

## DEGREE THEORY FOR SPHERICAL FIBRATIONS

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**0. Introduction.** Let  $X$  be a finite-dimensional  $CW$ -complex and  $\widetilde{KF}(X)$  the group of stable fibre homotopy classes of spherical Hurewicz fibrings over  $X$ . We define a degree-function  $q$  on the subgroup  $\widetilde{KF}^{\text{or}}(X)$  of  $\widetilde{KF}(X)$  generated by orientable fibrings (i.e. a function  $q: \widetilde{KF}^{\text{or}}(X) \rightarrow \mathbb{Z}^+$ ). The degree-function imposes on the group  $\widetilde{KF}^{\text{or}}(X)$  an additional structure and is related to the usual algebraic structure by the non-degeneracy condition:  $[\xi] = 0$  in  $\widetilde{KF}^{\text{or}}(X)$  if and only if  $q([\xi]) = 1$ . The geometric significance of the degree-function derives from the fact that some geometric problems can be reduced to a statement about  $q$  of the multiples of the Hopf bundle over the complex projective space (e.g. refer to [8, Proposition 4.3]).

The purpose of this paper is to study the relationship between the degree and algebraic order functions on the group  $\widetilde{KF}^{\text{or}}(X)$ . The main result is Theorem 4.14 which states that the degree and the order of an element of  $\widetilde{KF}^{\text{or}}(X)$  are divisible by the same set of primes.

The paper is divided into five sections. The first and the second contain preliminaries on function spaces and spherical Hurewicz fibrings. They are based upon and they extend the results of [1] and [9] respectively. In Section 3 we observe that Dold's theorem mod  $k$  of [1] holds not only for sphere bundles over finite complexes but more generally for spherical Hurewicz fibrings over finite-dimensional complexes. We then prove a converse to this generalized version. In Section 4 we define the degree-function  $q$  on  $\widetilde{KF}^{\text{or}}(X)$  and combine the generalized Dold theorem mod  $k$  and its converse in Theorem 4.14. Corollary 4.16 gives a  $p$ -primary decomposition for the degree-function. In Section 5, using the Adams conjecture, we deduce a corollary to the converse of Dold's theorem mod  $k$  which states that given an orientable vector-bundle  $\xi$  over a finite complex, there exists a fibrewise  $S$ -map,  $f: E_{S(\xi)} \rightarrow E_{S(\psi^k(\xi))}$  of degree  $k^t$  at each fibre ( $t \geq 0$ ) where  $\psi^k$  is the Adams operation. If  $L$  is a complex line bundle, this corollary generalizes the existence of the fibrewise map,  $f: E_L \rightarrow E_{L^k}$  of degree  $k$  at each fibre and given by  $f(x) = x \otimes \cdots \otimes x$ ,

for  $x \in E_L$ . We then compute  $q$  of certain multiples of the Hopf bundle over the complex projective space.

**1. Preliminaries on function spaces.** 1.1. *Notations and definitions.* We stick mainly to the notation of [1]. We set:

$H(n; k)$  = set of all maps from  $S^{n-1}$  to  $S^{n-1}$  of degree  $k$ .

$F(n; k)$  = subset of  $H(n; k)$  of maps which fix the north-pole.

$H(n) = \bigcup_{k \in \mathbb{Z}} H(n; k)$  and  $F(n) = \bigcup_{k \in \mathbb{Z}} F(n; k)$ .

Let  $r \leq n - 3$ . It follows from the exact homotopy sequence of the fibration,  $H(n; k) \xrightarrow{F(n; k)} S^{n-1}$ , and the  $(n - 2)$ -connectedness of  $S^{n-1}$  that for  $h \in F(n; k)$ , the groups  $\pi_r(H(n; k), h)$  and  $\pi_r(F(n; k), h)$  are isomorphic. Let  $u \in F(n; s)$  and  $v \in F(n; t)$ . The sum  $u + v \in F(n; s + t)$  is defined as the composite,

$$u + v : S^{n-1} \xrightarrow{\mu'} S^{n-1} \vee S^{n-1} \xrightarrow{u \vee v} S^{n-1} \vee S^{n-1} \xrightarrow{\Delta'} S^{n-1}$$

where  $\mu'$  is a base-point preserving map of bi-degree  $(1, 1)$  and  $\Delta'$  is the co-diagonal. This definition of addition turns  $F(n)$  into an  $H$ -space with homotopy unit the constant map  $c_0 \in F(n; 0)$  which sends  $S^{n-1}$  to the north-pole. Using the  $H$ -space structure on  $F(n)$ , we identify  $\pi_r(F(n; k), h)$  with  $\pi_r(F(n; 0), c_0)$  and the latter with the stable  $r$ -stem  $\pi_r^S$ . Hence  $\pi_r(H(n; k), h)$  is identified with  $\pi_r^S$  throughout. If  $f : (S^r, s_0) \rightarrow (H(n; k), h)$  is a map, then its homotopy class  $[f]$  is regarded as an element of  $\pi_r^S$ . If  $m \geq 0$  and  $h \in H(n; k)$ , then  $mh \in H(mn; k^m)$  denotes the  $m$ -fold join,  $h * h * \dots * h$ , of  $h$  with itself. If  $f : (S^r, s_0) \rightarrow (H(n; k), h)$  is a map, then  $mf : (S^r, s_0) \rightarrow (H(mn; k^m), mh)$  is defined by  $(mf)(x) = m(f(x))$ ,  $x \in S^r$ . We add three more lemmas to the list of preliminary lemmas of [1]. Lemma 1.2 is both an analogue of [1, Lemma 2.2] and also a generalization of [1, Lemma 2.1]. Let  $c : H(n; s) \times H(n; t) \rightarrow H(n; st)$ ,  $(s, t \in \mathbb{Z})$  be the composition map defined by  $c(u, v)(x) = u(v(x))$ ,  $u \in H(n; s)$ ,  $v \in H(n; t)$ ,  $x \in S^{n-1}$ . If  $h \in H(n; s)$ , then the map  $\bar{h} : H(n; t) \rightarrow H(n; st)$ , is defined by  $\bar{h}(u) = c(h, u)$ ,  $u \in H(n; t)$ .

**LEMMA 1.2.** *If  $\alpha \in \pi_r(H(n; s))$ ,  $\beta \in \pi_r(H(n; t))$  and  $r \leq n - 3$ , then  $c_*(\alpha + \beta) = t\alpha + s\beta$  in  $\pi_r^S$ .*

**PROOF.** Since  $c_*$  is a homomorphism, it suffices to prove that  $c_*(\alpha + 0) = t\alpha$  and  $c_*(0 + \beta) = s\beta$ . The case  $c_*(0 + \beta) = s\beta$  follows from [1, Lemma 2.1] and a parallel argument is required to do the case  $c_*(\alpha + 0) = t\alpha$ .

From now onwards we adopt the notation that if  $f, g: D^r \rightarrow X$  and  $H: S^{r-1} \times I \rightarrow X$  are such that  $H_0 = g|_{S^{r-1}}$  and  $H_1 = f|_{S^{r-1}}$  then  $\mu(g, H, f): S^r = S^{r-1} \times I \cup D^r \times \dot{I} \rightarrow X$  is the map defined by

$$\mu(g, H, f) = \begin{cases} g & \text{on } D^r \times (0) \\ H & \text{on } S^{r-1} \times I \\ f & \text{on } D^r \times (1) . \end{cases}$$

**LEMMA 1.3.** *Let  $f: D^r \rightarrow H(n; q)$ ,  $g: D^r \rightarrow H(nk; q^k)$  and  $H: S^{r-1} \times I \rightarrow H(nk; q^k)$  ( $r \leq n - 3$ ,  $k \geq 1$ ) be such that  $H_0 = g|_{S^{r-1}}$  and  $H_1 = k(f|_{S^{r-1}})$ . Then for any  $h \in F(n; kq^k)$  there exists  $f': D^r \rightarrow H(n; kq^{1+k})$  satisfying (i)  $f'|_{S^{r-1}} = \bar{h} \circ (f|_{S^{r-1}})$  and (ii)  $[\mu((\bar{k}\bar{h}) \circ g, (\bar{k}\bar{h}) \circ H, kf')] = 0$  in  $\pi_r^S$ .*

