Degrees of Freedom in Adaptive Modulation: A Unified View

Seong Taek Chung, *Student Member, IEEE,* and Andrea J. Goldsmith, *Senior Member, IEEE*

***Abstract*—We examine adaptive modulation schemes for flat-fading channels where the data rate, transmit power, and instantaneous BER are varied to maximize spectral efficiency, subject to an average power and BER constraint. Both continuous-rate and discrete-rate adaptation are considered, as well as average and instantaneous BER constraints. We find the general form of power, BER, and data rate adaptation that maximizes spectral efficiency for a large class of modulation techniques and fading distributions. The optimal adaptation of these parameters is to increase the power and data rate and decrease the BER as the channel quality improves. Surprisingly, little spectral efficiency is lost when the power or rate is constrained to be constant. Hence, the spectral efficiency of adaptive modulation is relatively insensitive to which degrees of freedom are adapted.**

***Index Terms*—Adaptive modulation, communication systems, fading channels, spectral efficiency.**

I. INTRODUCTION

ADAPTIVE MODULATION is a promising technique to increase the data rate that can be reliably transmitted over fading channels. For this reason some form of adaptive modulation is being proposed or implemented in many next generation wireless systems. The basic premise of adaptive modulation is a real-time balancing of the link budget in flat fading through adaptive variation of the transmitted power level, symbol transmission rate, constellation size, BER, coding rate/scheme, or any combination of these parameters [1]–[7], [9], [13], [16]. Thus, without wasting power or sacrificing BER, these schemes provide a higher average link spectral efficiency (bps/Hz) by taking advantage of flat fading through adaptation. Good performance of adaptive modulation requires accurate channel estimation at the receiver and a reliable feedback path between the receiver and transmitter. The impact of estimation error and delay on adaptive modulation schemes has been studied in [7]–[9].

Adaptive modulation provides many parameters that can be adjusted relative to the channel fading, including data rate, transmit power, instantaneous BER, symbol rate, and channel code rate or scheme. The question therefore arises as to which of these parameters should be adapted to obtain the best performance. Results from [6] indicate that the Shannon capacity of a flat-fading channel is achieved by varying both transmission rate and power, and this capacity can also be achieved by varying the transmit power alone [14]. Moreover, [6] also indicates that varying both power and rate leads to a negligibly higher capacity over varying just the rate alone. However, Shannon capacity assumes that the BER is arbitrarily small, coding schemes are random and of unbounded length and complexity, and there is no delay constraint. Therefore capacity results do not necessarily yield insight into the best adaptive schemes to use under more practical constraints. There is much recent work on adaptive modulation that varies one or two modulation parameters. In particular, [1]–[3], [6], and [7] investigate adapting power and/or rate, [4] and [9] investigate adapting rate and coding, and [5] investigates adapting power, rate, and instantaneous BER. However, no unified study on the tradeoffs in adapting all combinations of different modulation parameters has been previously undertaken.

In this paper we provide a systematic study on the increase in spectral efficiency obtained by optimally varying combinations of the transmission rate, power, and instantaneous BER. We assume that the resulting adaptive modulation schemes are subject to an average power and BER constraint. We do not consider symbol rate adaptation since it is difficult to implement in real systems. The effect of adaptive channel coding is also not considered. We first analyze adaptive modulation with continuous rate adaptation, where the set of signal constellations is unrestricted, and then consider the more practical scenario where only a discrete finite set of constellations is available. Analysis is done for both an average and an instantaneous BER constraint. Our goal is to determine the impact on spectral efficiency of adapting various modulation parameters under different constellation restrictions and BER constraints, for a large class of modulation techniques and fading distributions.

The remainder of this paper is organized as follows. The Section II describes the system model, including the average power and BER constraints. Section III presents the BER approximations used to derive the optimal adaptive modulation scheme. We derive the optimal rate, power, and BER adaptation strategies under different constellation restrictions and BER constraints in Section IV. Numerical results and plots of spectral efficiency, optimal power adaptation, optimal BER adaptation, and optimal rate adaptation are presented in Section V. We examine constant power and constant rate adaptation in Section VI. Conclusions will be given in Section VII.

II. SYSTEM MODEL

In this section we present our system model and notation, following that of [7]. The system model is illustrated in Fig. 1. We assume a discrete-time channel with stationary and ergodic time-varying gain $\sqrt{g[i]}$ and additive white Gaussian noise $n[i]$.

