Degrees of Freedom for the MIMO Interference Channel

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***Abstract*— We show that the exact number of spatial degrees of freedom for a two user nondegenerate (full rank channel matrices) MIMO Gaussian interference channel with M_1, M_2 (respectively) antennas at transmitters $1, 2$ and N_1, N_2 antennas at the corresponding receivers, and perfect channel knowledge at all transmitters and receivers, is $\min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1)\}$. A constructive achievability proof shows that zero forcing is sufficient to achieve all the available degrees of freedom on the two user MIMO interference channel. This is in contrast to the MIMO X channel where the combination of zero forcing, dirty paper coding, and successive decoding schemes is shown to achieve more degrees of freedom than are possible with spatial zero forcing [1] alone. We also study a share-and-transmit scheme and show how the gains of transmitter cooperation are entirely offset by the cost of enabling that cooperation so that the available DoF are not increased.**

I. Introduction

Multiple input multiple output (MIMO) systems have assumed great importance in recent times because of their remarkably higher capacity compared to single input single output systems. It is well known [2]–[4] that capacity of a point to point (PTP) MIMO system with M inputs and N outputs increases linearly as $\min(M, N)$ at high SNR. For power and bandwidth limited wireless systems, this opens up another dimension - "space" that can be exploited in a similar way as time and frequency. Similar to time division and frequency division multiplexing, MIMO systems present the possibility of multiplexing signals in space. Spatial dimensions are especially interesting for how they may be limited by distributed processing as well the amount of channel knowledge. Previous work has shown that in the absence of channel knowledge, spatial DoF are lost [5], [6]. Multiuser systems, with constrained cooperation between inputs/outputs distributed among multiple users, are especially challenging since, unlike PTP case, joint processing is not possible at inputs/outputs. The available spatial DoF are affected by the inability to jointly process the signals at the distributed inputs and outputs. [**?**] investigated DoF as a function of distributed and partial side information for multiple access (MAC) and broadcast (BC) channels. The two user interference channel with single antennas at all nodes is considered by Host-Madsen [7], [8]. It is shown that the maximum multiplexing gain is only equal to one even if cooperation between the two transmitters or the two receivers is allowed via a noisy communication link. Nosratinia and Host-Madsen [9] show that even if communication links are introduced between the two transmitters as well as between the two receivers the highest multiplexing gain achievable is equal to one. These results are somewhat surprising as it can be shown that with ideal cooperation between transmitters (broadcast channel) or with ideal cooperation between receivers (multiple access channel) the maximum multiplexing gain is equal to 2. A number of challenging questions arise in a wireless network with distributed nodes and with multiple (possibly varying across users) antennas at each transmitter and receiver. For example:

- What is the maximum multiplexing gain in distributed MIMO systems?
- How can this multiplexing gain be shared among users?
- Is spatial zero forcing optimal for achieving all the available multiplexing gain, or is it possible to use dirty paper coding and successive decoding principles to achieve more multiplexing gain than is possible with spatial zero forcing alone?
- How does the multiplexing gain depend on the number of messages in the system ?
- How does limited cooperation between distributed nodes affect the spatial degrees of freedom?

In this paper, we focus on the two user (M_1, N_1, M_2, N_2) MIMO interference channel where transmitter 1 with M_1 antennas has a message for receiver 1 with N_1 antennas, and transmitter 2 with M_2 antennas has a message for receiver 2 with N_2 antennas. We develop a MIMO multiple access channel (MAC) outerbound on the sum capacity of this MIMO interference channel. The outerbound is used to prove a converse result for the maximum number of degrees of freedom. We also provide a constructive proof of achievability of the degrees of freedom based on zero forcing. We show that the innerbound and the outerbound are tight, thereby establishing the precise number of degrees of freedom on the MIMO interference channel as $\min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_2)\}$. We also consider a simple cooperative scheme to understand why transmitter cooperation may not increase DoF. Through this simple scheme, we are able to show how the benefits of cooperation are completely offset by the cost of enabling it.