**PROOF.** Let  $h \in F(n; kq^k)$  and  $\alpha: (D^r, S^{r-1}) \rightarrow (H(n; kq^k), h)$ . Define  $f': D^r \rightarrow H(n; kq^{1+k})$  by  $f'(x) = \alpha(x) \circ f(x)$ ,  $x \in D^r$ . Let  $\alpha': S^r \rightarrow H(n; kq^k)$  be given by

$$\alpha' = \begin{cases} h & \text{on } S^{r-1} \times I \cup D^r \times (0) \\ \alpha & \text{on } D^r \times (1) . \end{cases}$$

Then  $[\alpha'] = [\alpha]$  in  $\pi_r^S$ . If  $c: H(nk; q^k) \times H(nk; k^k q^{k^2}) \rightarrow H(nk; k^k q^{k(1+k)})$  denotes composition, then  $\mu((\bar{k}\bar{h}) \circ g, (\bar{k}\bar{h}) \circ H, kf')(x) = c((k\alpha')(x), \mu(g, H, kf')(x))$ ,  $x \in S^r$ . Thus

$$\begin{aligned} & [\mu((\bar{k}\bar{h}) \circ g, (\bar{k}\bar{h}) \circ H, kf')] \\ &= q^k[k\alpha'] + k^k q^{k^2}[\mu(g, H, kf')] \quad \text{by [1, Lemma 1.2]} \\ &= q^k k(kq^k)^{k-1}[\alpha'] + k^k q^{k^2}[\mu(g, H, kf')] \quad \text{by [1, Lemma 2.4]} \\ &= k^k q^{k^2}([\alpha] + [\mu(g, H, kf)]) . \end{aligned}$$

Choose  $[\alpha] = -[\mu(g, H, kf)]$  and this makes  $[\mu((\bar{k}\bar{h}) \circ g, (\bar{k}\bar{h}) \circ H, kf')] = 0$ .

**LEMMA 1.4.** *Let  $g: D^r \rightarrow H(m+n; q)$ ,  $f: D^r \rightarrow H(n; q)$  and  $H: S^{r-1} \times I \rightarrow H(m+n; q)$ , ( $r \leq n - 2$ ,  $m \geq 0$ ) be such that  $H(x, 0) = f(x) * 1_m$  and  $H(x, 1) = g(x)$ ,  $x \in S^{r-1}$ . Then there exists  $f': D^r \rightarrow H(n; q)$  such that (i)  $f'|_{S^{r-1}} = f|_{S^{r-1}}$  and (ii)  $[\mu(g, H, f' * 1_m)] = 0$ .*

**PROOF.** Let  $h = f(0) * 1_m \in H(m+n; q)$  and choose a map  $\theta: (D^r, S^{r-1}) \rightarrow (H(m+n; q), h)$  representing the element  $-[\mu(g, H, f * 1_m)] \in \pi_r(H(m+n; q))$ . Define  $\hat{f}: D^r \rightarrow H(m+n; q)$  by

$$\hat{f}(x) = \begin{cases} \theta(2x) & \text{if } \|x\| \leq 1/2 \\ f((2\|x\| - 1)x) * 1_m & \text{if } \|x\| \geq 1/2 . \end{cases}$$

Then clearly  $\hat{f}(x) = f(x) * 1_m$  for  $x \in S^{r-1}$  and it follows from the definition

of addition in the group  $\pi_r(H(m+n; q))$  that  $[\mu(g, H, \hat{f})] = [\mu(g, H, f*1_m)] + [\theta] = 0$ . Regard  $[\hat{f}] \in \pi_r(H(m+n; q), H(n; q)) = 0$ , i.e. there exists  $f': D^r \rightarrow H(n; q)$  such that  $\hat{f} \underset{\text{rel } S^{r-1}}{\cong} f'*1_m$ . Thus  $\mu(g, H, f'*1_m) \cong \mu(g, H, \hat{f})$  and hence the result.

**2. Preliminaries on spherical Hurewicz fibrings.** 2.1. *Notations and definitions.* Let  $X$  be a  $CW$ -complex. A map  $p: E \rightarrow X$ , is said to define a Hurewicz fibring if and only if it is locally fibre homotopy equivalent to a product or equivalently if and only if it has the weak covering homotopy property (e.g. see [9, Definition 5.1, Theorem 6.4 and Proposition 6.7]). The sum  $\xi \oplus \eta$  of two Hurewicz fibrings  $\xi, \eta$  whose fibres are of the homotopy type of  $S^{m-1}$  and  $S^{n-1}$  respectively, is constructed by taking joins at each fibre and is thus a Hurewicz fibring whose fibre is of the homotopy type of  $S^{m+n-1}$ . The total space of a Hurewicz fibring  $\xi$  is denoted by  $E_\xi$  and the projection by  $p_\xi$ . If  $f_1: E_{\xi_1} \rightarrow E_{\eta_1}$  and  $f_2: E_{\xi_2} \rightarrow E_{\eta_2}$  are fibrewise maps between spherical Hurewicz fibrings, then their sum  $f_1 \oplus f_2$  is also defined by taking joins at each fibre. If  $f: E_\xi \rightarrow E_\eta$  is a fibrewise map, define  $kf: E_{k\xi} \rightarrow E_{k\eta}$  by  $kf = f \oplus \dots \oplus f$  ( $k$  times).  $m$  denotes the trivial spherical fibration with fibre  $S^{m-1}$ .

2.2. *The group  $\widetilde{KF}(X)$ .* Let  $X$  be a finite dimensional  $CW$ -complex and  $\text{Hur}_s(X)$  the set of all Hurewicz fibrings over  $X$  whose fibres are of the homotopy type of spheres. Define an equivalence relation on  $\text{Hur}_s(X)$  as follows: For  $\xi, \eta \in \text{Hur}_s(X)$ ,  $\xi \sim \eta$  if and only if  $\xi \oplus m$  and  $\eta \oplus n$  are fibre homotopy equivalent for some  $m, n \geq 0$ . Let  $\widetilde{KF}(X)$  denote the set of equivalence classes. Then  $\widetilde{KF}(X)$  is a group. The existence of inverses follows from [7, Theorem 4.7]. Let  $\widetilde{KF}^{\text{or}}(X)$  be the subgroup generated by orientable fibrings. The following are well-known:

(i) There exist classifying spaces  $B_F$  and  $B_F^{\text{or}}$  such that  $\widetilde{KF}(X) = [X; B_F]$  and  $\widetilde{KF}^{\text{or}}(X) = [X; B_F^{\text{or}}]$ . The spaces  $B_F$  and  $B_F^{\text{or}}$  are constructed as in [17, p. 60]. Let  $H_n = H(n; 1) \cup H(n; -1)$  and  $F_n = F(n; 1) \cup F(n; -1)$  be the associative  $H$ -spaces (with respect to composition of maps) of all homotopy equivalences and distinguished point preserving homotopy equivalences of  $S^{n-1}$  respectively. Let  $B_{F(n;1)}, B_{F_n}, B_{H(n;1)}, B_{H_n}$  be the classifying spaces of the corresponding  $H$ -spaces. The set of fibre homotopy classes of  $S^{n-1}$ -fibrations over  $X$  is classified by a homotopy set  $[X; B_{H_n}]$  and orientable ones by  $[X; B_{H(n;1)}]$ . There are natural inclusions,  $F(n; 1) \rightarrow F(n+1; 1)$ ,  $F_n \rightarrow F_{n+1}$ ,  $H(n; 1) \rightarrow H(n+1; 1)$  and  $H_n \rightarrow H_{n+1}$  and the induced inclusions,  $B_{F(n;1)} \rightarrow B_{F(n+1;1)}$ ,  $B_{F_n} \rightarrow B_{F_{n+1}}$ ,  $B_{H(n;1)} \rightarrow B_{H(n+1;1)}$  and  $B_{H_n} \rightarrow B_{H_{n+1}}$ . Define  $B_F = \text{dir lim}_{n \rightarrow \infty} B_{F_n}$ ,  $B_F^{\text{or}} = \text{dir lim}_{n \rightarrow \infty} B_{F(n;1)}$ ,  $B_H =$

$\text{dir lim}_{n \rightarrow \infty} B_{H_n}$  and  $B_H^{\text{or}} = \text{dir lim}_{n \rightarrow \infty} B_{H(n;1)}$ . Then as in [17, p. 60] we have  $\widetilde{KF}(X) = [X; B_H]$  and  $\widetilde{KF}^{\text{or}}(X) = [X; B_H^{\text{or}}]$ . The natural inclusions,  $F_n \rightarrow H_n \rightarrow F_{n+1}$  and  $F(n; 1) \rightarrow H(n; 1) \rightarrow F(n+1; 1)$  give rise to inclusions,  $B_{F_n} \rightarrow B_{H_n} \rightarrow B_{F_{n+1}}$  and  $B_{F(n;1)} \rightarrow B_{H(n;1)} \rightarrow B_{F(n+1;1)}$  and hence  $B_H = B_F$  and  $B_H^{\text{or}} = B_F^{\text{or}}$ .