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The authors are with the Department of Electrical Engineering, Stanford University, Stanford, CA 94305 USA (e-mail: stchung@dsl.stanford.edu; andrea@systems.stanford.edu).

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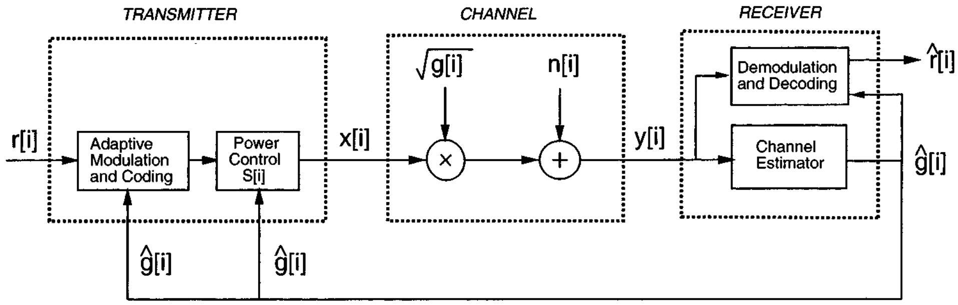

Fig. 1. System model.

Let $\overline{S}$ denote the average transmit signal power, σ^2 denote the variance of $n[i]$, B denote the received signal bandwidth, and $\overline{g}$ denote the average channel power gain. With appropriate scaling of $\overline{S}$, we can assume that $\overline{g} = 1$. There is a feedback path from the receiver to the transmitter for sending channel estimates. This path is assumed to be instantaneous and error-free, so the channel gain estimate $\hat{g}[i] = g[i]$. The impact of feedback delay and channel estimation error on the performance of adaptive modulation has been studied in [7]. For a constant transmit power $\overline{S}$, the instantaneous received SNR is $\gamma[i] = \overline{S}g[i]/\sigma^2$. We denote the transmit power at time i, which is a function of $\gamma[i]$, by $S\big(\gamma[i]\big)$. The received SNR at time i is then $\gamma[i]\big(S(\gamma[i])/\overline{S}\big)$. Since $g[i]$ is stationary, the distribution of $\gamma[i]$ is independent of i, and we denote this distribution by $p(\gamma)$. When the context is clear, we will omit the time reference i relative to γ and $S(\gamma)$. We also assume ideal coherent phase detection.

The parameters that can be adapted at the transmitter include the transmission rate, power, and instantaneous BER. We will consider both continuous rate adaptation (C-Rate), where the set of signal constellations is unrestricted, as well as discrete rate adaptation (D-Rate), where only a discrete finite set of N constellations is available. For the D-Rate case the rate region boundaries $\{\gamma_i\}_{i=0}^{N-1}$ define the range of γ values over which the different constellations are transmitted. Specifically, we assign one signal constellation and a corresponding data rate of k_i bits/symbol to each rate region $[\gamma_i, \gamma_{i+1})$ $(0 \le i \le N-1)$, where $\gamma_N = \infty$. When the instantaneous SNR γ falls within a given region, the associated signal constellation is transmitted. No signal is transmitted if $\gamma \le \gamma_0$. Thus, γ_0 serves as a cutoff SNR below which the channel is not used. We will find that in the C-Rate case there is also an optimized cutoff value, γ_0, below which the channel is not used. Thus, for both continuous and discrete rate adaptation, when the channel quality is significantly degraded, the channel should not be used.

The spectral efficiency of our modulation scheme equals its average data rate per unit bandwidth (R/B). When we send $k(\gamma) = \log_2[M(\gamma]$ (bits/symbol), the instantaneous data rate is $k(\gamma)/T_s$ (bps), where T_s is the symbol time. Assuming Nyquist data pulses $(B = 1/T_s)$, for continuous rate adaptation the spectral efficiency is given by

$$\frac{R}{B} = \int_0^\infty k(\gamma)p(\gamma)d\gamma \text{ bits/s/Hz} \tag{1}$$

and for discrete rate adaptation it is given by

$$\frac{R}{B} = \sum_{i=0}^{N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma)d\gamma \text{ bps/Hz.} \tag{2}$$

The rate adaptation $k(\gamma)$ is typically parameterized by the average transmit power $\overline{S}$ and the BER of the modulation technique, as discussed in more detail in Section IV.

