TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, December 1969

DEGREE OF SYMMETRY OF A PRODUCT MANIFOLD

BY
H. T. KU, L. N. MANN, J. L. SICKS AND J. C. SU[1]

0. **Introduction.** Suppose M^m is a compact connected differentiable m-manifold. Following [7], we define $N(M)$, *the degree of symmetry* of M, as the maximum of the dimensions of the isometry groups of all possible Riemannian structures on M. Of course, $N(M)$ is the maximum of the dimensions of the compact Lie groups which can act effectively and differentiably on M. It is well known that

$$N(M^m) \leqq m(m+1)/2,$$

and that if $N(M^m)=m(m+1)/2$, then M is diffeomorphic to the standard sphere S^m or the real projective space RP^m [5]. In this paper we generalize this result by considering a product manifold

$$M^m = M_1^{m_1} \times M_2^{m_2} \qquad (m \geqq 19),$$

where M_i is a compact connected differentiable manifold of dimension m_i. We show that

$$N(M) \leqq m_1(m_1+1)/2+m_2(m_2+1)/2,$$

and that if

$$N(M) = m_1(m_1+1)/2+m_2(m_2+1)/2,$$

then M is a product of two spheres, two real projective spaces or one of each.

We establish the above result by first showing that if M^m ($m \geqq 19$) is a compact connected differentiable m-manifold, then, except when M is diffeomorphic to the complex projective space CP^k ($m=2k$),

$$N(M^m) \leqq \alpha(\alpha+1)/2+(m-\alpha)(m-\alpha+1)/2,$$

for all α such that $H^\alpha(M; Q)\neq 0$ ($Q=$rationals). This bound on $N(M^m)$ is of course sharper than $m(m+1)/2$, especially if α can be chosen near $[m/2]$. It follows as an easy consequence of our results that

$$N(CP^k) = k^2+2k,$$

at least for $k \geqq 10$.

The authors are particularly indebted to Professor W. Y. Hsiang for several significant suggestions.

Received by the editors March 7, 1969.

(1) The second and fourth authors were supported in part by NSF GP-8929; the third author by NSF GP-8747.

1. **Preliminaries.** Consider a transformation group K acting on a space X. The subset K_0 of all elements of K which act as the identity on X form a normal subgroup of K, and K/K_0, in a natural fashion, acts effectively on X. The action of K on X is said to be *almost effective* if K_0 is finite. An almost effective action is said to be *almost free* if K/K_0 acts freely on X. The following lemma may be found in [11].

LEMMA 1. *Let $K=K_1\times K_2$ act almost effectively on X. If K_1 acts transitively on X, then K_2 acts almost freely on X.*

Following Jänich [9] we let $m(H)$, for a compact connected Lie group H, denote the minimal dimension of the connected manifolds upon which H acts almost effectively, e.g.,

$$m(\mathrm{SO}\,(n)) = n-1.$$

The values of $m(H)$ for all compact simply-connected simple Lie groups H are listed in [9, p. 68].

If a compact Lie group G acts differentiably on a manifold M, it is known that the orbit space M/G is a finite complex [15]. Furthermore it is well known that the dimension of M/G is equal to the dimension of M minus the dimension of a *principal orbit*.

A compact connected Lie group G can be expressed in the form:

$$\text{(A)} \qquad G = (T^q\times S_1\times S_2\times\cdots\times S_u)/N = \bar{G}/N,$$

where T^q is a q-torus, $q\geqq 0$, each S_j is a compact, connected, simply-connected simple Lie group and N is a finite normal subgroup of $\bar{G}$. Each S_j of dimension 3 is isomorphic to Spin (3). We employ the isomorphism

$$\mathrm{Spin}\,(4) \cong \mathrm{Spin}\,(3)\times\mathrm{Spin}\,(3)$$

to combine pairs of 3-dimensional S_j's. With this convention, we may rewrite G in the form

$$\text{(B)} \qquad G = (T^q\times S_1'\times S_2'\times\cdots\times S_v')/N = \bar{G}/N,$$

where each S_j' is either simple or isomorphic to Spin (4) and where there is at most one S_j' of dimension 3. We shall need later the following result [11, Theorem 1], [9, p. 68].

For a nonnegative integer n, let

$$\langle n\rangle = n(n+1)/2.$$

PROPOSITION A. *Let G be a compact connected Lie group acting effectively on a connected m-dimensional manifold M. If G is of the form* (B), *then there exist integers $t_1, t_2, \ldots, t_v$ such that*

$$\dim S_j' \leqq \langle t_j\rangle, \qquad j = 1, 2, \ldots, v,$$

and

$$\sum_{j=1}^{v} t_j \leqq m-q.$$

The following lemma is obvious.

LEMMA 2. *If* $n_1 \geqq n_2 \geqq w \geqq 0$,

(a) $\langle n_1\rangle + \langle n_2\rangle \leqq \langle n_1+n_2\rangle$,

(b) $\langle n_1\rangle + \langle n_2\rangle \leqq \langle n_1+w\rangle + \langle n_2-w\rangle$.

The next proposition is essentially due to W. Y. Hsiang [8] and related to Proposition A. The version presented here is easily derivable from [9, p. 74].

PROPOSITION B. *Suppose G acts effectively on a homogeneous space G/H and*

$$\dim G > r(\dim (G/H)), \qquad r \geqq 13/4.$$

Then if G is of the form (B), *there is at least one normal factor, say S_1', which is isomorphic to one of the following groups*:

(i) Spin (n), $n > 2r$,

(ii) SU (n), $n > 2r-1$,

(iii) Sp (n), $n > 2r-2$.

REMARK. The restriction that $r \geqq 13/4$ in Proposition B guarantees that the dominant normal factor S_1' is a classical Lie group. If we remove this restriction, we obtain an expanded version of Proposition B which has *exceptional* simple Lie groups as possibilities for S_1'.