(ii) The image of the  $J$ -homomorphism,  $J: \widetilde{K}_R(X) \rightarrow \widetilde{KF}(X)$ , which assigns to a vector-bundle  $\xi$  the underlying sphere-bundle  $S(\xi)$ , is the group  $J(X)$ .

(iii)  $\widetilde{KF}(S^r) = \pi_{r-1}^S$  for  $r > 1$  and  $\widetilde{KF}(S^1) = Z_2$ .

(iv) For each sequence,  $A \xrightarrow{j} X \xrightarrow{c} X/A$ , where  $j$  is the inclusion and  $c$  the collapsing map, is associated an exact sequence,  $\widetilde{KF}(X/A) \xrightarrow{c^!} \widetilde{KF}(X) \xrightarrow{j^!} \widetilde{KF}(A)$ .

(v)  $\widetilde{KF}(X)$  is a torsion group with the following finiteness property: For each  $k \in Z$  there exists an integer  $a_k$  depending only on the dimension of  $X$  such that if an element  $x \in \widetilde{KF}(X)$  belongs to the  $k$ -torsion, then  $k^{a_k}x = 0$ . (This follows by induction from (iii), (iv) and the finiteness of the stable stems.)

**DEFINITION 2.3.** Let  $X$  be a connected topological space and  $r: X \rightarrow A$  a deformation retraction onto a subspace  $A$ . Let  $\xi$  and  $\eta$  be Hurewicz fibrations over  $X$ . Then for each fibrewise map  $f: p_{\xi}^{-1}(A) \rightarrow p_{\eta}^{-1}(A)$ , we define a fibrewise map  $\theta_{\xi, \eta}^{X, A}(f): E_{\xi} \rightarrow E_{\eta}$ , as follows: Let  $j: A \subset X$  be the inclusion map and  $H: X \times I \rightarrow X$  be the deformation,  $H_0 = 1_X$  and  $H_1 = jr$ . By [9, Lemma 6.5] there exists a homotopy,  $R_{\xi}: E_{H^*(\xi)} \times I \rightarrow E_{H^*(\xi)}$ , such that the maps  $\pi_{\xi}: E_{H^*(\xi)} \rightarrow X$ ,  $\rho_{\xi}: E_{H^*(\xi)} \rightarrow I$ ,  $r_{\xi}: E_{H^*(\xi)} \rightarrow E_{H^*(\xi)}$  defined by  $\pi_{\xi} = \Delta_1 \circ p_{H^*(\xi)}$ ,  $\rho_{\xi} = \Delta_2 \circ p_{H^*(\xi)}$ ,  $r_{\xi} = R_{\xi} \circ (1 \times \rho_{\xi})$  (where  $\Delta_1: X \times I \rightarrow X$ ,  $\Delta_2: X \times I \rightarrow I$  denote the projections) have the following properties:

(i)  $p_{H^*(\xi)} \circ R_{\xi}(e, t) = (\pi_{\xi}(e), t)$ ,  $e \in E_{H^*(\xi)}$ ,  $t \in I$

and

(ii)  $r_{\xi} \cong 1$ .

Define  $h_{\xi}^1: E_{\xi} \xrightarrow{\text{fibre}} E_{r^*(\xi)}$  by  $h_{\xi}^1(e) = R_{\xi}((e, p_{\xi}(e), 0), 1)$ ,  $e \in E_{\xi}$ .  $h_{\xi}^0: E_{r^*(\xi)} \rightarrow E_{\xi}$  by  $h_{\xi}^0(e, x) = R_{\xi}((e, x, 1), 0)$ ,  $(e, x) \in E_{r^*(\xi)}$ . Similarly define  $R_{\eta}$ ,  $r_{\eta}$ ,  $h_{\eta}^1$ ,  $h_{\eta}^0$ .  $\theta_{\xi, \eta}^{X, A}(f)$  is then defined to the composite,

$$\theta_{\xi, \eta}^{X, A}(f): E_{\xi} \xrightarrow{h_{\xi}^1} E_{r^*(\xi)} \xrightarrow{r^*(f)} E_{r^*(\eta)} \xrightarrow{h_{\eta}^0} E_{\eta}.$$

**LEMMA 2.4** (Extension lemma).  $\theta_{\xi, \eta}^{X, A}(f)$  has the following properties:

(i) Let  $B \subset A$  and  $Y = r^{-1}(B)$ . Then

$$(\theta_{\xi, \eta}^{X, A}(f))|_{p_{\xi}^{-1}(Y)} = \theta_{\xi|_Y, \eta|_Y}^{Y, B}(f|_{p_{\xi}^{-1}(B)}) .$$

(ii) If  $f, g: p_{\xi}^{-1}(A) \rightarrow p_{\eta}^{-1}(A)$  are fibrewise maps and  $H: p_{\xi}^{-1}(A) \times I \rightarrow p_{\eta}^{-1}(A)$  is a fibrewise homotopy such that  $f \underset{H}{\cong} g$ , then

$$\theta_{\xi, \eta}^{X, A}(f) \underset{\theta_{\xi \times I, \eta \times I}^{X \times I, A \times I}(H')}{\cong} \theta_{\xi, \eta}^{X, A}(g)$$

where  $H': p_{\xi}^{-1}(A) \times I \rightarrow p_{\eta}^{-1}(A) \times I$  is defined by  $H'(e, t) = (H(e, t), t)$ ,  $e \in p_{\xi}^{-1}(A)$ ,  $t \in I$ .

(iii) For a fibrewise map  $f: E_{\xi} \rightarrow E_{\eta}$ , there exists a fibrewise homotopy  $H_f: E_{\xi} \times I \rightarrow E_{\eta}$ , such that  $f \underset{H_f}{\cong} \theta_{\xi, \eta}^{X, A}(f|_{p_{\xi}^{-1}(A)})$ .

(iv) For a fibrewise map  $g: p_{\xi}^{-1}(A) \rightarrow p_{\eta}^{-1}(A)$ , there exists a fibrewise homotopy  $K_g: p_{\xi}^{-1}(A) \times I \rightarrow p_{\eta}^{-1}(A)$ , such that  $g \underset{K_g}{\cong} (\theta_{\xi, \eta}^{X, A}(g)|_{p_{\xi}^{-1}(A)})$ .

(v) If  $f: p_{\xi}^{-1}(A) \rightarrow p_{\eta}^{-1}(A)$  and  $g: p_{\eta}^{-1}(A) \rightarrow p_{\xi}^{-1}(A)$  are fibre homotopy inverses, so are  $\theta_{\xi, \eta}^{X, A}(f)$  and  $\theta_{\eta, \xi}^{X, A}(g)$ .

(vi) Suppose the fibres of  $\xi$  and  $\eta$  are both of the homotopy type of  $S^{n-1}$ .

(a) For a fibrewise map  $f: p_{\xi}^{-1}(A) \rightarrow p_{\eta}^{-1}(A)$ ,  $f$  and  $\theta_{\xi, \eta}^{X, A}(f)$  are of the same degree.

(b) 1.  $\theta_{k\xi \oplus m, k\eta \oplus m}^{X, A}(kf \oplus 1_m) = k\theta_{\xi, \eta}^{X, A}(f) \oplus 1_m$ .

2. For a fibrewise map  $f: E_{\xi} \rightarrow E_{\eta}$ ,  $H_{kf \oplus 1_m} = kH_f \oplus 1_m$  (where  $H_f$  is as defined in (iii)).

3. For a fibrewise map  $g: p_{\xi}^{-1}(A) \rightarrow p_{\eta}^{-1}(A)$ ,  $K_{kg \oplus 1_m} = kK_g \oplus 1_m$  (where  $K_g$  is as defined in (iv)).

