3$.

If $q > 3$, then by the character table of $\mathrm{GL}_4(q)$ constructed by F. Lübeck for the CHEVIE system [6], $\mathrm{PGL}_4(q)$ has an irreducible character of degree $q\Phi_2\Phi_3\Phi_4$. We have $d = 1$, 2, or 4 and $\mathrm{PSL}_4(q)$ has an irreducible character whose degree is divisible by $(q\Phi_2\Phi_3\Phi_4)/d$, hence divisible by p and the odd primes dividing Φ_2. Therefore, p is adjacent to the odd primes dividing Φ_2, and $\Delta(G)$ is complete when $q > 3$. ∎

3.2. Unitary Groups

In this section G is the projective special unitary group $\mathrm{PSU}_{\ell+1}(q^2)$ of type ${}^2A_\ell(q^2)$, where q is a power of a prime p. Because $\mathrm{PSU}_2(q^2) \cong \mathrm{PSL}_2(q)$, we take $\ell \geqslant 2$. If $\ell = 2$, then we take $q > 2$, as $\mathrm{PSU}_3(2^2)$ is not simple.

Theorem 3.4. *Let $G \cong \mathrm{PSU}_3(q^2)$, where $q > 2$ is a power of a prime p.*

1. *The graph $\Delta(G)$ is complete if and only if q satisfies $q+1 = 2^i3^j$ for some $i \geqslant 0$, $j \geqslant 0$.*

2. *If $q > 2$, then $\rho(G) = \{p\} \cup \pi\big((q-1)(q+1)(q^2-q+1)\big)$. The subgraph of $\Delta(G)$ corresponding to $\pi\big((q-1)(q+1)(q^2-q+1)\big)$ is complete, and p is adjacent to precisely those primes dividing $q - 1$ or $q^2 - q + 1$.*

Proof. Since $q > 2$, the character table of G in either [11] or the CHEVIE system [6] shows that every character degree of G divides one of the degrees in the subset

$$\{q^3, q(q-1), (q-1)(q^2-q+1), q(q^2-q+1), (q+1)(q^2-q+1), (q-1)(q+1)^2\}$$

of $\mathrm{cd}(G)$. Hence all primes dividing $q-1$, $q+1$, or q^2-q+1 are adjacent in $\Delta(G)$ and p is adjacent to precisely those primes dividing $q-1$ or q^2-q+1. Therefore, statement 2 of the theorem follows.

It also follows from the list of degrees above that $\Delta(G)$ is complete if and only if p is adjacent to all primes dividing $q+1$. This holds if and only if every prime dividing $q+1$ also divides either $q-1$ or q^2-q+1.

We have

$$(q+1, q-1) = \begin{cases} 1 & \text{if } q \text{ is even} \\ 2 & \text{if } q \text{ is odd} \end{cases}$$

and

$$(q+1, q^2-q+1) = \begin{cases} 1 & \text{if } q \not\equiv -1 (\mathrm{mod}\ 3) \\ 3 & \text{if } q \equiv -1 (\mathrm{mod}\ 3). \end{cases}$$

Hence $\Delta(G)$ is complete if and only if no primes other than 2 or 3 divide $q+1$, that is, if and only if $q+1 = 2^i 3^j$ for some $i \geqslant 0$, $j \geqslant 0$. Statement 1 of the theorem therefore holds. ∎

Finally, we determine the graph $\Delta(G)$ for the simple groups of type ${}^2A_\ell$ for all $\ell \geqslant 3$.

Theorem 3.5. *If $G \cong \mathrm{PSU}_{\ell+1}(q^2)$, where $\ell \geqslant 3$ and q is a power of a prime p, then the graph $\Delta(G)$ is complete.*

Proof. It is easy to check using the character tables in the Atlas [3] that if (ℓ, q) is in $\{(3,2),(3,3),(4,2)\}$, then the graph $\Delta(G)$ is complete. In fact, in each case, there is an irreducible character whose degree is divisible by all primes in $\rho(G)$. Using Atlas notation for the characters, we have $\rho(\mathrm{PSU}_4(2^2)) = \{2,3,5\}$ and $\chi_{11}(1) = 30 = 2\cdot 3\cdot 5$, $\rho(\mathrm{PSU}_4(3^2)) = \{2,3,5,7\}$ and $\chi_8(1) = 210 = 2\cdot 3\cdot 5\cdot 7$, and $\rho(\mathrm{PSU}_5(2^2)) = \{2,3,5,11\}$ and $\chi_{26}(1) = 330 = 2\cdot 3\cdot 5\cdot 11$. We therefore assume for the remainder of the proof that $(\ell, q) \notin \{(3,2),(3,3),(4,2)\}$.