II. Degrees of Freedom Measure

In order to isolate the impact of distributed processing from channel uncertainty, we assume that channel state is fixed and perfectly known at all transmitters and receivers. Also, we assume that the channel matrices are sampled from a rich scattering environment. Therefore we can ignore the measure zero event that some channel matrices are rank deficient. It is well known that the capacity of a *scalar* additive white Gaussian noise (AWGN) channel scales as $\log(SNR)$ at high SNR. On the other hand, for a single user MIMO channel with M inputs and N outputs, the capacity growth rate can be shown to be $\min(M,N)\log(SNR)$ at high SNR. This motivates the natural definition of spatial DoF as:

$$\eta \triangleq \lim_{\rho\to\infty} \frac{C_\Sigma(\rho)}{\log(\rho)}, \tag{1}$$

where $C_\Sigma(\rho)$ is the sum capacity (just capacity in case of PTP channels) at SNR ρ. In other words, DoF η represent the maximum *multiplexing gain* [4] of the generalized MIMO system. For PTP case, (M,N) DoF are easily seen to correspond to the parallel channels that can be isolated using SVD, involving joint processing at the M inputs and N outputs, i.e.

$$\eta(\text{PTP}) \quad = \quad \min(M,N) \tag{2}$$

A. The Multiple Access Channel

The MAC channel is an example of a MIMO system where cooperation is allowed only between the channel outputs. Let the MAC consist of N outputs controlled by the same receiver and 2 users, each controlling M_1 and M_2 inputs for a total of $M = M_1 + M_2$ inputs. For the MAC, the available DoF are the same as with perfect cooperation between all users.

$$\eta(\text{MAC}) = \eta(\text{PTP}) = \min(M_1 + M_2, N). \tag{3}$$

While the capacity region of the MIMO MAC is well known and the spatial multiplexing gain has also been explored in previous work, we include the following constructive proof to introduce zero forcing (ZF) notation which will be useful in the derivation of our main result for the interference channel. ZF, which is normally a suboptimal strategy, is sufficient in this case (as well as in MIMO BC channel) to utilize all DoF.
Converse: The converse is straightforward because, for the same number of inputs and outputs, $\eta(\text{MAC}) \le \eta(\text{PTP}) = \min(M_1+M_2,N)$. In other words, the lack of cooperation at the inputs can not increase DoF.
Achievability: The $N\times 1$ received signal $\mathbf{Y}$ at the MAC receiver

$$\mathbf{Y} = \sum_{k=1}^{2} \mathbf{H}^{(k)}\mathbf{X}^{(k)} + \mathbf{N} = \mathbf{V_H}^\dagger \mathbf{V_X} + \mathbf{Z}, \tag{4}$$

where $\mathbf{N}$ is the $N\times 1$ AWGN vector, $\mathbf{H}^{(k)}$ is the $N\times M_k$ channel matrix for user k, and $\mathbf{X}^{(k)}$ is the $M_k \times 1$ transmitted vector for user k. $\mathbf{V_H} = V(\mathbf{H}^{(\cdot)^\dagger})$ is the $(M_1+M_2)\times N$ matrix obtained by vertically stacking the matrices $\mathbf{H}^{(1)^\dagger}$ and $\mathbf{H}^{(2)^\dagger}$. Similarly, $\mathbf{V_X} = V(\mathbf{X}^{(\cdot)})$ is the $(M_1+M_2)\times 1$ matrix obtained by vertically stacking $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$. Transforming the output vector

$$\mathbf{Y}^{\text{new}} = \left(\mathbf{V_H}\mathbf{V_H}^\dagger\right)^{-1} \mathbf{V_H}\mathbf{Y}$$

(using generalized Moore-Penrose inverse) and ignoring the zero gain channels result in the $\min(M,N)$ parallel channels

$$\mathbf{Y}^{\text{new}}(i) = \mathbf{V_X}(i) + \mathbf{N}^{\text{new}}(i), \qquad 1 \le i \le \min(M,N), \tag{5}$$

where $\mathbf{N}^{\text{new}}(i) \sim \mathcal{N}(0,\lambda_i)$ are Gaussian noise terms and λ_i is the i^{th} diagonal term of $\left(\mathbf{V_H}\mathbf{V_H}^\dagger\right)^{-1}$. The noise terms may be correlated across different channels but the correlations are inconsequential since each channel is encoded and decoded separately. Dividing power equally among the $\min(M,N)$ channels, we can achieve

$$\eta(\text{MAC}) \ge \lim_{\rho\to\infty} \frac{1}{\log(\rho)} \sum_{i=1}^{\min(M,N)} \log\left(1 + \frac{\rho}{\min(M,N)}\frac{1}{\lambda_i^2}\right)$$

$$= \lim_{\rho\to\infty} \frac{1}{\log(\rho)} [\min(M,N)\log(\rho) + \sum_{i=1}^{\min(M,N)} \log\left(\frac{1}{\lambda_i^2 \min(M,N)}\right)] = \min(M,N)$$

Note that the channel gains or the exact power allocation does not affect the DoF as long as the SNR on each channel is proportional to ρ.