PROOF. Properties (i), (ii) immediately follow from Definition 2.3 and (v) from [9, Corollary 6.6] which states that  $h_{\xi}^0$  and  $h_{\eta}^0$  are fibre homotopy inverses of  $h_{\xi}^1$  and  $h_{\eta}^1$  respectively. To prove (iii), let  $T(\xi): E_{H^*(\xi)} \times I \rightarrow E_{H^*(\xi)}$  be a homotopy such that  $T_0(\xi) = 1$  and  $T_1(\xi) = r_{\xi}$ . Let  $T(\eta)$  be similarly defined. Define  $H_f: E_{\xi} \times I \rightarrow E_{\eta}$  by

$$H_f(e, t) = \begin{cases} T(\eta)(H^*(f) \circ T(\xi)(e, p_{\xi}(e), 0), 2t, 2t) & \text{if } 0 \leq t \leq 1/2 \\ R_{\eta}(H^*(f) \circ R_{\xi}((e, p_{\xi}(e), 0), 2t - 1), 0) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

(iv) is proved similarly. (vi) (a) follows from (iv) and (vi) (b) from the fact that  $R_{k\xi \oplus m} = kR_{\xi} \oplus 1_m$ .

2.5. *Obstruction theory for spherical Hurewicz fibrings.* Suppose  $\xi$  and  $\eta$  are fibre bundles (i.e. locally trivial) over a CW-complex  $X$  with fibre  $F$  and  $f: E_{\xi} \rightarrow E_{\eta}$  is a fibrewise map defined over the  $(r-1)$ -skeleton  $X^{r-1}$ . For each  $r$ -cell  $e_i^r$  of  $X$ ,  $f$  defines an obstruction  $[f_i] \in \pi_{r-1}(F)$  and  $f$  extends to a fibrewise map over  $X^r$  if and only if all the  $[f_i]$  vanish. This straightforward obstruction theory for the inductive extension of

fibrewise maps between fibre bundles does not go through for spherical Hurewicz fibrings. The vanishing of the obstructions is necessary but no longer sufficient for the extension of  $f$  over  $X^r$ . For spherical Hurewicz fibrings, we can only develop an obstruction theory for the extension of  $f$  in the homotopy category; i.e. for the existence of a fibrewise map  $\tilde{f}$  defined over  $X^r$  such that  $\tilde{f}|_{p_{\xi}^{-1}(X^{r-1})} \underset{\text{fibre}}{\cong} f$ . To do this we define a canonical neighbourhood  $N^r$  of  $X^{r-1}$  and using (iv) of the Extension lemma, we extend  $f$  in the homotopy category to a fibrewise map  $\hat{f}$  defined over  $N^r$ . For each  $r$ -cell  $e_i^r$ ,  $\hat{f}$  defines an obstruction  $[\hat{f}_i] \in \pi_{r-1}^S$  and  $\hat{f}|_{p_{\xi}^{-1}(X^{r-1})}$  extends over  $X^r$  if and only if all the  $[\hat{f}_i]$  vanish (e.g. (i) of the Obstruction lemma). Hence the vanishing of the  $[\hat{f}_i]$  is a necessary and sufficient condition for the extension of  $f$  over  $X^r$  in the homotopy category.

We then introduce analogous modifications for the obstruction theory for extending fibrewise homotopies between fibrewise maps (e.g. (ii) of the Obstruction lemma).

Let  $(e_i^r)_{i \in I_r}$  be the  $r$ -cells of  $X$  given by characteristic maps  $c_i: (D^r, S^{r-1}) \rightarrow (X, X^{r-1})$  where we assume  $X$  to be of dimension  $r$ . Let  $A^r = \{x \in D^r \mid 1/2 \leq \|x\| \leq 1\}$ ,  $D_{1/2}^r = \{x \in D^r \mid \|x\| \leq 1/2\}$ ,  $S_{1/2}^{r-1} = A^r \cap D_{1/2}^r$ .

I. Define  $N^r = X^{r-1} \cup \bigcup_{i \in I_r} c_i(A^r)$ . Then  $N^r$  is a closed neighbourhood of  $X^{r-1}$  in  $X$  of which  $X^{r-1}$  is a strong deformation retract.

Suppose  $\xi, \eta$  are Hurewicz fibrings over  $X$  whose fibres are of the homotopy type of  $S^{n-1}$  ( $r \leq n-3$ ). Choose fibre homotopy equivalences

$$D^r \times S^{n-1} \xrightleftharpoons[\psi_{\xi}(\xi)]{\phi_{\xi}(\xi)} E_{c_i^*(\xi)}^*$$

and

$$D^r \times S^{n-1} \xrightleftharpoons[\psi_{\eta}(\eta)]{\phi_{\eta}(\eta)} E_{c_i^*(\eta)}^*$$

and fibrewise homotopies  $M^i(\xi), M^i(\eta): D^r \times S^{n-1} \times I \rightarrow D^r \times S^{n-1}$ ,  $N^i(\xi): E_{c_i^*(\xi)}^* \times I \rightarrow E_{c_i^*(\xi)}^*$  and  $N^i(\eta): E_{c_i^*(\eta)}^* \times I \rightarrow E_{c_i^*(\eta)}^*$  such that  $M_0^i(\xi) = 1$ ,  $M_1^i(\xi) = \psi_{\xi}(\xi)\phi_{\xi}(\xi)$ ,  $N_0^i(\xi) = 1$ ,  $N_1^i(\xi) = \phi_{\xi}(\xi)\psi_{\xi}(\xi)$  and similarly for  $\eta$ . Suppose  $f: p_{\xi}^{-1}(N^r) \rightarrow p_{\eta}^{-1}(N^r)$  is a fibrewise map of degree  $q$  at each fibre.

II. Define  $f_i: S_{1/2}^{r-1} \rightarrow H(n; q)$  as follows: Let

$$F_i: S_{1/2}^{r-1} \times S^{n-1} \xrightarrow{\phi_i(\xi)} E_{c_i^*(\xi)}^* \xrightarrow{c_i^*(f)} E_{c_i^*(\eta)}^* \xrightarrow{\psi_i(\eta)} S_{1/2}^{r-1} \times S^{n-1}.$$

Then  $f_i$  is the map obtained from  $F_i$  by fixing the first coordinate. Its homotopy class  $[f_i] \in \pi_{r-1}^S$  is called the  $i^{\text{th}}$ -obstruction to extending  $f$ . Note that  $[(kf \oplus 1_m)_i] = [kf_i \oplus 1_m] = [kf_i]$  and  $[kf_i] = kq^{k-1}[f_i]$  by [1, Lemma 2.4].

III. If (i)  $f_i$  has an extension  $\hat{f}_i$  to  $D_{1/2}^r$  and (ii)  $g: E_\xi \rightarrow E_\eta$  is a fibre-wise map of degree  $q$  at each fibre such that  $g|_{p_\xi^{-1}(Nr)} \cong_H f$ , then define  $o(g, H, \hat{f}_i) \in \pi_r^S$  as follows: Let

$$\bar{g}_i: D_{1/2}^r \times S^{n-1} \xrightarrow{\phi_i(\xi)} E_{c_i^*(\xi)}^* \xrightarrow{c_i^*(g)} E_{c_i^*(\eta)}^* \xrightarrow{\psi_i(\eta)} D_{1/2}^r \times S^{n-1}$$

and  $g_i: D_{1/2}^r \rightarrow H(n; q)$  be the map obtained from  $\bar{g}_i$  by fixing the first coordinate. Let

$$\bar{H}_i: S_{1/2}^{r-1} \times S^{n-1} \times I \xrightarrow{\phi_i(\xi) \times 1} E_{c_i^*(\xi)}^* \times I \xrightarrow{c_i^*(H)} E_{c_i^*(\eta)}^* \xrightarrow{\psi_i(\eta)} S_{1/2}^{r-1} \times S^{n-1}$$

and  $H_i: S_{1/2}^{r-1} \times I \rightarrow H(n; q)$  be the map obtained from  $\bar{H}_i$  by fixing the first and the third coordinates. Then define

$$o(g, H, \hat{f}_i) = [\mu(g_i, H_i, \hat{f}_i)] \in \pi_r^S.$$

$o(g, H, \hat{f}_i)$  is called the  $i^{\text{th}}$ -obstruction to extending the homotopy  $H$ . The terminology is justified by the following:

LEMMA 2.6 (Obstruction lemma). (i) Any family of extensions  $(\hat{f}_i)$  of  $(f_i)$  to  $D_{1/2}^r$  determines an extension  $\phi_{\xi, \eta}^X(f, (\hat{f}_i))$  of  $f|_{p_\xi^{-1}(X^{r-1})}$  to  $E_\xi$  which is also of degree  $q$  at each fibre.