2. The main lemma.

MAIN LEMMA. *Let G be a compact connected Lie group acting differentiably and effectively on a compact connected m-manifold M, $m \geqq 19$. Then if*

$$\dim G \geqq m^2/4 + m/2,$$

exactly one of the following holds:

(α) *M is diffeomorphic to CP^k $(m=2k)$, G acts transitively on M and G is locally isomorphic to* SU $(k+1)$.

(β) *M is diffeomorphic to $CP^k \times S^1$ $(m=2k+1)$, G acts transitively on M and G is locally isomorphic to* U $(k+1)$.

(γ) *M is a simple lens space finitely covered by S^{2k+1} $(m=2k+1)$, G acts transitively on M and G is locally isomorphic to* U $(k+1)$.

(δ) *G contains a normal factor $S_1' \cong$* Spin (n) (*see* §1 *for terminology*) *where*

(a) $n \geqq m/2+1$,

(b) *S_1' acts almost effectively on M with principal isotropy subgroup H whose identity component H^0 is a standardly imbedded* Spin $(n-1)$ *in* Spin (n).

Proof. Let $G(x)$ be a *principal orbit* for the action of G on M. Then G acts effectively on the homogeneous space $G(x) = G/G_x$, where G_x is the *principal isotropy subgroup* of G at x. Now

$$\dim G \geqq \frac{m^2}{4}+\frac{m}{2} = \left(\frac{m}{4}+\frac{1}{2}\right)m \geqq \left(\frac{m}{4}+\frac{1}{2}\right)\left(\dim \frac{G}{G_x}\right).$$

Let

$$(1) \qquad r = m/4+1/2-\varepsilon,$$

where ε is a small positive quantity. Then

$$(2) \qquad \dim G > r \dim (G/G_x).$$

[Note that r can be chosen equal to $m/4+1/2$ if $\dim G(x)<m$.] Since $m \geqq 19$, it follows from (1) that $r>13/4$. Suppose now that G is decomposed into the form (B). Then we may use (2) to apply Proposition B and obtain a normal factor group S_1' of $\bar{G}$ as described in the proposition. Since $\bar{G}$ acts almost effectively on M, so does S_1'. We consider the possibilities for S_1'.

Case I. $S_1' \cong \mathrm{Sp}\,(n)$.

By Proposition B and (1),

$$n > 2r-2 = m/2-1-2\varepsilon.$$

Hence

$$(3) \qquad n \geqq m/2-1.$$

On the other hand,

$$4n-4 = m(\mathrm{Sp}\,(n)) \leqq m$$

or

$$(4) \qquad 4n \leqq m+4.$$

It follows from (3) and (4) that

$$2m-4 \leqq 4n \leqq m+4$$

or $m \leqq 8$. Hence Case I is eliminated.

Case II. $S_1' \cong \mathrm{SU}\,(n)$.

By Proposition B,

$$n > 2r-1 = m/2-2\varepsilon.$$

Hence

$$(5) \qquad n \geqq m/2.$$

Let H denote a principal isotropy subgroup for the almost effective action of S_1' on M. We consider various cases for $\dim H$.

Subcase (a). $\dim H \leqq [\dim \mathrm{SU}\,(n-1)]-2$. Now,

$$2n+1 \leqq \dim (S_1'/H) \leqq m.$$

Hence

$$(6) \qquad 2n \leqq m-1.$$

We obtain an immediate contradiction from (5) and (6). Hence

$$\dim H \geqq [\dim \mathrm{SU}\,(n-1)]-1.$$

On the other hand if $n \geqq 8$, which will always be the case from (5) if $m \geqq 15$, it follows automatically that

$$\dim H \geqq \dim \mathrm{SU}(n-1).$$

(See, for example, [6, Theorem 1.19].)

Subcase (b). $\dim H = \dim \mathrm{SU}(n-1)$.

First let us show that S_1' is transitive on M. If not,

$$2n-1 = \dim (S_1'/H) \leqq m-1,$$

or $2n \leqq m$. It follows from (5) that $2n = m$. Suppose $\bar{G} = S_1' \times K$, where

$$K = S_2' \times \cdots \times S_v' \times T^q,$$

in the decomposition (B). We shall show that $\dim K \leqq 3$, from which it follows that

$$\dim G = \dim \bar{G} = \dim \mathrm{SU}(n) + \dim K \leqq n^2+2.$$

However since $m = 2n$,

$$\dim G \leqq n^2+2 < n^2+n = m^2/4+m/2,$$

which contradicts our original assumption on the size of $\dim G$.

We proceed to show $\dim K \leqq 3$. Now $\bar{G} = S_1' \times K$ acts transitively and almost effectively on a principal orbit $M_1 = \bar{G}(x)$. Then S_1' acts transitively on the orbit space M_1/K which must be a compact connected manifold. If $\dim K \geqq 4$, then K acts on M_1 with principal orbits of dimension at least 3 and

$$\dim (M_1/K) \leqq m-3 = 2n-3.$$

Since $S_1' \cong \mathrm{SU}(n)$ is simple, S_1' acts either almost effectively or trivially on M_1/K. However, since

$$m(\mathrm{SU}(n)) = 2n-2,$$

the action must be trivial. Hence M_1/K is a point and K acts transitively on M_1. By Lemma 1, $S_1' \cong \mathrm{SU}(n)$ acts almost freely on M_1 and

$$n^2-1 = \dim \mathrm{SU}(n) \leqq m = 2n,$$

which is an obvious contradiction. Hence $S_1' \cong \mathrm{SU}(n)$ must act transitively on M.