Combining the converse and the achievability, we have established that $\eta(MAC) = \min(M_1+M_2,N)$.

B. The Broadcast Channel

The BC channel is an example of a MIMO system where cooperation is allowed only between the channel inputs. Let the BC consist of M inputs controlled by the same transmitter and 2 users, each controlling N_1 and N_2 outputs for a total of $N = N_1 + N_2$ outputs. In a similar fashion as the MAC, it is possible to show that by ZF at the BC transmitter, $\min(M,N)$ parallel channels can be created, so that the total DoF are the same as with perfect cooperation between all the users.

$$\eta(\text{BC}) = \eta(\text{MAC}) = \eta(\text{PTP}) = \min(M,N). \tag{6}$$

III. Interference Channel

Consider an $(M_1,N_1),(M_2,N_2)$ interference channel with two transmitters T_1 and T_2, and two receivers R_1 and R_2, where T_1 has a message for R_1 only and T_2 has a message for R_2 only. T_1 and T_2 have M_1 and M_2 antennas respectively. R_1 and R_2 have N_1 and N_2 antennas respectively. We denote the channels for link 1 with N_1xM_1 channel gain matrix $\mathrm{H}^{(1)}$, for link 2 by N_2xM_2 matrix $\mathrm{H}^{(2)}$, for the channel between T_1 and R_2 by N_2xM_1 channel matrix $\mathrm{Z}^{(2)}$, and between T_2 and R_1 by N_1xM_2 matrix $\mathrm{Z}^{(1)}$. We assume that the channels are non-degenerate, i.e., all channel matrices are full rank. Figure 1 shows an illustration of this interference channel. Without loss of generality we arrange the links so that link 1 always has the most number of antennas either at its transmitter or receiver, i.e. $\max(M_1,N_1) \ge \max(M_2,N_2)$.

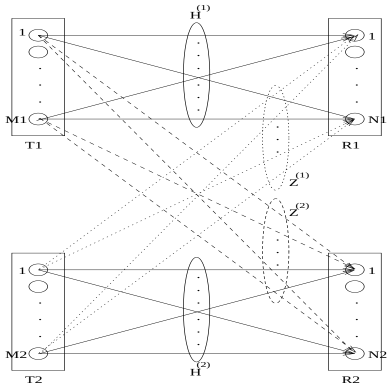


Fig. 1. (M_1,N_1) , (M_2,N_2) Interference channel

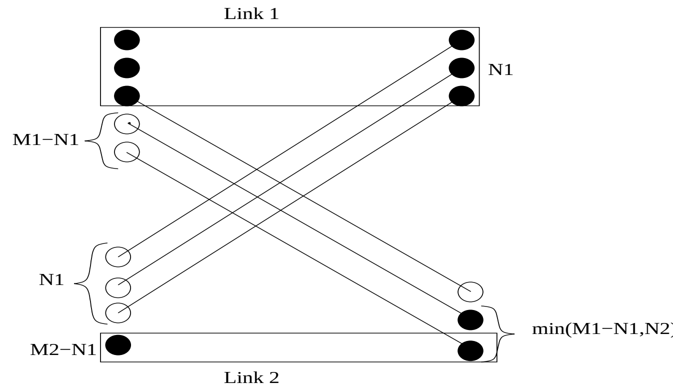


Fig. 2. Achievability proof for (M_1,N_1) , (M_2,N_2) Interference channel when $M_1 \geq M_2, N_1, N_2$

A. Achievability: Innerbound on the Degrees of Freedom

For the $(M_1, N_1), (M_2, N_2)$ interference channel we prove the following innerbound on the available degrees of freedom.

$$\begin{aligned} \eta(\text{INT}) &\geq \min(M_1, N_1) \\ &+ \min(M_2 - N_1, N_2)^+ \ 1(M_1 > N_1) \\ &+ \min(M_2, N_2 - M_1)^+ \ 1(M_1 < N_1), \end{aligned} \quad (7)$$

where 1(.) is the indicator function and $(x)^+ = \max(0, x)$.

1) Sketch of Achievability Proof: According to our model, either $M_1 \geq N_1, M_2, N_2$ or $N_1 \geq M_1, M_2, N_2$. First, we consider the case when $M_1 \geq N_1, M_2, N_2$.

Step 1: From SVD, $Z^{(2)} = U\Lambda V^H$, where U and V are N_2xN_2 and M_1xM_1 unitary matrices respectively and Λ is the diagonal matrix of singular values of $Z^{(2)}$. By applying SVD to $Z^{(2)}$, we decompose the channel into $min(M_1, N_2)$ parallel channels. Therefore, there are $M_1 - N_2$ effective inputs at T_1 that are not connected to R_2, and do not cause any interference to R_2.