(ii) If  $o(g, H, \hat{f}_i) = 0$  for all  $i \in I_r$ , then there exists an extension  $\Phi_{\xi, \eta}^X(g, H, (\hat{f}_i))$  of  $H|_{p_\xi^{-1}(X^{r-1}) \times I}$  to  $E_\xi \times I$  which is also of degree  $q$  at each fibre and such that  $(\Phi_{\xi, \eta}^X(g, H, (\hat{f}_i)))_0 = g$ ,  $(\Phi_{\xi, \eta}^X(g, H, (\hat{f}_i)))_1 = \phi_{\xi, \eta}^X(f, (\hat{f}_i))$ .

(iii) (a)  $\phi_{k\xi \oplus m, k\eta \oplus m}^X(kf \oplus 1_m, (k\hat{f}_i \oplus 1_m)) = k\phi_{\xi, \eta}^X(f, (\hat{f}_i)) \oplus 1_m$ .

(b)  $\Phi_{k\xi \oplus m, k\eta \oplus m}^X(kg \oplus 1_m, kH \oplus 1_m, (k\hat{f}_i \oplus 1_m)) = k\Phi_{\xi, \eta}^X(g, H, (\hat{f}_i)) \oplus 1_m$ .

(iv) If  $Y$  is a subcomplex of  $X$ , then

(a)  $(\phi_{\xi, \eta}^X(f, (\hat{f}_i)))|_{p_\xi^{-1}(Y)} = \phi_{\xi|Y, \eta|Y}^X(f|_{p_\xi^{-1}(N_Y^r)}, (\hat{f}_i))$ .

(b)  $(\Phi_{\xi, \eta}^X(g, H, (\hat{f}_i)))|_{p_\xi^{-1}(Y) \times I} = \Phi_{\xi|Y, \eta|Y}^X(g|_{p_\xi^{-1}(N_Y^r)}, H|_{p_\xi^{-1}(N_Y^r) \times I}, (\hat{f}_i))$ .

PROOF. (i) Let  $\hat{F}_i: D_{1/2}^r \times S^{n-1} \rightarrow D_{1/2}^r \times S^{n-1}$  be the corresponding evaluation map which extends  $F_i$ . Define  $f'_i: E_{c_i^*(\xi)}^* \rightarrow E_{c_i^*(\eta)}^*$  by

$$f'_i(e) = \begin{cases} N^i(\eta)(c_i^*(f) \circ N^i(\xi)(e, 2\alpha_i(e)), 2\alpha_i(e)) & \text{if } \alpha_i(e) \leq 1/2 \\ \phi_i(\eta) \circ \hat{F}_i \circ \psi_i(\xi)(e) & \text{if } 1/2 \leq \alpha_i(e) \leq 1 \end{cases}$$

for  $e \in E_{c_i^*(\xi)}^*$ , where  $\alpha_i(e) = 1 - \|p_{c_i^*(\xi)}(e)\|$ . Then  $f'_i|_{p_{c_i^*(\xi)}^{-1}(S^{r-1})} = c_i^*(f)$ . Hence the map  $f|_{p_\xi^{-1}(X^{r-1})}$  and the family  $(f'_i)_{i \in I_r}$  fit together to define a map  $\phi_{\xi, \eta}^X(f, (\hat{f}_i)): E_\xi \rightarrow E_\eta$  of degree  $q$  at each fibre and which extends  $f|_{p_\xi^{-1}(X^{r-1})}$ .

(ii) Choose an extension  $\hat{\mu}_i$  of  $\mu(g_i, H_i, \hat{f}_i)$  to  $D_{1/2}^r \times I$  and let  $\hat{U}_i: D_{1/2}^r \times S^{n-1} \times I \rightarrow D_{1/2}^r \times S^{n-1}$  be the corresponding evaluation map. Define a homotopy  $H'_i: E_{c_i^*(\xi)}^* \times I \rightarrow E_{c_i^*(\eta)}^*$  by



$$H'_i(e, t) = \begin{cases} N^i(\eta)(c_i^*(g) \circ N^i(\xi)(e, 2t), 2t) & \text{if } \min(1/2, \alpha_i(e)) \geq t \\ N^i(\eta)(c_i^*(H)(N^i(\xi)(e, 2\alpha_i(e)), \beta_i(e, t)), 2\alpha_i(e)) & \text{if } \min(1/2, t) \geq \alpha_i(e) \\ \phi_i(\eta) \circ \hat{U}_i(\psi_i(\xi)(e), 2t - 1) & \text{if } \min(t, \alpha_i(e)) \geq 1/2 \end{cases}$$

for  $e \in E_{c_i^*(\xi)}$ ,  $t \in I$ , where  $\beta_i(e, t) = 1 - (1 - t)/\|p_{c_i^*(\xi)}(e)\|$ . Then  $H'_i|_{p_{c_i^*(\xi)}^{-1}(S^{r-1})} = c_i^*(H)$ . Hence the homotopy  $H|_{p_{\xi}^{-1}(X^{r-1}) \times I}$  and the family  $(H'_i)_{i \in I_r}$  fit together to define a homotopy  $\Phi_{\xi, \gamma}^X(g, H, (\hat{f}_i)): E_{\xi} \times I \rightarrow E_{\gamma}$  which is of degree  $q$  at each fibre, extends  $H|_{p_{\xi}^{-1}(X^{r-1}) \times I}$  and such that  $(\Phi_{\xi, \gamma}^X(g, H, (\hat{f}_i)))_0 = g$  and  $(\Phi_{\xi, \gamma}^X(g, H, (\hat{f}_i)))_1 = \phi_{\xi, \gamma}^X(f, (\hat{f}_i))$ . Properties (iii) and (iv) are readily verified from the definition.

**3. Converse to Dold's theorem mod  $k$ .** We first state the generalized form of Dold's theorem mod  $k$  as indicated in the introduction.

**THEOREM 3.1** (Dold's theorem mod  $k$ ). *Let  $\xi$  be a Hurewicz fibring over a finite-dimensional CW-complex  $X$  and whose fibre is of the homotopy type of  $S^{n-1}$  and suppose that  $\dim X \leq n - 3$ . The existence of a fibrewise map  $f: E_{\xi} \rightarrow X \times S^{n-1}$ , of degree  $k$  at each fibre implies the existence of fibre homotopy equivalence  $g: E_{k^t \xi} \rightarrow X \times S^{nk^t-1}$ , for some non-negative integer  $t$  such that the diagram:*

$$\begin{array}{ccc} E_{k^t \xi} & \xrightarrow{g} & X \times S^{nk^t-1} \\ & \searrow k^t f & \downarrow 1 \times h \\ & & X \times S^{n-1} \end{array}$$

for some  $h \in F(nk^t; k^t)$ , is fibre homotopy commutative.

We do not attempt to prove this generalized version since the proof in [1] can be adapted with the following modifications: The induction should be on dimension rather than the number of cells attached. The theorem then becomes true for finite dimensional CW-complexes instead of finite ones. In this way it is necessary to give the order a certain increase at each dimension while the other way one gives the order this same increase over each cell of the same dimension. Hence by shifting the induction, one can obtain a smaller value for the exponent  $t$ . Also the straightforward obstruction theory for fibre bundles used in [1] should be modified for Hurewicz fibrings as explained in 2.5.

**THEOREM 3.2** (Converse to Dold's theorem mod  $k$ ). *Let  $\xi$  be a Hurewicz fibring over a finite-dimensional CW-complex  $X$  and whose fibre is of the homotopy type of  $S^{n-1}$  and suppose further that  $\xi$  is orientable and*

that  $\dim X \leq n - 3$ . The existence of a fibre homotopy equivalence  $g: E_{k\xi} \rightarrow X \times S^{nk-1}$ , implies the existence of a fibrewise map  $f: E_{\xi} \rightarrow X \times S^{n-1}$ , of degree  $k^t$  ( $t \geq 0$ ) at each fibre and such that the diagram:

$$\begin{array}{ccc} E_{k\xi} & \xrightarrow{g} & X \times S^{nk-1} \\ & \searrow kf & \downarrow 1 \times h \\ & & X \times S^{n-1} \end{array}$$

for some  $h \in F(nk; k^t)$ , is fibre homotopy commutative.