Since

$$\dim H = \dim \mathrm{SU}(n-1),$$

it follows, at least for $n \geqq 8$, that H^0, the identity component of H, is a standardly imbedded $\mathrm{SU}(n-1)$ in $\mathrm{SU}(n)$ [6, Theorem 1.19]. We have the covering

$$\begin{array}{l} \tilde{M} = S_1'/H^0 = \mathrm{SU}(n)/\mathrm{SU}(n-1) = S^{2n-1} \\ \downarrow \\ M = S_1'/H \end{array}$$

and, therefore M is finitely covered by S^{2n-1}. Moreover the group of covering transformations H/H^0 is a subgroup of

$$\frac{N(\mathrm{SU}\,(n-1), \mathrm{SU}\,(n))}{\mathrm{SU}\,(n-1)} \cong S^1,$$

where $N(\mathrm{SU}\,(n-1), \mathrm{SU}\,(n))$ denotes the normalizer of $\mathrm{SU}\,(n-1)$ in $\mathrm{SU}\,(n)$. Hence H/H^0 is a finite *cyclic* group and the action of H/H^0 on $\tilde{M}=S^{2n-1}$ is the restriction of the standard free action of

$$\frac{N(\mathrm{SU}\,(n-1), \mathrm{SU}\,(n))}{\mathrm{SU}\,(n-1)} \cong S^1$$

on S^{2n-1}. It follows that M is a simple lens space. To show that we have possibility (γ) of the Main Lemma we must now verify that G is locally isomorphic to $\mathrm{U}(n)$. Recall that $\bar{G}=S_1'\times K$. Now $S_1'\cong \mathrm{SU}\,(n)$ acts transitively on M and by Lemma 1, K acts almost freely on M. Furthermore, S_1' acts transitively on the compact connected manifold $M_1=M/K$. This action must also be nontrivial, and therefore almost effective, for otherwise K would act transitively on M and by Lemma 1 again, S_1' would act almost freely on M which is impossible due to dimensional considerations. Now

$$\dim M_1 = \dim M - \dim K = 2n-1-\dim K,$$

and since

$$m(\mathrm{SU}\,(n)) = 2n-2,$$

it follows that $\dim K \leqq 1$. On the other hand,

$$\dim \mathrm{SU}\,(n) = n^2-1 = m^2/4+m/2-3/4,$$

since $m=2n-1$. It follows due to our dimensional restriction on G that $\dim K=1$, and K is a circle group. Since $\bar{G}=\mathrm{SU}\,(n)\times S^1$, G is locally isomorphic to $\mathrm{U}\,(n)$ and we have possibility (γ) of the Main Lemma.

Since $m(\mathrm{SU}\,(n))=2n-2$, if $\dim H>\dim \mathrm{SU}\,(n-1)$, then we must have

$$\dim H = \dim \mathrm{U}\,(n-1).$$

Subcase (c). $\dim H=\dim \mathrm{U}\,(n-1)$.

First we show that S_1' must act either transitively or with an $(m-1)$-dimensional orbit on M. If not,

$$2n-2 = \dim (S_1'/H) \leqq m-2,$$

or $2n\leqq m$. It follows from (5) that $2n=m$, and we can proceed exactly as in Subcase (b) to obtain a contradiction.

It follows, at least for $n\geqq 8$, that H^0, the identity component of H, is a standardly imbedded U for consistency $(n-1)$ in $\mathrm{SU}\,(n)$. Since the normalizer of $\mathrm{U}\,(n-1)$ in $\mathrm{SU}\,(n)$ is $\mathrm{U}\,(n-1)$ itself, $H=H^0$. Suppose first that S_1' is transitive on M. Then

$$M = \frac{S_1'}{H} = \frac{\mathrm{SU}\,(n)}{\mathrm{U}\,(n-1)} = CP^{n-1}.$$

As usual let $\bar{G}=S_1'\times K$. Since $S_1'\cong \mathrm{SU}(n)$ acts transitively on M, K acts almost freely on M. Using our standard argument we may show dim $K=0$, and, hence, that G is locally isomorphic to SU (n). This gives us possibility (α) of the Main Lemma. By the way, in this case, since the Euler characteristic of $M=CP^{n-1}$ is positive, it follows directly that rank $K=0$.

We are left with the case that S_1' acts on M with an $(m-1)$-dimensional orbit. It follows that

(i) $M/S_1'=S^1$ or I (closed interval),

(ii) All orbits are principal and diffeomorphic to CP^{n-1}.

(i) is a well-known result due to P. S. Mostert [13] and G. E. Bredon [3]. (ii) is reasoned out as follows. Since we have already agreed that the normalizer of U $(n-1)$ in SU (n) is U $(n-1)$ itself, all orbits of maximal dimension are principal and diffeomorphic to CP^{n-1}. Moreover since $H=\mathrm{U}(n-1)$, the only other possible orbits are fixed points. However, if the action has fixed points, then the orbit space is a closed interval with the two end points corresponding to the two exceptional orbits, each of which are fixed points. It follows that M is homeomorphic to the suspension of CP^{n-1} and therefore not a manifold since $n>2$.

Since all orbits are principal, M is a fibre bundle over M/S_1' with structural group

$$\frac{N[\mathrm{U}(n-1), \mathrm{SU}(n)]}{\mathrm{U}(n-1)},$$

which is trivial. Hence $M=CP^{n-1}\times M/S_1'$. Clearly $M/S_1'=I$ is impossible. It follows that

$$M = CP^{n-1}\times S^1.$$

Let $\bar{G}=S_1'\times K$ and suppose that $M_1=\bar{G}(x)$ is a principal orbit of the action of $\bar{G}$ on M. Since

$$\dim M_1 \leqq \dim M = 2n-1,$$

and $S_1'\cong \mathrm{SU}(n)$ acts almost effectively on $M_2=M_1/K$, it follows using our standard techniques that dim $K\leqq 1$. On the other hand, due to our dimensional restriction on G, dim $K=1$, and G is locally isomorphic to U (n). This gives us possibility (β) of the Main Lemma. (Clearly $\bar{G}=\mathrm{SU}(n)\times S^1$ acts transitively on M.)