We assume an average transmit power constraint given by

$$\int_0^\infty S(\gamma)p(\gamma)d\gamma \le \overline{S}. \tag{3}$$

For the BER, we assume either an average (A-BER) or an instantaneous (I-BER) constraint.[1] The instantaneous BER constraint implies that the system must maintain a constant probability of bit error for each fading value. This is more restrictive than the average constraint. There are two possible definitions for the average BER constraint:

$$\overline{\text{BER}} = \frac{E[\text{number of error bits per transmission}]}{E[\text{number of bits per transmission}]} \tag{4}$$

or

$$\overline{\text{BER}} = E\left[\frac{\text{number of error bits per transmission}}{\text{number of bits per transmission}}\right]. \tag{5}$$

Definition (4) is slightly better than (5) since, for a stationary and ergodic fading process, (4) gives a more accurate measure of the total number of bits received in error divided by the total number of bits received. We will therefore consider only the first definition in deriving the optimal power and rate adaptation.

When applied to continuous rate adaptation (4) becomes

$$\overline{\text{BER}} = \frac{\int_0^\infty \text{BER}(\gamma)k(\gamma)p(\gamma)d\gamma}{\int_0^\infty k(\gamma)p(\gamma)d\gamma} \tag{6}$$

and when applied to discrete rate adaptation (4) becomes

$$\overline{\text{BER}} = \frac{\sum_{i=0}^{N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} \text{BER}(\gamma)p(\gamma)d\gamma}{\sum_{i=0}^{N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma)d\gamma}. \tag{7}$$

[1]The average BER is typically used to characterize quality for voice communications in fast fading. For data systems, the frame error rate is typically the quantity of interest, and this can be computed from the instantaneous BER.

III. BER APPROXIMATIONS

In order to obtain the optimal power and rate adaptation for different modulation schemes, for each modulation technique we need an expression for its BER in AWGN that is easily inverted with respect to rate and power. Unfortunately, for most nonbinary modulation techniques (e.g., MQAM and MPSK) an exact expression for BER is hard to find. Often the BER with Gray bit mapping [10] at high SNR's is approximated as the symbol error rate (SER) divided by the number of bits per symbol $(\log_2 M)$. Closed-form expressions for SER of MQAM and MPSK as functions of transmission rate and power can be found in [10]. But these expressions are neither easily invertible nor easily differentiable in their arguments. These properties are needed for adaptive modulation design. Therefore, we now introduce new tight BER approximations for several modulation techniques in AWGN that can be easily differentiated and inverted. We later use these approximations to derive the optimal power and rate adaptation of the corresponding adaptive modulation techniques.

A. BER Approximation for MQAM

The expression for the BER of square MQAM with Gray bit mapping in AWGN as a function of received SNR $\gamma(S(\gamma)/\overline{S})$ and constellation size $M = 2^{k(\gamma)}$ is approximately [10], [11]

$$\mathrm{BER}_{\mathrm{MQAM}}(\gamma) \approx \frac{2}{k(\gamma)}\left(1 - \frac{1}{\sqrt{2^{k(\gamma)}}}\right) \times \mathrm{erfc}\left(\sqrt{1.5\frac{\gamma\frac{S(\gamma)}{\overline{S}}}{2^{k(\gamma)}-1}}\right) \quad (8)$$

where the approximation is tightest at high SNRs. This expression is not easily differentiable or invertible in its power $S(\gamma)$ or rate $k(\gamma)$, so we now consider a different approximation with these properties. We find an approximation for BER tight to within 1 dB for $k(\gamma) \geq 2$ and BER $\leq 10^{-3}$ as[2]

$$\mathrm{BER}_{\mathrm{MQAM}}(\gamma) \approx 0.2\exp\left[\frac{-1.6\gamma\frac{S(\gamma)}{\overline{S}}}{2^{k(\gamma)}-1}\right]. \quad (9)$$

In Fig. 2 the tightness of this BER approximation (9) to the standard formula (8) is shown.

B. BER Approximation for MPSK

The BER expression for MPSK in AWGN with Gray bit mapping and $M = 2^{k(\gamma)}$ is commonly approximated as [10]

$$\mathrm{BER}_{\mathrm{MPSK}}(\gamma) \approx \frac{1}{k(\gamma)}\mathrm{erfc}\left(\sqrt{\gamma\frac{S(\gamma)}{\overline{S}}}\sin\left(\frac{\pi}{2^{k(\gamma)}}\right)\right). \quad (10)$$

In Fig. 3 we show that the approximation (10) provides an excellent fit to the exact BER for MPSK given in [12, Table II].

[2]In [7, eq. (17)] a similar expression was used with 1.5 in the exponent instead of 1.6. This approximation is looser.