PROOF. By induction on  $r = \dim X$ . It is true for  $r = 0$  with  $t = 0$ . Let  $r > 1$  and assume it for  $(r - 1)$ . By the induction hypothesis, there exists a fibrewise map  $f_0: p_{\xi}^{-1}(X^{r-1}) \rightarrow X^{r-1} \times S^{n-1}$ , of degree  $k^t$  at each fibre and such that  $kf_0 \cong (1 \times h_0) \circ (g|_{p_{\xi}^{-1}(X^{r-1})})$  for some  $h_0 \in F(nk; k^t)$ . By (ii), (iii) and (vi) (b) 1 of the Extension lemma, this implies the existence of a fibrewise map  $\hat{f}: p_{\xi}^{-1}(N^r) \rightarrow N^r \times S^{n-1}$ , also of degree  $k^t$  at each fibre and such that

I.  $k\hat{f} \cong (1 \times h_0) \circ (g|_{p_{\xi}^{-1}(N^r)})$ . Let  $\hat{h} \in F(n; k^{1+t(k-1)})$ . Put

$$u = \begin{cases} 1 & \text{if } r = 1 \\ 1 + tk & \text{if } r > 1. \end{cases}$$

Define  $h \in F(nk; k^{ku})$  and  $f: p_{\xi}^{-1}(N^r) \rightarrow N^r \times S^{n-1}$  by

$$h = \begin{cases} h_0 & \text{if } r = 1 \\ (k\hat{h}) \circ h_0 & \text{if } r > 1 \end{cases} \quad \text{and} \quad f = \begin{cases} \hat{f} & \text{if } r = 1 \\ (1 \times \hat{h}) \circ \hat{f} & \text{if } r > 1. \end{cases}$$

II.  $kf \cong (1 \times h) \circ (g|_{p_{\xi}^{-1}(N^r)})$ . Let  $H$  denote the homotopy in between. Let the obstructions  $[f_i], [\hat{f}_i], [g_i] \in \pi_{r-1}^S$  to extending  $f, \hat{f}, g$ , respectively, be defined with respect to orientation preserving fibre homotopy equivalences

$$D^r \times S^{n-1} \xrightleftharpoons[\psi_i(\xi)]{\phi_i(\xi)} E_{e_i^r(\xi)}^*$$

over each  $r$ -cell  $e_i^r$  ( $i \in I_r$ ). For  $r = 1$ ,  $t = 0$  and, since  $\xi$  is orientable,  $\text{Image}(f_i)$  lies in a single path component  $H(n; 1)$  and hence extends to  $D_{1/2}^r$ . For  $r > 1$ , it follows from I, the definition of  $f$  and [1, Lemmas 2.1 and 2.4] that

$$[f_i] = k^{1+t(k-1)}[\hat{f}_i] = [k\hat{f}_i] = \bar{h}_*[g_i] = k^{tk}[g_i] = 0.$$

Hence  $f_i$  has an extension  $\tilde{f}_i$  to  $D_{1/2}^r$ . Let  $h_1 \in F(n; k^{1+uk})$  and define  $h' \in F(nk; k^{k(1+u+uk)})$  and fibrewise  $f': p_{\xi}^{-1}(N^r) \rightarrow N^r \times S^{n-1}$  by  $h' = (kh_1) \circ h$  and  $f' = (1 \times h_1) \circ f$ .

III.  $kf' \cong (1 \times h') \circ (g|_{p_{\xi}^{-1}(N^r)})$  where the homotopy in between is  $H' = (1 \times kh_1) \circ H$ . Then  $f'_i = \bar{h}_1 \circ f_i$  and  $\tilde{f}'_i = \bar{h}_1 \circ \tilde{f}_i$  defines an extension of  $f'_i$  to  $D_{1/2}^r$ . We deduce from Lemma 1.3 (with  $q = k^n$ ) that  $f'_i$  has another extension  $\hat{f}'_i$  to  $D_{1/2}^r$  such that  $o((1 \times h') \circ g, H', k\hat{f}'_i) = 0$  in  $\pi_r^S$ . Hence by (i), (ii) and (iii) (a) of the Obstruction lemma, there exists a fibrewise map  $\hat{f}': E_{\xi} \rightarrow X \times S^{n-1}$ , of degree  $k^{1+u+uk}$  at each fibre and such that  $k\hat{f}' \cong (1 \times h') \circ g$ .

**4. Degree-function  $q$  on the group  $\widetilde{KF}^{\text{or}}(X)$ .** 4.1. *Notations and definitions.* Let  $X$  be a connected  $CW$ -complex of dimension  $r$  and  $\xi, \eta$  be Hurewicz fibrings over  $X$  whose fibres are of the homotopy type of  $S^{n-1}$ . Let  $M(\xi, \eta)$  and  $M_q(\xi, \eta)$  denote fibre homotopy classes of all fibrewise maps from  $E_{\xi}$  to  $E_{\eta}$  and those of degree  $q$  at each fibre respectively ( $q \in \mathbb{Z}$ ). Then  $M(\xi, \eta) = \bigcup_{q \in \mathbb{Z}} M_q(\xi, \eta)$ . For  $m \geq 0$ , the suspension map  $\Sigma: M(\xi, \eta) \rightarrow M(\xi \oplus m, \eta \oplus m)$ , is defined by  $\Sigma([f]) = [f \oplus 1_m]$ .  $\Sigma$  preserves degrees and its restriction to  $M_q(\xi, \eta)$  thus defines  $\Sigma_q: M_q(\xi, \eta) \rightarrow M_q(\xi \oplus m, \eta \oplus m)$ . Let  $M^S(\xi, \eta) = \text{dir lim}_{m \rightarrow \infty} M(\xi \oplus m, \eta \oplus m)$ ,  $M_q^S(\xi, \eta) = \text{dir lim}_{m \rightarrow \infty} M_q(\xi \oplus m, \eta \oplus m)$ . Then  $M^S(\xi, \eta) = \bigcup_{q \in \mathbb{Z}} M_q^S(\xi, \eta)$ .

**OBSERVATION 4.2.** *Let  $\xi, \eta$  and  $\gamma$  be spherical Hurewicz fibrings over a finite dimensional  $CW$ -complex  $X$  and whose fibres are of the homotopy types of  $S^{m-1}$ ,  $S^{n-1}$  and  $S^{p-1}$  respectively and such that  $\eta \oplus \gamma \oplus r$  is fibre homotopy equivalent to the trivial fibration  $(n + p + r)$ . Then there exist maps*

$$\begin{aligned} \alpha_q: M_q(\xi \oplus n, \eta \oplus m) &\rightarrow M_q((\xi \oplus \gamma) \oplus (n + r), (m + p + n + r)), \\ \beta_q: M_q(\xi \oplus \gamma, (m + p)) &\rightarrow M_q(\xi \oplus (n + p + r), \eta \oplus (m + p + r)) \end{aligned}$$

such that  $(\beta_q \oplus \Sigma_{n+r}) \circ \alpha_q = 1 \oplus \Sigma_{p+r}$  and  $(\alpha_q \oplus \Sigma_{p+r}) \circ \beta_q = 1 \oplus \Sigma_{n+r}$ , hence inducing  $\alpha_q^S: M_q^S(\xi \oplus n, \eta \oplus m) \rightarrow M_q^S(\xi \oplus \gamma, (m + p))$  and  $\beta_q^S: M_q^S(\xi \oplus \gamma, (m + p)) \rightarrow M_q^S(\xi \oplus n, \eta \oplus m)$  such that  $\alpha_q^S \circ \beta_q^S = \beta_q^S \circ \alpha_q^S = 1$  and similarly for  $\alpha^S$  and  $\beta^S$  defined by  $\alpha^S = \bigcup_{q \in \mathbb{Z}} \alpha_q^S$  and  $\beta^S = \bigcup_{q \in \mathbb{Z}} \beta_q^S$ .

**PROOF.** Straightforward verification.

**DEFINITION 4.3.** Let  $\xi$  be a spherical Hurewicz fibring over a finite dimensional  $CW$ -complex  $X$  and whose fibre is of the homotopy type of  $S^{n-1}$ . The Thom complex  $X^{\xi}$  of  $\xi$  is defined by  $X^{\xi} = E_{\xi \oplus 1}/s(X)$ , where  $s: X \rightarrow E_{\xi \oplus 1}$ , is the section at infinity. Then  $X^{\xi \oplus 1} = SX^{\xi}$ . Let  $U_{\xi} \in H^n(X^{\xi})$  be the cohomology Thom class,  $i_n \in H^n(S^n) = \mathbb{Z}$  be the generator. A map  $f: X^{\xi} \rightarrow S^n$  is called a map of degree  $q$  if and only if the induced map  $f^*$  on the cohomology level satisfies:  $f^*(i_n) = qU_{\xi}$ . Let  $[X^{\xi}; S^n]$  and  $[X^{\xi}; S^n]_q$  denote the homotopy classes of all base-point preserving maps from  $X^{\xi}$

to  $S^n$  and those of degree  $q$  respectively. Then  $[X^\xi; S^n] = \bigcup_{q \in \mathbb{Z}} [X^\xi; S^n]_q$ . Define

$$\begin{aligned} \{X^\xi; S^n\} &= \varinjlim_{m \rightarrow \infty} [S^m X^\xi; S^{m+n}] = \varinjlim_{m \rightarrow \infty} [X^{\xi \oplus m}; S^{m+n}] \\ \{X^\xi; S^n\}_q &= \varinjlim_{m \rightarrow \infty} [X^{\xi \oplus m}; S^{m+n}]_q. \end{aligned}$$

Then  $\{X^\xi; S^n\} = \bigcup_{q \in \mathbb{Z}} \{X^\xi; S^n\}_q$ . We now quote the following result which appears in [19, Proposition 3.7].