Case III. $S_1'\cong \mathrm{Spin}(n)$.

By Proposition B,

$$n > 2r = m/2+1-2\varepsilon.$$

Hence

$$n \geqq m/2+1. \tag{7}$$

Let H denote a principal isotropy subgroup of the almost effective action of S_1' on M. Suppose first that

$$\dim H \leqq \dim [\mathrm{SO}(n-2)\times \mathrm{SO}(2)].$$

Now

$$2n-4 \leqq \dim (S_1'/H) \leqq m,$$

or

$$(8) \qquad 2n \leqq m+4.$$

Let $\bar{G}=S_1'\times K$. We shall show that $\dim K \leqq m+6$, from which it will follow that

$$\dim G = \dim \bar{G} = \dim \text{Spin}\,(n)+\dim K \leqq n(n-1)/2+m+6.$$

However, since from (8)

$$n \leqq (m+4)/2,$$

$$\dim G \leqq \frac{\left(\frac{m+4}{2}\right)^2-\left(\frac{m+4}{2}\right)}{2}+m+6 = \frac{m^2}{8}+\frac{7m}{4}+7 < \frac{m^2}{4}+\frac{m}{2},$$

for $m \geqq 12$. This, of course, is a contradiction.

We proceed to show $\dim K \leqq m+6$. Now $\bar{G}=S_1'\times K$ acts transitively and almost effectively on a principal orbit $M_1=\bar{G}(x)$. Hence K acts transitively on the orbit space M_1/S_1' which must be a compact connected manifold. Let $M_2=M_1/S_1'$. Now S_1' must act on M with principal orbits of dimension at least $m-2$. For otherwise

$$2n-4 \leqq \dim (S_1'/H) \leqq m-3,$$

which contradicts (7). Hence

$$\dim M_2 = \dim (M_1/S_1') \leqq \dim (M/S_1') \leqq 2.$$

Now either M_2 is a point or K acts nontrivially on M_2 since

$$M_2/K = M_1/\bar{G}$$

is a point. If M_2 is a point, S_1' is transitive on M_1 and by Lemma 1, K acts almost freely on M_1. Therefore $\dim K \leqq \dim M_1 \leqq m$. Suppose then that K acts nontrivially on M_2. Now

$$K = S_2'\times S_3'\times \cdots \times S_v'\times T^q$$

and $1 \leqq \dim M_2 \leqq 2$. Now either T^q or S_j', $2 \leqq j \leqq v$, acts nontrivially on M_2. Suppose, say, S_2' acts nontrivially on M_2. Then S_2' acts transitively on M_2 and $\dim S_2' \leqq 6$. [Note that conceivably $S_2' \cong \text{Spin}\,(4)$.] Let

$$K_1 = S_3'\times \cdots \times S_v'\times T^q.$$

Now $\bar{G}=S_1'\times S_2'\times K_1$, and $S_1'\times S_2'$ acts transitively on M_1. Hence by Lemma 1, K_1 acts almost freely on M_1 and $\dim K_1 \leqq \dim M_1 \leqq m$. It follows that

$$\dim K = \dim S_2'+\dim K_1 \leqq m+6.$$

If each S'_j, $2 \leqq j \leqq v$, acts trivially on M_2, then T^q acts transitively and nontrivially on M_2. Therefore T^q contains a subgroup T^2 which acts transitively on M_2. Let

$$K_2 = S'_2 \times S'_3 \times \cdots \times S'_v \times T^{q-2}$$

where $T^{q-2} = T^q/T^2$. Now $\bar{G} = S'_1 \times T^2 \times K_2$ and $S'_1 \times T^2$ acts transitively on M_1. Hence by Lemma 1 again, K_2 acts almost freely on M_1 and

$$\dim K = \dim T^2 + \dim K_2 \leqq 2 + m.$$

(Actually the above argument applies to the case where $\dim M_2 = 2$; for $\dim M_2 = 1$, an obvious alteration is required.) It follows that

$$\dim H > \dim [\mathrm{SO}(n-2) \times \mathrm{SO}(2)].$$

Hence if we assume $n \geqq 11$, which from (7) will always be the case for $m \geqq 19$, then $\dim H = \dim \mathrm{SO}(n-1)$ and H^0 is a standardly imbedded Spin $(n-1)$ in Spin (n) [6, Theorem 1.18]. Thus we have the last possibility (δ) of the Main Lemma.

3. Main results.

THEOREM 1. *Let M^m be a compact connected differentiable m-manifold* ($m \geqq 19$). *Then precisely one of the following holds*:

(i) *$N(M^m) \leqq \langle \alpha \rangle + \langle m-\alpha \rangle$ for all α such that $H^\alpha(M; Q) \neq 0$.*

(ii) *M is diffeomorphic to CP^k ($m = 2k$), and $N(M) = \dim \mathrm{SU}(k+1)$.*

Proof. Suppose first that M is diffeomorphic to CP^k. Now

$$H^k(CP^k; Q) \neq 0, \qquad k \text{ even};$$
$$H^{k+1}(CP^k; Q) \neq 0, \qquad k \text{ odd}.$$

On the other hand SU $(k+1)$ acts almost effectively on CP^k. Consequently

$$N(CP^k) \geqq \dim \mathrm{SU}(k+1) = k^2 + 2k > \langle k \rangle + \langle k \rangle, \qquad k \text{ even};$$
$$> \langle k+1 \rangle + \langle k-1 \rangle, \qquad k \text{ odd}.$$

We proceed to show that CP^k is the only manifold which violates (i). Suppose $N(M^m) > \langle \alpha \rangle + \langle m-\alpha \rangle$, for some α where $H^\alpha(M; Q) \neq 0$. Then there exists a compact connected Lie group G acting effectively and differentiably on M with

$$N(M^m) = \dim G > \langle \alpha \rangle + \langle m-\alpha \rangle.$$

Now

$$\langle \alpha \rangle + \langle m-\alpha \rangle = \frac{\alpha^2 + (m-\alpha)^2}{2} + \frac{m}{2} \geqq \frac{m^2}{4} + \frac{m}{2}.$$

Hence $\dim G > m^2/4 + m/2$, and we must be in one of the four possibilities of the Main Lemma. If possibility (α) occurs, M is diffeomorphic to CP^k and G is locally isomorphic to SU $(k+1)$. Hence $N(M) = \dim G = \dim \mathrm{SU}(k+1)$, and we have statement (ii).