Step 2: Similarly, applying SVD to $Z^{(1)}$ creates $min(M_2, N_1)$ parallel connections. There are $(M_2 - N_1)^+$ effective inputs at T_2 that are not connected to R_1, and therefore do not cause any interference with R_1.

Step 3: For link 1, all N_1 effective outputs are used by R_1.

Step 4: T_1 transmits to R_1 using N_1 effective inputs such that at most $(N_1 + N_2 - M_1)^+$ effective inputs that are active are also connected to R_2.

Step 5: Link 2 uses only those effective inputs/outputs that are not connected to an active effective input/output of link 1.

Step 6: Link 1 is left with N_1 effective inputs and N_1 effective outputs, i.e. the number of DoF for link 1 $= N_1$.

Step 7: For link 2, T_2 is left with $(M_2 - N_1)^+$ effective inputs while R_2 is left with $min(M_1 - N_1, N_2)$ effective outputs, i.e. the number of DoF for link 2 $= min(M_2 - N_1, min(M_1 - N_1, N_2))^+ = min(M_2 - N_1, N_2)^+$ since $M_1 \geq M_2$ by assumption. Hence proved.

For the case when $N_1 \geq M_1, M_2, N_2$, the same logic is followed. Then, the total number of DoF is $min(M_1, N_1) + min(M_2, N_2 - M_1)^+$. By adding the results from the two cases, we obtain a general achievable proof of (8). An illustration of this proof is shown in figure 2.

B. Converse: Outerbounds on the Degrees of Freedom

For the $(M_1, N_1), (M_2, N_2)$ interference channel we prove the following outerbound on the available degrees of freedom.

$$\eta(\text{INT}) \leq \min\{_{M_1+M_2, N_1+N_2, \max(M_1,N_2), \max(M_2,N_1)}\}$$

To start with, notice that a trivial outerbound is obtained from the PTP case, i.e. $\eta(\text{INT}) \leq \min(M_1 + M_2, N_1 + N_2)$. Indeed this outerbound coincides with the innerbound when either $\min(M_1, M_2) \geq N_1 + N_2$ or $\min(N_1, N_2) \geq M_1 + M_2$. In general, while the capacity region of the interference channel is not known even with single antennas at all nodes, various outerbounds have been obtained [10]–[12] that have been useful in finding the capacity region in some special cases [13], [14]. Most of the existing outerbounds are for single antenna systems.

For our purpose, we develop a genie based outerbound for MIMO interference channel where the number of antennas at either receiver is $\geq$ the number of transmit antennas at the interfering transmitter, i.e. either $N_1 \geq M_2$ or $N_2 \geq M_1$. This outerbound is the key to the tight converse needed to establish the number of DoF. Note that for this section, since we do not need the assumption that $\max(M_1, N_1) \geq \max(M_2, N_2)$, the proof for the cases $N_1 \geq M_2$ or $N_2 \geq M_1$ is identical.

Theorem 1: For the $(M_1, N_1), (M_2, N_2)$ interference channel with $N_1 \geq M_2$, the sum capacity is bounded above by that of the corresponding (M_1, M_2, N_1) MAC channel with additive noise $\mathbf{N^{(1)}} \sim \mathcal{N}(\mathbf{0}, \mathbf{I_N})$ modified to $\mathbf{N^{(1)}}' \sim \mathcal{N}(\mathbf{0}, \mathbf{K}')$ where

$$\begin{aligned} \mathbf{K}' &= \mathbf{I_N} - \mathbf{Z^{(1)}} \left(\mathbf{Z^{(1)\dagger}} \mathbf{Z^{(1)}}\right)^{-1} \mathbf{Z^{(1)\dagger}} + \alpha \mathbf{Z^{(1)}} \mathbf{Z^{(1)}}^{\dagger}, \\ \alpha &= \min\left(\frac{1}{\sigma^2_{\max}(\mathbf{Z^{(1)}})}, \frac{1}{\sigma^2_{\max}(\mathbf{H^{(2)}})}\right). \end{aligned}$$

Proof:

Let us define

$$\begin{aligned}
\mathbf{N_a^{(1)}} &\sim \mathcal{N}\left(\mathbf{0}, \mathbf{I_N} - \mathbf{Z^{(1)}}\left(\mathbf{Z^{(1)\dagger}}\mathbf{Z^{(1)}}\right)^{-1}\mathbf{Z^{(1)\dagger}}\right) \\
\mathbf{N_b^{(1)}} &\sim \mathcal{N}\left(\mathbf{0}, \mathbf{Z^{(1)}}\left(\mathbf{Z^{(1)\dagger}}\mathbf{Z^{(1)}}\right)^{-1}\mathbf{Z^{(1)\dagger}} - \alpha\mathbf{Z^{(1)}}\mathbf{Z^{(1)}}^{\dagger}\right) \\
\mathbf{N_c^{(1)}} &\sim \mathcal{N}\left(\mathbf{0}, \alpha\mathbf{Z^{(1)}}\mathbf{Z^{(1)}}^{\dagger}\right),
\end{aligned}$$

as three $N \times 1$ jointly Gaussian and mutually independent random vectors. The positive semidefinite property of the respective covariance matrices is easily established from the definition of α.

Without loss of generality we assume

$$\begin{aligned}
\mathbf{N^{(1)}} &= \mathbf{N_a^{(1)}} + \mathbf{N_b^{(1)}} + \mathbf{N_c^{(1)}} \\
\mathbf{N^{(1)}}' &= \mathbf{N_a^{(1)}} + \mathbf{N_c^{(1)}}
\end{aligned}$$

Furthermore, because $\mathbf{N^{(1)}}$ and $\mathbf{N^{(2)}}$ have the same marginal distributions and the capacity of the interference channel does not depend on the correlation between $\mathbf{N^{(1)}}$ and $\mathbf{N^{(2)}}$, the capacity region is not affected if we assume

$$\mathbf{N^{(1)}} = \mathbf{N^{(2)}}.$$

Since a part of the proof is similar to the corresponding proof for the single antenna case, we will summarize the common steps, and emphasize only the part that is unique to MIMO interference channel. Consider any achievable scheme for any rate point within the capacity region of the interference channel, so that R_1 and R_2 can correctly decode their intended messages from their received signals with sufficiently high probability.

Step 1: We replace the original additive noise $\mathbf{N^{(1)}}$ at R_1 with $\mathbf{N^{(1)}}'$ as defined in Theorem 1. We argue that this does not make the capacity region smaller because the original noise statistics can easily be obtained by locally generating and adding noise $\mathbf{N_b^{(1)}}$ at R_1. Therefore, since R_1 was originally capable of decoding its intended message with noise $\mathbf{N^{(1)}}$, it is still capable of decoding its intended message with $\mathbf{N^{(1)}}'$.

Step 2: Suppose that a genie provides R_2 with side information containing the entire codeword $\mathbf{X}^{(1)}$. Since $\mathbf{X}^{(2)}$ is independent of $\mathbf{X}^{(1)}$, R_2 simply subtracts out the interference from its received signal. Thus, the channel $\mathbf{Z}^{(2)}$ can be eliminated without making the capacity region smaller.

Step 3: By our assumption, R_1 can decode its own message and therefore it can subtract $\mathbf{X}^{(1)}$ from its own received signal as well. In this manner, after the interfering signals have been subtracted out we have

$$\begin{aligned}
\mathbf{Y}^{(1)} &= \mathbf{Z^{(1)}}\mathbf{X^{(2)}} + \mathbf{N^{(1)}}', \\
\mathbf{Y}^{(2)} &= \mathbf{H^{(2)}}\mathbf{X^{(2)}} + \mathbf{N^{(2)}}.
\end{aligned}$$

To complete the proof we need to show that if R_2 can decode $\mathbf{X}^{(2)}$ then so can R_1. This would imply that R_1 can decode both messages, hence giving us the MAC outer bound.

Step 4: Without loss of generality, let us perform SVD $\mathbf{H}^{(2)} = \mathbf{U}^{(2)}\mathbf{\Lambda^{(2)}}\mathbf{V^{(2)}}$ on the channel between T_2 and R_2. This is a lossless operation that leads to:

$$\mathbf{Y}^{(2)\text{new}} = \mathbf{X^{(2)}}^{\text{new}} + \left(\mathbf{\Lambda^{(2)}}\right)^{-1}\mathbf{N^{(2)}}, \tag{8}$$

where $\mathbf{X^{(2)}}^{\text{new}} = \mathbf{V^{(2)}}\mathbf{X^{(2)}}$.

To save space we allow some notation abuse as we use generalized inverse and ignore the terms that correspond to zero diagonal channel gains in $\mathbf{\Lambda^{(2)}}$. Note that these channels are useless for R_2. Also, we use the same symbol for rotated versions of noise that are statistically equivalent.