**LEMMA 4.4.** *Let  $\xi$  be a spherical Hurewicz fibring over the suspension  $SX$  of an  $r$ -dimensional CW-complex  $X$  whose fibre is of the homotopy type of  $S^{n-1}$  with  $n > r + 1$ . Let  $\alpha: X \rightarrow F(n+1)$  be the classifying map of  $\xi$  and  $\theta(\alpha): S^n X \rightarrow S^n$  be its adjoint. Then  $S^n X \xrightarrow{\theta(\alpha)} S^n \xrightarrow{j} (SX)^\xi$  is a cofibration.*

**COROLLARY 4.5.** *A degree  $q$  map  $f: (SX)^\xi \rightarrow S^n$ , exists if and only if  $q([\xi]) = 0$  in  $\widetilde{KF}(SX)$  ( $n > r + 3$ ).*

**PROOF.** Take the exact cofibration sequence

$$[S^n X; S^n] \xleftarrow{\theta(\alpha)^*} [S^n; S^n] \xleftarrow{j^*} [(SX)^\xi; S^n]$$

and use the identity:  $\widetilde{KF}(SX) = [S^n X; S^n]$ .

We now slightly generalize Lemma 4.4;

**LEMMA 4.6.** *Let  $f: X \rightarrow Y$  be a base-point preserving map of CW-complexes with mapping cone  $Z = CX \vee_f Y$ . Let  $\xi$  be a spherical Hurewicz fibring over  $Z$  whose fibre is of the homotopy type of  $S^{n-1}$  with  $n > \dim X + 1$  and which is fibre homotopy trivial over  $Y$ . Let  $\alpha: X \rightarrow Y \times F(n+1)$  be the classifying map of  $\xi$  and  $\theta(\alpha): S^n X \rightarrow S^n Y$  be its adjoint. Then the sequence  $S^n X \xrightarrow{\theta(\alpha)} S^n Y \xrightarrow{j} Z^\xi$  is a cofibration.*

**PROOF.**  $p_{\xi \oplus 1}^{-1}(CX) \cong_{\text{fibre}} CX \times S^n$  and  $p_{\xi \oplus 1}^{-1}(Y) \cong_{\text{fibre}} Y \times S^n$  and the two pieces are identified by the clutching-function  $\alpha: X \rightarrow Y \times F(n+1)$ . Hence  $E_{\xi \oplus 1} = CX \times S^n \cup_\alpha Y \times S^n$ , where the map  $\Delta: X \times S^n \rightarrow Y \times S^n$ , is the evaluation-map corresponding to  $\alpha$ . Thus  $Z^\xi = E_{\xi \oplus 1}/s(Z) = C(S^n X) \vee_{\theta(\alpha)} S^n Y$ .

**COROLLARY 4.7.** *For  $n > r + 3$ , a degree  $q$  map  $f: Z^\xi \rightarrow S^n$ , exists if and only if  $q([\xi]) = 0$  in  $\widetilde{KF}(Z)$ .*

**PROOF.** Take the exact cofibration sequence

$$[S^n X; S^n] \xleftarrow{\theta(\alpha)^*} [S^n Y; S^n] \xleftarrow{j^*} [Z^\xi; S^n],$$

and use the identity: subgroup of  $\widetilde{KF}(Z)$  of fibrations fibre homotopy trivial over  $Y = [S^n X; S^n]$ .

**COROLLARY 4.8.** *Let  $\xi$  be a spherical Hurewicz fibring over an  $r$ -dimensional CW-complex  $X$  whose fibre is of the homotopy type of  $S^{n-1}$  with  $n > r + 2$  such that the restriction of  $\xi$  to  $X^{r-1}$  is fibre homotopy trivial. Then a degree  $q$  map  $f: X^\xi \rightarrow S^n$  exists if and only if  $q([\xi]) = 0$  in  $\widetilde{KF}(X)$ .*

**PROOF.** It immediately follows from Corollary 4.7.

**LEMMA 4.9.** *There exist bijections  $\psi: \{X^\xi; S^n\} \rightarrow M^S(\xi, n)$  and  $\psi_q: \{X^\xi; S^n\}_q \rightarrow M_q^S(\xi, n)$ .*

**PROOF.** Let  $\psi_\xi: E_{\xi \oplus 1} \rightarrow X^\xi$  be the quotient projection and  $\psi_\xi^*: [X^\xi; S^n] \rightarrow M(\xi \oplus 1, n + 1)$  be the induced map. The map  $\Sigma(\xi): M(\xi, n) \rightarrow M(\xi \oplus 1, n + 1)$  factors through  $[X^\xi; S^n]$ . Let  $\phi_\xi: M(\xi, n) \rightarrow [X^\xi; S^n]$  be the unique map such that

$$(i) \quad \Sigma(\xi) = \psi_\xi^* \circ \phi_\xi.$$

If  $S(\xi): [X^\xi; S^n] \rightarrow [X^{\xi \oplus 1}; S^{n+1}]$  is the suspension map, then

$$(ii) \quad S(\xi) = \phi_{\xi \oplus 1} \psi_\xi^*.$$

Define  $\psi: \{X^\xi; S^n\} \rightarrow M^S(\xi, n)$  and  $\phi: M^S(\xi, n) \rightarrow \{X^\xi; S^n\}$  by

$$\psi = \varinjlim_{m \rightarrow \infty} \psi_{\xi \oplus m}^* \quad \text{and} \quad \phi = \varinjlim_{m \rightarrow \infty} \phi_{\xi \oplus m}.$$

It follows from (i) and (ii) that  $\psi\phi = 1$  and  $\phi\psi = 1$ .

**LEMMA 4.10.** *Let  $X$  be a finite dimensional CW-complex and  $\xi$  be a Hurewicz fibring over  $X$  whose fibre is of the homotopy type of  $S^{n-1}$  and suppose that  $\xi$  is orientable. Then there exists  $q \geq 1$  such that  $M_q^S(\xi, n) \neq \emptyset$ .*

**PROOF.** It is an immediate consequence of the converse to Dold's theorem mod  $k$  and the fact that  $\widetilde{KF}(X)$  is a torsion group.

**COROLLARY 4.11.** *Let  $\xi, \eta$  be spherical Hurewicz fibrings over a finite dimensional CW-complex  $X$  and whose fibres are of the homotopy type of  $S^{m-1}$  and  $S^{n-1}$  respectively and such that  $[\xi] - [\eta] \in \widetilde{KF}^{\text{or}}(X)$ . Then  $M_q^S(\xi \oplus n, \eta \oplus m) \neq \emptyset$ .*

**PROOF.** It is an immediate consequence of Lemma 4.10 and Observation 4.2.

Motivated by Corollaries 4.5, 4.7, 4.8 and Lemma 4.10, we define a degree-function  $q$  on the subgroup  $\widetilde{KF}^{\text{or}}(X)$  of  $\widetilde{KF}(X)$ .

DEFINITION 4.12. Let  $X$  be a finite dimensional  $CW$ -complex and  $\xi$  be a Hurewicz fibring over  $X$  whose fibre is of the homotopy type of  $S^{n-1}$  and suppose that  $\xi$  is orientable.  $q(\xi)$  is defined to be the least positive integer  $q$  such that  $\widetilde{M}_q^S(\xi, n) \neq \emptyset$  (or, equivalently,  $[X^\xi; S^n]_q \neq \emptyset$ ). This defines a function  $q: \widetilde{KF}^{\text{or}}(X) \rightarrow \mathbb{Z}^+$ , which we shall call the degree-function on  $\widetilde{KF}^{\text{or}}(X)$ .