We show that the remaining possibilities of the Main Lemma cannot occur.

In the case of possibility (β), we have

$$\dim G = \dim \mathrm{U}(k+1) = (k+1)^2.$$

On the other hand,

$$\langle\alpha\rangle+\langle m-\alpha\rangle \geqq \langle k+1\rangle+\langle k\rangle = (k+1)^2.$$

Hence $\dim G \leqq \langle\alpha\rangle+\langle m-\alpha\rangle$.

If M is finitely covered by S^m (possibility (γ)), then M is a rational cohomology sphere. (See, for example, [1, p. 38].) Hence $\alpha=0$ or m and $\langle\alpha\rangle+\langle m-\alpha\rangle=\langle m\rangle$. However, $N(M^m)\leqq\langle m\rangle$ as is the case for every m-dimensional manifold. We are left with possibility (δ) of the Main Lemma. Now

$$\bar{G} = S_1'\times\cdots\times S_v'\times T^q$$

acts almost effectively on M with

$$S_1' \cong \mathrm{Spin}(n), \qquad n \geqq m/2+1.$$

Hence

$$\dim S_1' = \dim \mathrm{Spin}(n) = \langle n-1\rangle \geqq \langle[(m+1)/2]\rangle$$

and by Proposition A,

$$\dim S_1' \geqq \dim S_j', \qquad 2 \leqq j \leqq v.$$

Let $\beta=\max(\alpha, m-\alpha)$. We claim it is sufficient to show that $n\leqq\beta+1$. For if

$$\dim S_1' = \dim \mathrm{Spin}(n) = \langle n-1\rangle \leqq \langle\beta\rangle,$$

we know that

$$\beta \geqq t_1 \geqq t_j, \qquad 2 \leqq j \leqq v$$

from Proposition A. (Note we assume that the t_j's are chosen minimally.) Let

$$t_1 = \beta-u, \qquad u \geqq 0.$$

We know that

$$\dim G = \dim \bar{G} \leqq \langle\beta-u\rangle+\sum_{j=2}^{v}\langle t_j\rangle+q \tag{9}$$

where

$$\sum_{j=2}^{v} t_j \leqq m-q-(\beta-u) = m-\beta+u-q. \tag{10}$$

We consider two cases.

(i) $\sum_{j=2}^{v} t_j+q\leqq u$. Then by Lemma 2(a)

$$\sum_{j=2}^{v}\langle t_j\rangle+q \leqq \left\langle \sum_{j=2}^{v} t_j+q \right\rangle \leqq \langle u\rangle,$$

and from (9) it follows that

$$\dim G = \dim \bar{G} \leqq \langle\beta-u\rangle+\langle u\rangle \leqq \langle\beta\rangle \leqq \langle\alpha\rangle+\langle m-\alpha\rangle.$$

(ii) $\sum_{j=2}^{v} t_j+q>u$. By repeated use of Lemma 2(b), since $\beta-u=t_1\geqq t_j$ for all j,

$$\langle\beta-u\rangle+\sum_{j=2}^{v}\langle t_j\rangle+q \leqq \langle\beta\rangle+\sum_{j=2}^{v}\langle \bar{t}_j\rangle+\tilde{q}$$

where

(a) $0\leqq \bar{t}_j\leqq t_j$,

(b) $0\leqq\tilde{q}\leqq q$,

(c) $\sum_{j=2}^{v}\bar{t}_j+\tilde{q}=\sum_{j=2}^{v}t_j+q-u$.

From (9) it follows that

$$\begin{aligned}\dim G = \dim \bar{G} &\leqq \langle\beta\rangle+\sum_{j=2}^{v}\langle\bar{t}_j\rangle+\tilde{q}\\ &\leqq \langle\beta\rangle+\left\langle\sum_{j=2}^{v}\bar{t}_j+\tilde{q}\right\rangle\\ &\leqq \langle\beta\rangle+\left\langle\sum_{j=2}^{v}t_j+q-u\right\rangle\\ &\leqq \langle\beta\rangle+\langle m-\beta\rangle \qquad \text{(from (10))}\\ &\leqq \langle\alpha\rangle+\langle m-\alpha\rangle.\end{aligned}$$

It remains to show that $n\leqq\beta+1$. Suppose, on the contrary, that

$$(11) \qquad n \geqq \beta+2.$$

We know from the Main Lemma that $S_1'\cong\text{Spin}\,(n)$ acts almost effectively on M with principal isotropy subgroup H whose identity component H^0 is a standardly imbedded Spin $(n-1)$. It follows that the only possible orbits of the action of S_1' on M are S^{n-1}, RP^{n-1} and fixed points, all of which are acyclic over Q up to and including dimension $n-2$.

Let X denote the orbit space M/S_1'. Then

$$(12) \qquad \dim X = m-(n-1) \leqq m-\beta-1 < \alpha$$

from (11). Consider the projection π: $M\to X$. Since $\pi^{-1}(x)$ is acyclic over Q up to dimension $n-2$ for each x, it follows from the Vietoris-Begle Mapping Theorem [14] that

$$H^i(M;Q)\cong H^i(X;Q), \qquad i\leqq n-2.$$

However, since $\alpha\leqq\beta\leqq n-2$, it follows that $H^\alpha(X;Q)\neq 0$ which contradicts (12). Hence $n\leqq\beta+1$ and we are finished.