Step 5: Next, we show that R_1 can obtain a stronger channel to $\mathbf{X^{(2)}}^{\text{new}}$ so that if R_2 can decode it, so can R_1. To this end, let R_1 use ZF to obtain:

$$\begin{aligned}
\mathbf{Y}^{(1)\text{new}} &= \mathbf{X^{(2)}}^{\text{new}} + \mathbf{V^{(2)}}\left(\mathbf{Z^{(1)\dagger}}\mathbf{Z^{(1)}}\right)^{-1}\mathbf{Z^{(1)\dagger}}\mathbf{N^{(1)}}', \\
&= \mathbf{X^{(2)}}^{\text{new}} + \alpha\mathbf{N^{(2)}}
\end{aligned}$$

Now both R_1 and R_2 have a diagonal channel with input $\mathbf{X^{(2)}}^{\text{new}}$ and uncorrelated additive white noise components on each diagonal channel. Moreover, the strongest channel for R_2 has noise $\frac{1}{\sigma^2_{\max}(\mathbf{H^{(2)}})}$. However the noise on any channel for R_1 is only α which is smaller. Thus, we argue once again that R_1 can locally generate noise and add it to its received signal to create a statistically equivalent noise signal as seen by R_2. In other words, R_1 has a less noisy channel to T_2 and therefore can decode any signal that R_2 can. Since R_1 can decode T_1's message by assumption, we have the MAC outerbound. ■

The previous theorem leads directly to the following corollary:

Corollary 1: For the $(M_1, N_1), (M_2, N_2)$ interference channel the number of spatial degrees of freedom $\eta(\text{INT}) \leq \max(M_2, N_1)$.

Proof: If $M_2 \leq N_1$ the sum capacity of the interference channel is upperbounded by the multiple access channel with N_1 receive antennas. Therefore, for $M_2 \leq N_1$ we must have $\eta(\text{INT}) \leq N_1$. Now, if $M_2 > N_1$, then let us add more antennas to receiver 1 so that it has a total of M_2 receive antennas. Additional receive antennas cannot hurt, so the converse argument is not violated. However, with M_2 receive antennas at receiver 1, once again the multiple access upperbound applies to the new interference channel. The number of degrees of freedom is therefore upperbounded as $\eta(\text{INT}) \leq M_2$ when $M_2 > N_1$. Combining the two cases, we have the result of the corollary $\eta(\text{INT}) \leq \max(M_2, N_1)$. ■

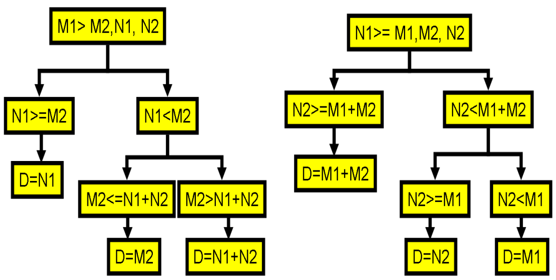
Simply by switching the arguments to user 2 instead of user 1, Corollary 1 leads to another upperbound: $\eta(\text{INT}) \leq \max(M_1, N_2)$ that holds for all M_1, M_2, N_1, N_2. Combining the two upperbounds of the Corollary and the trivial PTP upperbounds we have the converse result. ■

Finally we show that the achievable innerbound and the converse outerbound are always tight. The following theorem presents the main result of the paper.

Theorem 2: For the $(M_1, N_1), (M_2, N_2)$ interference channel the number of spatial degrees of freedom

$$\begin{aligned}\eta(\text{INT}) &= \min(M_1, N_1) \\ &+ \min(M_2 - N_1, N_2)^+ \ 1(M_1 > N_1) \\ &+ \min(M_2, N_2 - M_1)^+ \ 1(M_1 < N_1) \\ &= \min\{M_1+M_2, N_1+N_2, \max(M_1,N_2), \max(M_2,N_1)\}\end{aligned}$$

Proof: The proof is found by verifying directly that the number of degrees of freedom obtained from the inner and outerbounds always match. The resulting number D from the $\eta(\text{INT})$ inner and outerbounds is listed for all cases in the following figure. ■



Thus we have the exact number of degrees of freedom for all possible M_1, M_2, N_1, N_2. Some examples are provided in the following table.