REMARK 4.13. Let  $S = \{(u, v) \in \widetilde{KF}(X) \times \widetilde{KF}(X) \mid u - v \in \widetilde{KF}^{\text{or}}(X)\}$ . One is inclined to define a seemingly more general degree-function  $\tilde{q}: S \rightarrow \mathbb{Z}^+$  by letting  $\tilde{q}(\xi, \eta)$ , for an  $S^{m-1}$ -fibration  $\xi$  and an  $S^{n-1}$ -fibration  $\eta$ , to be the least positive integer  $\tilde{q}$  such that  $M_{\tilde{q}}^S(\xi \oplus n, \eta \oplus m) \neq \emptyset$ . However an immediate consequence of Observation 4.2 is the following:

*Translation property.*

$$\tilde{q}(u, v) = q(u - v)$$

which makes clear that it suffices to study  $q$ .

Let  $\pi_i: X^\xi \vee X^\xi \rightarrow X^\xi$ ,  $i = 1, 2$  denote the projections. Let  $\mu': X^\xi \rightarrow X^\xi \vee X^\xi$  be a base-point preserving map such that the composites  $\pi_i \mu'$  are homotopic to the identity for  $i = 1, 2$ . If  $f_1, f_2: X^\xi \rightarrow S^n$  are base-point preserving maps of degrees  $q_1$  and  $q_2$ , respectively, at each fibre, then the composite  $X^\xi \xrightarrow{\mu'} X^\xi \vee X^\xi \xrightarrow{f_1 \vee f_2} S^n \vee S^n \xrightarrow{\Delta'} S^n$ , where  $\Delta'$  is the co-diagonal, is a base-point preserving map of degree  $(q_1 + q_2)$  at each fibre. If  $f: X^\xi \rightarrow S^n$  is a base-point preserving map of degree  $q$  at each fibre and  $h \in F(n; -1)$ , then the composite  $hf$  is a base-point preserving map of degree  $-q$  at each fibre. Hence the degrees of maps  $f: X^{\xi \oplus m} \rightarrow S^{m+n}$  (over all  $m \geq 0$ ) form a group and thus if  $f$  is such a map of degree  $q$ , then  $q$  is a multiple of  $q(\xi)$ .

The degree-function has the following properties:

- (i) Non-degeneracy condition: For  $u \in \widetilde{KF}^{\text{or}}(X)$ ,  $u = 0$  if and only if  $q(u) = 1$ .
- (ii) Multiplicative property: For  $u, v \in \widetilde{KF}^{\text{or}}(X)$ ,  $q(u + v) \mid q(u)q(v)$ .
- (iii) Naturality: If  $f: X \rightarrow Y$  is a map, then  $q(f^!(u)) \mid q(u)$ .

THEOREM 4.14. *The degree and the order of an element of  $\widetilde{KF}^{\text{or}}(X)$  are divisible by the same set of primes.*

PROOF. It is an immediate consequence of the generalized Dold theorem mod  $k$  and its converse.

From now onwards  $d$  denotes the order function in  $\widetilde{KF}(X)$ , i.e.; for  $\xi \in \widetilde{KF}(X)$ ,  $d(\xi)$  denotes its order is  $\widetilde{KF}(X)$ .

**LEMMA 4.15.** *Suppose  $(p_i)_{1 \leq i \leq k}$  is a set of distinct primes and  $\xi = \sum_{i=1}^k \xi_i$  is a decomposition in the group  $\widetilde{KF}^{\text{or}}(X)$  such that  $d(\xi_i)$  is a power of  $p_i$ . Then the  $p_i$ -primary component of  $q(\xi)$  is  $q(\xi_i)$  and  $q(\xi) = \prod_{i=1}^k q(\xi_i)$ .*

**PROOF.**  $q(\xi) | \prod_{i=1}^k q(\xi_i)$  by the multiplicative property.  $d(\xi_i)$  is a power of  $p_i$  and hence by Theorem 4.14,  $q(\xi_i)$  is a power of  $p_i$ . Thus the  $p_i$ -primary component of  $q(\xi)$  divides  $q(\xi_i)$ .  $\xi_i = \xi - \sum_{j \neq i} \xi_j$ , and by the multiplicative property,  $q(\xi_i) | q(\xi) \prod_{j \neq i} q(-\xi_j)$ . For  $j \neq i$ ,  $d(-\xi_j)$ , and hence by Theorem 4.14,  $q(-\xi_j)$ , is a power of  $p_j$  and, consequently,  $p_i$  does not divide  $\prod_{j \neq i} q(-\xi_j)$ . Thus  $q(\xi_i)$  divides the  $p_i$ -primary component of  $q(\xi)$ .

Let us recall that if  $n = \prod_{i=1}^k p_i^{a_i}$ , for distinct primes  $(p_i)_{1 \leq i \leq k}$  ( $a_i > 0$ ), then the finite cyclic group  $Z_n$  of order  $n$  decomposes into a direct sum  $Z_n = \bigoplus_{i=1}^k Z_{p_i^{a_i}}$ , where  $Z_{p_i^{a_i}}$  is the cyclic subgroup of  $Z_n$  generated by  $x_i = np_i^{-a_i}$ .

**COROLLARY 4.16.** *Let  $\xi \in \widetilde{KF}^{\text{or}}(X)$  and write  $1 = \sum_{i=1}^k n_i x_i$  in the group  $Z_{d(\xi)}$  relative to the above decomposition. Put  $\xi_i = n_i x_i \xi$  so that  $\xi = \sum_{i=1}^k \xi_i$  in the group  $\widetilde{KF}^{\text{or}}(X)$ . Then the  $p_i$ -primary component of  $q(\xi)$  is  $q(\xi_i)$  and  $q(\xi) = \prod_{i=1}^k q(\xi_i)$ .*

**5. Applications.** Using the Adams conjecture (for the proof refer to either [6] or [13]), we obtain the following corollary to the converse of Dold's theorem mod  $k$ .

**COROLLARY 5.1.** *Let  $\xi$  be an orientable vector-bundle over a finite CW-complex  $X$  and  $k \in \mathbb{Z}$ . Then there exist  $m, t \geq 0$  and a fibrewise map  $f: E_{S(\xi+m)} \rightarrow E_{S(\psi^k(\xi+m))}$  of degree  $k^t$  at each fibre.*

Let us note that Corollary 5.1 is also proved by different methods in [18, Chapter 11].

Let  $\eta_{k-1}$  be the Hopf line bundle over complex projective space  $P_{k-1}(C)$ . Let  $\partial_{n,k}^C: \pi_{2n-1}(S^{2n-2}) \rightarrow \pi_{2n-2}(W_{n-1,k-1})$  denote the boundary operator of complex Stiefel fibering

$$p_{n,k}^C: W_{n,k} \xrightarrow{W_{n-1,k-1}} S^{2n-1}.$$

Let  $i_r \in \pi_r(S^r) = \mathbb{Z}$  be the generator. Then  $w_{n,k} = \partial_{n,k}^C(i_{2n-1})$  is the obstruction to cross-sectioning  $p_{n,k}^C$ . Let  $d(w_{n,k})$  denote its order in the group  $\pi_{2n-2}(W_{n-1,k-1})$ .

**PROPOSITION 5.2.** *For  $n \geq 2k - 1$ ,  $d(w_{n,k}) = q(-n\eta_{k-1})$ .*

**PROOF.** Straightforward application of the Duality theorem of [4] and the Freudenthal suspension theorem.

Let  $M_k = \prod_p p^{v_p(M_k)}$  be the Atiyah-Todd number defined in [5] by

$$v_p(M_k) = \begin{cases} \sup \{r + v_p(r); 1 \leq r \leq [(k-1)/(p-1)]\} & \text{if } p \leq k \\ 0 & \text{if } p > k. \end{cases}$$

PROPOSITION 5.3. *If  $v_p(n) \geq v_p(M_{k-1})$ , then*

$$v_p(q(n\eta_{k-1})) = \begin{cases} v_p(M_k) - v_p(n) & \text{if } v_p(n) \leq v_p(M_k) \\ 0 & \text{if } v_p(n) > v_p(M_k). \end{cases}$$

PROOF. It is an immediate consequence of Corollary 4.8 and the result of [3] that  $J(\eta_{k-1})$  is of order  $M_k$  in  $J(P_{k-1}(C))$ .

COROLLARY 5.4. *If  $v_p(n) \geq v_p(M_{k-1})$ , then*

$$v_p(d(w_{n,k})) = \begin{cases} v_p(M_k) - v_p(n) & \text{if } v_p(n) \leq v_p(M_k) \\ 0 & \text{if } v_p(n) > v_p(M_k). \end{cases}$$

PROOF. It is an immediate consequence of Propositions 5.2 and 5.3.

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