THEOREM 2. *Suppose*

$$M^m = M_1^{m_1}\times M_2^{m_2}, \qquad (m_1\geqq m_2\geqq 1;\, m\geqq 19)$$

where $M_i^{m_i}$, $i=1,2$, *is a compact connected differentiable manifold of dimension* m_i. *Then* $N(M^m)\leqq\langle m_1\rangle+\langle m_2\rangle$. *Moreover if* $N(M^m)=\langle m_1\rangle+\langle m_2\rangle$, *then* M *is diffeomorphic to one of the following*:

(i) $S^{m_1}\times S^{m_2}$,

(ii) $RP^{m_1}\times RP^{m_2}$,

(iii) $S^{m_1}\times RP^{m_2}$,

(iv) $RP^{m_1}\times S^{m_2}$.

Proof. For the first part of the theorem we may as well suppose that M_1 and M_2 are each orientable. If not, let

$$\tilde{M}^m = \tilde{M}_1^{m_1} \times \tilde{M}_2^{m_2},$$

where $\tilde{M}_i$, $i=1,2$ is either the orientable double covering of M_i or M_i itself depending upon whether or not M_i is orientable. Now $\tilde{M}$ is either a two or four fold covering of M and $N(\tilde{M}) \geqq N(M)$, since every group G acting almost effectively on M can be lifted to a covering group $\tilde{G}$ acting almost effectively on $\tilde{M}$. (See, for example, [10, Lemma 2].)

Now $H^{m_1}(M; Q) \neq 0$, and since CP^k does not split as the product of two manifolds, as is immediately obvious from its cohomology ring structure, it follows from Theorem 1 that $N(M^m) \leqq \langle m_1 \rangle + \langle m_2 \rangle$.

Suppose now that $N(M^m) = \langle m_1 \rangle + \langle m_2 \rangle$, and let G be a compact connected Lie group of dimension $\langle m_1 \rangle + \langle m_2 \rangle$ acting effectively on M. If we refer to the Main Lemma, possibility (α) is out since CP^k does not split. Possibility (γ) is out for a similar reason. In possibility (β) if

$$CP^k \times S^1 = M^{m_1} \times M^{m_2},$$

then $m_1 = 2k$ and $m_2 = 1$ due to the cohomology ring structure of $CP^k \times S^1$. However,

$$\dim G = \dim \mathrm{U}(k+1) < \langle 2k \rangle + \langle 1 \rangle = \langle m_1 \rangle + \langle m_2 \rangle.$$

Consequently we are dealing with possibility (δ) of the Main Lemma.

Let $\bar{G} = S_1' \times K$. We know $S_1' \cong \mathrm{Spin}(n)$, where $n \geqq m/2 + 1$. Following the proof of Theorem 1 we can show that $n \leqq m_1 + 1$, lifting the action if necessary to the orientable covering $\tilde{M} = \tilde{M}_1^{m_1} \times \tilde{M}_2^{m_2}$. Let us first show that $n = m_1 + 1$. Suppose, on the contrary, that $n \leqq m_1$. Then $n - 1 \leqq m_1 - 1$ and $\dim S_1' \leqq \langle m_1 - 1 \rangle$. Assume that G is in the form (B) and apply Proposition A. Then

$$\dim G = \dim \bar{G} = \dim S_1' + \sum_{j=2}^{v} \dim S_j' + q$$

$$\leqq \langle t_1 \rangle + \sum_{j=2}^{v} \langle t_j \rangle + q.$$

We know

$$t_j \leqq t_1 \leqq m_1 - 1, \qquad j \geqq 2.$$

If $m_2 \leqq m_1 - 2$, we may use Lemma 2 to show

$$\dim G \leqq \langle m_1 - 1 \rangle + \langle m_2 + 1 \rangle < \langle m_1 \rangle + \langle m_2 \rangle.$$

If $m_2 = m_1 - 1$ we can conclude by Lemma 2 that

$$\dim G \leqq \langle m_1 - 1 \rangle + \langle m_2 \rangle + \langle 1 \rangle < \langle m_1 \rangle + \langle m_2 \rangle.$$

Finally, if $m_2 = m_1$,

$$\dim G \leqq \langle m_1 - 1 \rangle + \langle m_2 - 1 \rangle + \langle 2 \rangle < \langle m_1 \rangle + \langle m_2 \rangle.$$

We proceed with $n = m_1 + 1$.

Choose $x \in M$ so that it lies simultaneously on principal orbits of the actions of S_1', K and $\bar{G}$ on M, and let H_1, H_2 and H, respectively, denote the isotropy subgroups at x of the three actions. Then $N_1 = \bar{G}(x)$ is a principal orbit for the action of $\bar{G}$ on M and $\bar{G} = S_1' \times K$ acts transitively and almost effectively on N_1. We know that

$$\dim K = \dim \bar{G} - \dim S_1' = \langle m_2 \rangle,$$

and therefore that K acts almost effectively on N_1 with principal orbits of dimension at least m_2. On the other hand, if the dimension of the principal orbits of this action is greater than m_2, it follows that

$$\dim N_2 < m - m_2 = m_1,$$

where N_2 denotes the orbit space N_1/K. However S_1', which is simple, acts either trivially or almost effectively on the compact connected manifold N_2. If the action is trivial, then N_2 is a point and, therefore, K acts transitively on N_1. It then follows from Lemma 1 that S_1' acts almost freely on N_1 and that

$$\langle m_1 \rangle = \dim S_1' \leqq \dim N_1 \leqq m,$$

which is impossible. But if S_1' acts almost effectively on N_2 we again arrive at a contradiction since $\dim S_1' = \langle m_1 \rangle$ and $\dim N_2 < m_1$. Therefore K acts on N_1 with principal orbits precisely of dimension m_2. It follows that $K \cong \mathrm{Spin}\,(m_2+1)$, and that