The order of $G \cong \mathrm{PSU}_{\ell+1}(q^2)$ is given by

$$|G| = \frac{1}{d}\, q^{\ell(\ell+1)/2}(q^2-1)(q^3+1)\cdots(q^\ell-(-1)^\ell)(q^{\ell+1}-(-1)^{\ell+1}),$$

where $d = (\ell+1, q+1) = [\mathrm{PU}_{\ell+1}(q^2) : \mathrm{PSU}_{\ell+1}(q^2)]$. Therefore, $\rho(G)$ is precisely the set of primes dividing q or some $q^k-(-1)^k$ for $2 \leqslant k \leqslant \ell+1$.

The character degrees found in Lemma 2.3 are degrees of $\mathrm{PU}_{\ell+1}(q^2)$. As in the linear case, if χ is an irreducible character of $\mathrm{PU}_{\ell+1}(q^2)$ and μ is an irreducible constituent of the restriction of χ to $\mathrm{PSU}_{\ell+1}(q^2)$, then $\chi(1)/\mu(1)$ divides d and hence divides $q+1$ as well. In particular, if $q+1$ divides $\chi(1)$, then $\chi(1)/(q+1)$ divides $\mu(1)$.

First, let $\ell \geqslant 4$. By Lemma 2.3, $\mathrm{PU}_{\ell+1}(q^2)$ has an irreducible character χ_3 of degree

$$\chi_3(1) = q\cdot\frac{(q^2-1)(q^3+1)\cdots(q^\ell-(-1)^\ell)(q^{\ell+1}-(-1)^{\ell+1})}{(q^2-1)(q^{\ell-1}-(-1)^{\ell-1})}.$$

Since $\ell \geqslant 4$, $\chi_3(1)$ is divisible by at least $q^\ell - (-1)^\ell$ and $q^{\ell+1} - (-1)^{\ell+1}$, one of which is divisible by $q^2 - 1$ and the other by $q + 1$. Hence $(q+1)^2$ and $q - 1$ both divide $\chi_3(1)$, despite the division by $q^2 - 1$ in the degree formula. It follows that $\mathrm{PSU}_{\ell+1}(q^2)$ has an irreducible character whose degree is divisible by p and all $q^k - (-1)^k$ with $2 \leqslant k \leqslant \ell + 1$ except $q^{\ell-1} - (-1)^{\ell-1}$. Thus all primes in $\rho(G)$ except those dividing only $q^{\ell-1} - (-1)^{\ell-1}$ are adjacent in $\Delta(G)$.

By Lemma 2.3, $\mathrm{PU}_{\ell+1}(q^2)$ has irreducible characters of degrees

$$\chi_1(1) = (q+1)(q^2-1)(q^3+1)\cdots(q^{\ell-1} - (-1)^{\ell-1})(q^\ell - (-1)^\ell)$$

and

$$\chi_2(1) = (q^2-1)(q^3+1)\cdots(q^{\ell-1} - (-1)^{\ell-1})(q^{\ell+1} - (-1)^{\ell+1}).$$

Since $\ell \geqslant 4$, both degrees are divisible by $(q+1)^2$, hence $\mathrm{PSU}_{\ell+1}(q^2)$ has irreducible characters μ_1 and μ_2 whose degrees are divisible by the same primes as $\chi_1(1)$ and $\chi_2(1)$, respectively. These degrees show that the primes dividing $q^{\ell-1} - (-1)^{\ell-1}$ are adjacent in $\Delta(G)$ to all primes in $\rho(G)$ except possibly to p.