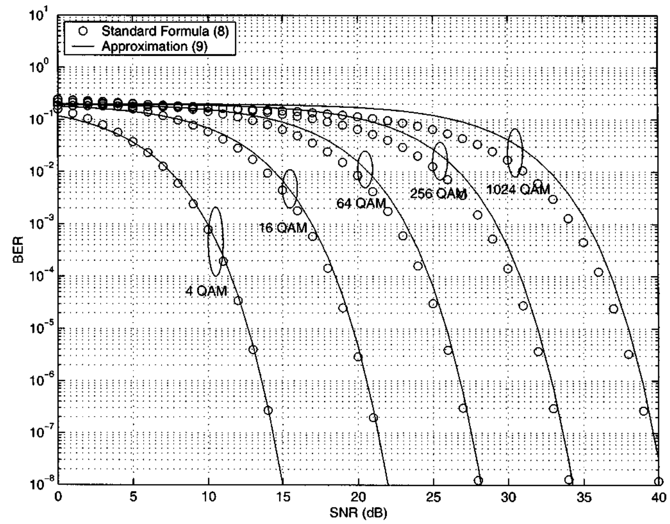

Fig. 2. BER approximations for MQAM.

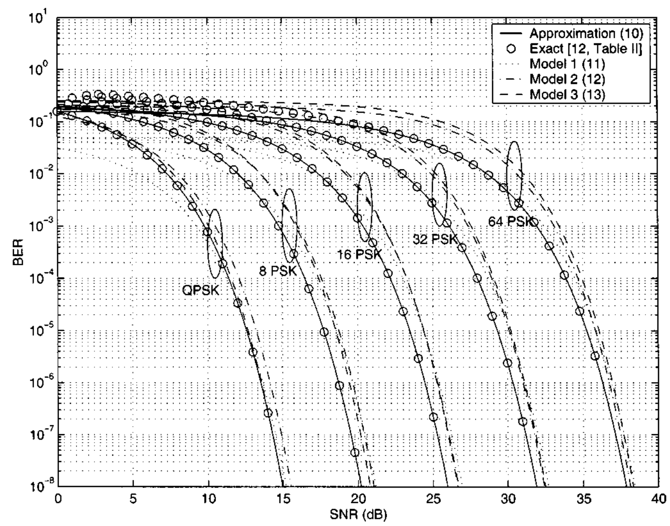

Fig. 3. BER approximations for MPSK using Models 1, 2, 3.

However, the approximation (10) is not easily invertible and differentiable in its arguments. Therefore we now consider BER approximations that have such properties.

By curve-fitting, we find three different BER approximations that are valid for $k(\gamma) \geq 2$ within 1.5 dB of error for $\mathrm{BER} \leq 10^{-3}$. The bounds are given as

$$\text{Model 1:}\ \mathrm{BER}_{\mathrm{MPSK}}(\gamma) \approx 0.05\exp\left[\frac{-6\gamma\frac{S(\gamma)}{\overline{S}}}{2^{1.9k(\gamma)}-1}\right]. \quad (11)$$

$$\text{Model 2:}\ \mathrm{BER}_{\mathrm{MPSK}}(\gamma) \approx 0.2\exp\left[\frac{-7\gamma\frac{S(\gamma)}{\overline{S}}}{2^{1.9k(\gamma)}+1}\right]. \quad (12)$$

$$\text{Model 3:}\ \mathrm{BER}_{\mathrm{MPSK}}(\gamma) \approx 0.25\exp\left[\frac{-8\gamma\frac{S(\gamma)}{\overline{S}}}{2^{1.94k(\gamma)}}\right]. \quad (13)$$

These approximations are plotted in Fig. 3 as "Model 1," "Model 2," and "Model 3," where they are shown to be close to the exact BER in [12, Table II], especially at high SNRs.

C. Generic Form of BER Approximations

The approximations (9) and (11)–(13) for all modulation techniques can be written in the following generic form:

$$\mathrm{BER}(\gamma) \approx c_1 \exp\left[\frac{-c_2\gamma \frac{S(\gamma)}{\overline{S}}}{f\big(k(\gamma)\big)}\right] \tag{14}$$

where

$$f\big(k(\gamma)\big) = 2^{c_3 k(\gamma)} - c_4 \tag{15}$$

c_1, c_2, and c_3 are positive fixed constants, and c_4 is a real constant. Note that for the BER bounds on MQAM and MPSK discussed above $c_4 = 1, -1$, or 0, and $1 \leq c_3 \leq 2$. The generic expression (14) is valid for MQAM and MPSK to within 1.5 dB of error for $k(\gamma) \geq 2$ and $\mathrm{BER} \leq 10^{-3}$.