$$\dim \bar{G}(x) = \dim N_1 = \dim N_2 + m_2 = m_1 + m_2 = m.$$

Hence $\bar{G}$ acts transitively on M and $M = \bar{G}/H$. We already know that H_1^0 is a standardly imbedded $\mathrm{Spin}\,(m_1)$ in S_1'. From our knowledge of the action of K on $N_1 = M$, it follows that

$$\dim H_2^0 = \dim \mathrm{Spin}\,(m_2),$$

and consequently for $m_2 \neq 3, 7$ that H_2^0 is a standardly imbedded $\mathrm{Spin}\,(m_2)$ in K [12, Lemma 7]. We proceed under the assumption that $m_2 \neq 3, 7$ acknowledging that separate arguments are required to handle these two possibilities. In any case $H_1^0 \times H_2^0 \subset H^0$, and since it is immediately checked that $\dim (H_1^0 \times H_2^0) = \dim H^0$, it follows that $H_1^0 \times H_2^0 = H^0$.

Now $H/H^0 \subset N(H^0, \bar{G})/H^0$ and

$$\frac{N(H^0, \bar{G})}{H^0} = \frac{N(\mathrm{Spin}\,(m_1), \mathrm{Spin}\,(m_1+1))}{\mathrm{Spin}\,(m_1)} \times \frac{N(\mathrm{Spin}\,(m_2), \mathrm{Spin}\,(m_2+1))}{\mathrm{Spin}\,(m_2)}$$
$$\cong Z_2 \oplus Z_2.$$

We obtain the possibilities (i), (ii), (iii) and (iv) listed in Theorem 2 by taking H/H^0 to be the subgroups 0, $Z_2 \oplus Z_2$, $0 \oplus Z_2$ and $Z_2 \oplus 0$ respectively. The only remaining possibility is that H/H^0 is a Z_2 imbedded diagonally in $Z_2 \oplus Z_2$. In this case M can be described as follows. Let Z_2 act on $S^{m_1} \times S^{m_2}$ by $(x, y) \to (-x, -y)$. Then

$$M = (S^{m_1} \times S^{m_2})/Z_2$$

is the orbit space of this action. We shall show that, depending upon m_1 and m_2, M may or may not split as a product but when M does split, then it is diffeomorphic to $S^{m_1}\times RP^{m_2}$. This would complete the proof of Theorem 2.

Recall [2] that $(KO)^{\sim}(RP^{m_2})$ is a cyclic group of order $2^{\phi(m_2)}$ where

$$\phi(m_2) = \text{the number of integers } p$$

such that

$$0 < p \leqq m_2 \tag{13}$$

$$p \equiv 0, 1, 2, 4 \pmod 8. \tag{14}$$

Moreover a generator of $(KO)^{\sim}(RP^{m_2})$ is given by the element $\alpha=\xi-1$, where ξ is the usual Hopf bundle

$$\xi: Z_2 \to S^{m_2} \to RP^{m_2}$$

defined by the antipodal involution of Z_2 on S^{m_2}. We complete the proof of Theorem 2 by proving the following proposition.

PROPOSITION C. *Let*

$$M = (S^{m_1}\times S^{m_2})/Z_2$$

be the orbit space of the diagonal involution $(x, y)\to(-x, -y)$ *on* $S^{m_1}\times S^{m_2}$, $m_1\geqq m_2$. *Then M splits as a product if and only if*

$$m_1+1 \equiv 0 \pmod{2^{\phi(m_2)}}.$$

Moreover, if M splits, then it is diffeomorphic to $S^{m_1}\times RP^{m_2}$.

Proof. First observe that we have the fibration

$$\eta: S^{m_1} \to M \to RP^{m_2},$$

which is the bundle associated with ξ having fiber S^{m_1}. Thus as vector bundles, $\eta=(m_1+1)\xi$. If

$$m_1+1 \equiv 0 \pmod{2^{\phi(m_2)}},$$

then

$$\eta-(m_1+1) = (m_1+1)\alpha = 0$$

in $(KO)^{\sim}(RP^{m_2})$. This means that η is stably trivial. But since $m_2<m_1+1$, we are in the stable range and therefore η itself is a trivial bundle. In particular, M is diffeomorphic to $S^{m_1}\times RP^{m_2}$.

Suppose now that $M=M_1^{m_1}\times M_2^{m_2}$. We wish to show that

$$m_1+1 \equiv 0 \pmod{2^{\phi(m_2)}}.$$

We shall consider only the case $m_2>1$ as the argument for $m_2=1$ is similar. If $m_2>1$, then $\pi_1(M)=Z_2$, and hence one of the two factors M_1 and M_2 is simply connected. Denote the simply connected factor by M_β and the other by M_γ. Then $\pi_1(M_\gamma)=Z_2$. Let $\tilde{M}_\gamma$ be the universal covering space of M_γ. Then we must have

$$M_\beta\times\tilde{M}_\gamma \approx S^{m_1}\times S^{m_2}.$$

From this it follows that M_β and $\tilde{M}_\gamma$ must be homotopy spheres of dimensions m_1 or m_2. If $m_1 = m_2$, it does not matter of course which is which. But if $m_1 > m_2$, we claim that

$$\dim M_\beta = m_1, \qquad \dim M_\gamma = m_2.$$

For if it were the reverse way, then M would have the mod 2 cohomology

$$H^*(M; Z_2) = H^*(RP^{m_1} \times S^{m_2}; Z_2).$$

However, a simple spectral sequence computation (the spectral sequence is trivial because $m_1 > m_2$) shows that

$$H^*(M; Z_2) = H^*(S^{m_1} \times RP^{m_2}; Z_2),$$

at least additively. But this gives an immediate contradiction by looking at $H^{m_2}(M; Z_2)$.