(M_1, N_1)	(M_2, N_2)	$\eta(INT)$
$(1, 1)$	$(1, 1)$	1
$(1, 2)$	$(1, 2)$	2
$(2, 1)$	$(2, 1)$	2
$(1, 2)$	$(2, 1)$	1
$(3, 2)$	$(2, 3)$	2
$(2, 3)$	$(2, 3)$	3
$(2, 3)$	$(1, 3)$	3
$(2, 2)$	$(3, 2)$	2
(n, m)	(m, n)	$\min(m, n)$
(m, n)	(m, n)	$\min(2m, n)(n \geq m)$

A couple of observations can be made about the spatial degrees of freedom. First, there is a reciprocity in that $\eta(INT)$ is unaffected if M_1 and M_2 are switched with N_1 and N_2 respectively. In other words, the degrees of freedom are unaffected if the directions of the messages are reversed. However, notice that $\eta(INT)$ may change if only M_1 and N_1 are switched while M_2 and N_2 are not switched. Finally from the constructive achievability proof one can see that the available degrees of freedom can be divided among the two users in all possible ways so that the sum is $\eta(INT)$ and the individual degree of freedom allocations are within the individual maxima of $\max(M_1, N_1)$ for user 1 and $\max(M_2, N_2)$ for user 2.

IV. Effect of Transmit Cooperation on the Number of Degrees of Freedom

Comparing the interference channel and the BC channel obtained by full cooperation between the transmitters, it is clear that the available DoF are severely limited by the lack of transmitter cooperation in the interference channel. As an example, consider the interference channel with $(M_1, N_1) = (n, 1)$ and $(M_2, N_2) = (1, n)$. From the preceding section we know there is only one available degree of freedom in this channel. However, if full cooperation between the transmitters is possible the resulting BC channel has $(M, N_1, N_2) = (n + 1, 1, n)$. The number of DoF is now $n + 1$. Therefore, transmitter cooperation would seem highly desirable. Rather surprisingly, it has been shown recently [7] that for the $(1, 1), (1, 1)$ interference channel, allowing the transmitters to cooperate through a wireless link between them (even with full duplex operation), does not increase DoF. For MIMO interference channels, as suggested by the example above, the potential benefits of cooperation are even stronger and it is not known if transmitter cooperation can increase DoF. The capacity results of [7] do not seem to allow direct extensions to MIMO interference channels.

To gain insights into the cost and benefits of cooperation in a MIMO interference channel, we consider a specific scheme where transmitters first share their information in a full duplex mode as a MIMO channel (step 1) and subsequently transmit together as BC channel. We will refer to this scheme as the share-and-transmit scheme.

A. Degrees of Freedom with Share-and-Transmit

Consider an $(M, N), (M, N)$ interference channel ($M \leq N$). Also assume that each transmitter is sending information with rate R. Note that while we make the preceding simplifying assumptions for simplicity of exposition, the following analysis and the main result extend directly to the general case of unequal number of antennas and unequal rates.

From (8), we know that the number of DoF for this intereference channel with no transmitter cooperation is $\min(M, N) + \min(M, N - M)^+ = M + \min(M, N - M)^+$. For the share-and-transmit scheme, we compute DoF as follows. We first find the capacity of the sharing link C_s and the capacity of transmission C_t. Then, we find the total capacity of the system C by evaluating the total amount of data transmitted divided by the total time it requires to transmit this data, i.e.

$$C = \frac{2R}{\frac{R}{C_s} + \frac{2R}{C_t}} \quad . \tag{9}$$

Dividing by log(SNR) where SNR is large, we obtain the total number of DoF as

$$\lim_{SNR \to \infty} \frac{C}{\log SNR} = \frac{2}{\frac{1}{DOF(sharing)} + \frac{2}{DOF(transmit)}} \ . \tag{10}$$

The number of DoF for the sharing link is that of MIMO PTP channel with M transmit and receive antennas $= min(M, M) = M$. After transmitters share their information, they can fully cooperate as a $(2M, N, N)$ BC channel. The number of DoF for this channel is $\min(2M, 2N) = 2\min(M, N)$. Therefore (10), which gives the total number of DoF for the share-and-transmit scheme, becomes

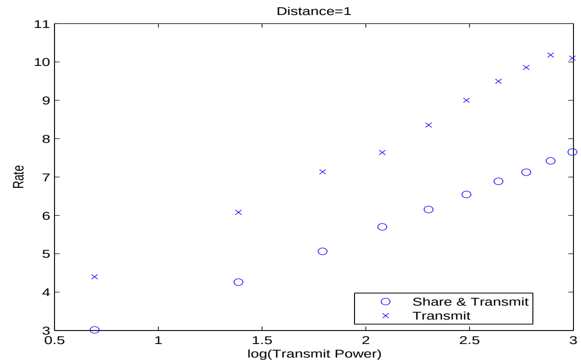


Fig. 3. Rate vs log(Transmit Power) with same distance.