It remains to show that p is adjacent to all primes dividing $q^{\ell-1} - (-1)^{\ell-1}$. The character $\chi^{(1,1,\ell-1)}$ in Lemma 2.3 is a unipotent character of degree

$$\chi^{(1,1,\ell-1)}(1) = q^3 \cdot \frac{(q^{\ell-1} - (-1)^{\ell-1})(q^\ell - (-1)^\ell)}{(q+1)(q^2-1)}.$$

As in the linear case, $\mathrm{PSU}_{\ell+1}(q^2)$ has a unipotent character of the same degree. Since $\ell \geqslant 4$, we already know that p is adjacent to all primes dividing $q^2 - 1$. Also because $\ell \geqslant 4$, both p and all other primes dividing $q^{\ell-1} - (-1)^{\ell-1}$ divide this degree. Hence $\Delta(G)$ is a complete graph for $\ell \geqslant 4$.

Now let $\ell = 3$, so that $G = \mathrm{PSU}_4(q^2)$, and recall that in this case we assume $q > 3$. We have $d = (4, q+1)$ and

$$|G| = \frac{1}{d}\, q^6(q^2-1)(q^3+1)(q^4-1) = \frac{1}{d}\, q^6 \Phi_1^2 \Phi_2^3 \Phi_4 \Phi_6,$$

where $\Phi_k = \Phi_k(q)$ denotes the value of the kth cyclotomic polynomial evaluated at q. The primes in $\rho(G)$ are those primes dividing one of q, Φ_1, Φ_2, Φ_4, or Φ_6. The degrees listed in Lemma 2.3 are

$$\chi^{(1,1,\ell-1)}(1) = q^3 \cdot \frac{(q^2-1)(q^3+1)}{(q+1)(q^2-1)} = q^3 \Phi_6,$$

$$\chi_1(1) = (q+1)(q^2-1)(q^3+1) = \Phi_1 \Phi_2^3 \Phi_6,$$
$$\chi_2(1) = (q^2-1)(q^4-1) = \Phi_1^2 \Phi_2^2 \Phi_4,$$

and

$$\chi_3(1) = q \cdot \frac{(q^2-1)(q^3+1)(q^4-1)}{(q^2-1)(q^2-1)} = q\Phi_2\Phi_4\Phi_6.$$

We have $d = (4, q+1) = (4, \Phi_2)$, and Φ_2^2 divides both $\chi_1(1)$ and $\chi_2(1)$. For χ_3, note that if q is odd, then $d = 2$ or $d = 4$, but d divides Φ_2 and $\Phi_4 = q^2 + 1$ is even. If q is even, then $d = 1$ and $\mathrm{PSU}_4(q^2) = \mathrm{PU}_4(q^2)$. Hence, in any case, the primes dividing the degree of an irreducible constituent of the restriction of χ_1, χ_2, or χ_3 to $\mathrm{PSU}_4(q^2)$ are the same as the primes dividing $\chi_1(1)$, $\chi_2(1)$, or $\chi_3(1)$, respectively.

Therefore, using $\chi_3(1)$, we see that all primes in $\rho(G)$ are adjacent in $\Delta(G)$ except possibly those dividing Φ_1. The degrees $\chi_1(1)$ and $\chi_2(1)$ show that the primes dividing Φ_1 are adjacent in $\Delta(G)$ to all primes dividing Φ_2, Φ_4, or Φ_6. It remains to show that p is adjacent to the primes dividing Φ_1.

By the character table of $\mathrm{GU}_4(q^2)$ constructed by F. Lübeck for the CHEVIE system [6], since $q > 3$, $\mathrm{PU}_4(q^2)$ has an irreducible character of degree $q\Phi_1^2\Phi_4$. Hence $\mathrm{PSU}_4(q^2)$ has an irreducible character whose degree is divisible by $(q\Phi_1^2\Phi_4)/d$. If q is even, then $d = 1$. If q is odd, then $d = 2$ or $d = 4$, and $q\Phi_1^2\Phi_4$ is divisible by 8. Therefore, this degree is divisible by p and all primes dividing Φ_1 in any case. Hence p is adjacent to all primes dividing Φ_1, and $\Delta(G)$ is complete when $\ell = 3$. ∎

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