Thus at any rate M is homotopy equivalent to

$$S^{m_1} \times RP^{m_2}.$$

Let

$$h: M \to S^{m_1} \times RP^{m_2}$$

be a homotopy equivalence and let

$$i: S^{m_1} \to M$$

be the inclusion of a fiber in η. It is easily checked that

$$S^{m_1} \xrightarrow{i} M \xrightarrow{h} S^{m_1} \times RP^{m_2} \xrightarrow{\rho_1} S^{m_1}$$

induces an isomorphism on the integral cohomology in dimension m_1. According to Dold [4], this means the bundle η is fiber homotopically trivial. However, it is known [2] that the J-homomorphism

$$J: (KO)^{\sim}(RP^{m_2}) \to J(RP^{m_2})$$

is an isomorphism for projective spaces. Hence the element

$$\eta - (m_1 + 1) = (m_1 + 1)(\xi - 1) = (m_1 + 1)\alpha$$

is zero in $(KO)^{\sim}(RP^{m_2})$ or

$$m_1 + 1 \equiv 0 \pmod{2^{\phi(m_2)}}.$$

Remarks. The assumption that $m \geqq 19$ throughout the paper appears to be technical and we are able to dispose of this restriction in most cases. The authors are currently attempting to generalize Theorem 2 to products of more than two manifolds. The key to the situation appears to involve increasing the scope of the Main Lemma. For example, to handle the product of three manifolds, we should consider manifolds M where

$$N(M) \geqq m^2/6 + m/2;$$

for the product of four manifolds, we should consider manifolds M where

$$N(M) \geqq m^2/8+m/2$$

and so on.

It is easily checked that differentiability is not really used in the proof of the Main Lemma. Consequently, Theorems 1 and 2 can be stated in terms of topological manifolds and *topological degree of symmetry* $N_T(M^m)$ where for a compact connected topological m-manifold M^m, $N_T(M^m)$ is the maximum of the dimensions of the compact Lie groups which can act effectively on M.

PROPOSITION D. *Suppose*

$$M = M_1 \times M_2 \times \cdots \times M_\lambda,$$

where M and the M_i are compact differentiable manifolds. Then

$$N(M) \geqq \sum_{i=1}^{\lambda} N(M_i).$$

Proof. Let $\tilde{M}_i$ be a Riemannian structure for M_i such that

$$N(M_i) = \dim \operatorname{Isom}(\tilde{M}_i), \qquad i = 1, 2, \ldots, \lambda.$$

Let $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_\lambda$ be the product Riemannian manifold. Clearly Isom $(\tilde{M})$ contains a subgroup isomorphic to the product

$$\operatorname{Isom}(\tilde{M}_1) \times \operatorname{Isom}(\tilde{M}_2) \times \cdots \times \operatorname{Isom}(\tilde{M}_\lambda).$$

Hence

$$N(M) \geqq \dim \operatorname{Isom}(\tilde{M}) \geqq \sum_{i=1}^{\lambda} N(M_i).$$

EXAMPLE. If Σ^n is an exotic sphere which bounds a π-manifold, it is known that $\Sigma^n \times S^2$ is diffeomorphic to $S^n \times S^2$. Furthermore, it is known [7] that

$$N(\Sigma^n) < \tfrac{1}{8}n^2+1.$$

It follows that

$$N(\Sigma^n \times S^2) > N(\Sigma^n)+N(S^2).$$

COROLLARY 1 (COROLLARY TO THEOREM 2).

$$N(M_1)+N(M_2) \leqq N(M_1^{m_1} \times M_2^{m_2}) \leqq \langle m_1 \rangle + \langle m_2 \rangle \qquad (m_1+m_2 \geqq 19).$$

REMARK. If the right-hand equality in the above corollary holds we know that $M_1 \times M_2$ is diffeomorphic to one of the four manifolds listed in Theorem 2. However, this does not imply that the left-hand equality holds as exhibited by the last example where we chose an exotic splitting $M_1 = \Sigma^n$, $M_2 = S^2$ of $S^n \times S^2$.

REFERENCES

1. A. Borel et al., *Seminar on transformation groups*, Ann. of Math. Studies, Princeton Univ. Press, Princeton, N. J., 1960.

2. R. Bott, *Lectures on $K(X)$*, Harvard Univ., Cambridge, Mass., 1963.

3. G. E. Bredon, *Some theorems on transformation groups*, Ann. of Math. **67** (1958), 104–118.

4. A. Dold, *Über fasernweise Homotopieäquivalenz von Faserräumen*, Math. Z. **62** (1955), 111–136.

5. L. P. Eisenhart, *Riemannian geometry*, Princeton Univ. Press, Princeton, N. J., 1949.

6. W. C. Hsiang and W. Y. Hsiang, *Differentiable actions of compact connected classical groups*. I, Amer. J. Math. **89** (1967), 705–786.

7. W. Y. Hsiang, *On the bound of the dimensions of the isometry groups of all possible Riemannian metrics on an exotic sphere*, Ann. of Math. **85** (1967), 351–358.

8. ———, *The natural metric on* SO (n)/SO $(n-2)$ *is the most symmetric metric*, Bull. Amer. Math. Soc. **73** (1967), 55–58.

9. K. Jänich, *Differenzierbare G-Mannigfaltigkeiten*, Springer-Verlag, Berlin, 1968.

10. J. M. Kister and L. N. Mann, *Isotropy structure of compact Lie groups on complexes*, Michigan Math. J. **9** (1962), 93–96.

11. L. N. Mann, *Gaps in the dimensions of transformation groups*, Illinois J. Math. **10** (1966), 532–546.

12. Deane Montgomery and Hans Samelson, *Transformation groups of spheres*, Ann. of Math. **44** (1943), 454–470.

13. Paul S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math. **65** (1957), 447–455.

14. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

15. C. T. Yang, *The triangulability of the orbit space of a differentiable transformation group*, Bull. Amer. Math. Soc. **69** (1963), 405–408.

University of Massachusetts,
Amherst, Massachusetts