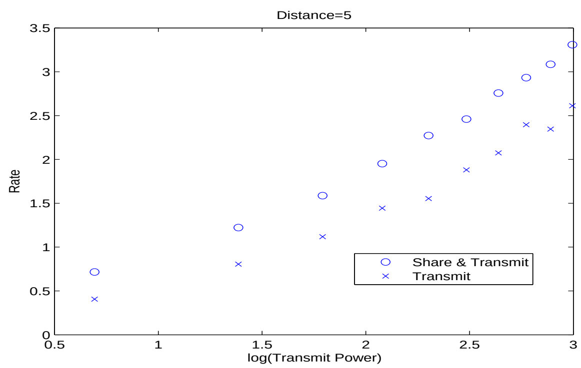


Fig. 4. Rate vs log(Transmit Power) with $5\times$ distance for transmitting.

$\frac{2M\min(M,N)}{M+\min(M,N)} = M$. Note that,

$$M + \min(M, N-M)^+ \geq M. \tag{11}$$

Therefore, we conclude that (for this specific scheme) transmitter cooperation in the high SNR regime does not provide any advantage to the number of DoF in the MIMO interference channel.

V. Simulation Results

In this section, we verify the result discussed in the previous section, and discuss the effect of transmitter cooperation when the sharing links between the transmitters are stronger than the transmission links. For simplicity, we consider a $(4,1),(4,1)$ interference channel, and plot the rate versus the logarithm of the transmit power. Note that we assume the noise to be 0-mean unit-variance Gaussian additive noise.

The share-and-transmit scheme is implemented as explained in section IV-A. For the no cooperation scheme, T_1 has a message for R_1 only and dedicates its available power to its link with R_1. The same is true for T_2 and R_2. Note that since the transmit signal space is much larger than the receive signal space, T_1 can decompose its channel with R_1 as well as its channel with R_2 to create one non-interfering link to R_1 and another to R_2. T_2 is able to achieve this as well, and each receiver can then decode its message without interference.

In fig. 3, we fix the distance between each transmitter and receiver to be equal to that between T_1 and T_2. In this case, the transmitters allocate the same resources to their sharing link as to their transmission links. Fig. 3 indicates that the share-and-transmit scheme always has a lower rate for the same transmit power than the no cooperation scheme, which agrees with our result in section IV.

In fig. 4, the distance between each transmitter and receiver is $5\times$ that between T_1 and T_2. Note that in this case, the sharing link is stronger than the transmission links since it does not suffer any path loss whereas the transmission links do. Fig. 4 shows that share-and-transmit scheme outperforms the no cooperation scheme. As expected, when the sharing link is stronger, cooperation between transmit nodes results in performance improvement over the no cooperation scheme. Note that while our simulations are for the interference channel, similar results have been obtained for the MAC in [15].

VI. Conclusions

We investigate the DoF for the MIMO interference channel. The distributed nature of the antennas significantly limits DoF. For an interference channel with a total of N transmit antennas and a total of N receive antennas, the available number of DoF can vary from N to 1 based on how the antennas are distributed among the two transmitters and receivers. Through an example of a share-and-transmit scheme, we show how the gains of transmitter cooperation are entirely offset by the cost of enabling that cooperation so that the available DoF are not increased. Our result is in a sense a negative result, because similar to [8] it shows that on the MIMO interference channel there is nothing beyond zero forcing as far as spatial multiplexing is concerned.

An exception to this pessimistic inference is recently shown by Maddah-Ali, Motahari and Khandani in [1] for the two user MIMO X channel with three antennas at all nodes. The MIMO X channel is physically identical to the MIMO interference channel. However, in the interference channel there are only two messages (M_{11} from transmitter 1 to receiver 1 and M_{22} from transmitter 2 to receiver 2) whereas in the X channel there are two additional messages, M_{21} from transmitter 1 to receiver 2 and M_{12} from transmitter 2 to receiver 1. Maddah-Ali, Motahari and Khandani propose a novel scheme, that we refer to as the MMK scheme. The MMK scheme combines zero forcing with dirty paper encoding and successive decoding and is shown to provide 4 degrees of freedom with only 3 antennas at all nodes. The result of [1] can be further strengthened to achieve 4 degrees of freedom on the $(2,3,2,3)$ MIMO X channel as well as on the $(3,2,3,2)$ MIMO X channels as well. The interesting conclusion is that while we prove that zero forcing is optimal in terms of degrees of freedom on the interference channel, it is not optimal on the two user MIMO X channel where additional degrees of freedom can be obtained by a combination of zero forcing with dirty paper coding and successive decoding [1].

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