IV. Optimal Rate, Power, and BER Adaptation

In this section we determine the optimal rate, power, and BER adaptation for maximizing spectral efficiency in the following four cases: continuous rate adaptation with an average BER constraint (C-Rate A-BER), continuous rate adaptation with an instantaneous BER constraint (C-Rate I-BER), discrete rate adaptation with an average BER constraint (D-Rate A-BER), and discrete rate adaptation with an instantaneous BER constraint (D-Rate I-BER). Our analysis here applies for any fading distribution. Clearly the instantaneous BER constraints are special cases of the average BER constraints, and will therefore have a lower spectral efficiency.

A. Continuous Rate and Average BER (C-Rate A-BER)

We now derive the optimal continuous rate, power, and BER adaptation to maximize spectral efficiency (1) subject to the average power constraint (3) and the average BER constraint (6). This is a standard constrained optimization problem, which we solve using the Lagrange method. The Lagrange equation is

$$\begin{aligned} J\big(k(\gamma), S(\gamma)\big) = &\int_0^\infty k(\gamma)p(\gamma)d\gamma \\ &+ \lambda_1\left[\int_0^\infty \mathrm{BER}(\gamma)k(\gamma)p(\gamma)d\gamma \right. \\ &\left. \qquad - \overline{\mathrm{BER}}\int_0^\infty k(\gamma)p(\gamma)d\gamma\right] \\ &+ \lambda_2\left[\int_0^\infty S(\gamma)p(\gamma)d\gamma - \overline{S}\right]. \end{aligned} \tag{16}$$

The optimal rate and power adaptation should satisfy

$$\frac{\partial J}{\partial k(\gamma)} = 0 \text{ and } \frac{\partial J}{\partial S(\gamma)} = 0 \tag{17}$$

with the additional constraint that $k(\gamma)$ and $S(\gamma)$ are nonnegative for all γ. Let $f\big(k(\gamma)\big)$ be as defined in (15). Then using the generic BER expression (14) in (16) and solving (17) we get that the power and BER adaptation that maximize spectral efficiency satisfy

$$\frac{S(\gamma)}{\overline{S}} = \max\left[\frac{f\big(k(\gamma)\big)}{\frac{\partial f(k(\gamma))}{\partial k(\gamma)}\lambda_2\overline{S}}\big(\lambda_1\overline{\mathrm{BER}} - 1\big) - \frac{f\big(k(\gamma)\big)^2}{c_2\gamma\frac{\partial f(k(\gamma))}{\partial k(\gamma)}k(\gamma)}, 0\right] \tag{18}$$

for nonnegative $k(\gamma)$, and

$$\mathrm{BER}(\gamma) = \frac{\lambda_2\overline{S}f\big(k(\gamma)\big)}{\lambda_1 c_2\gamma k(\gamma)}. \tag{19}$$

Moreover, from (14), (18), and (19) we get that the optimal rate adaptation $k(\gamma)$ is either zero or the nonnegative solution of

$$\frac{\lambda_1\overline{\mathrm{BER}} - 1}{\frac{\partial f(k(\gamma))}{\partial k(\gamma)}\lambda_2\overline{S}} - \frac{f\big(k(\gamma)\big)}{c_2\gamma\frac{\partial f(k(\gamma))}{\partial k(\gamma)}k(\gamma)} = \frac{1}{\gamma c_2}\ln\left[\frac{\lambda_1 c_1 c_2\gamma k(\gamma)}{\lambda_2\overline{S}f\big(k(\gamma)\big)}\right]. \tag{20}$$

The values of $k(\gamma)$ and the Lagrangians λ_1 and λ_2 are found through a numerical search such that the average power (3) and BER (6) constraints are satisfied. More details on the numerical search process are described in Appendix A. Numerical results for the resulting adaptive policies and corresponding spectral efficiency are given in Section V.

B. Continuous Rate and Instantaneous BER (C-Rate I-BER)

We now derive the optimal continuous rate and power adaptation to maximize spectral efficiency (1) subject to the average power constraint (3) and an instantaneous BER constraint $\mathrm{BER}(\gamma) = \overline{\mathrm{BER}}$. This case was investigated in [7] for MQAM, and we now extend that analysis to more general modulations using our generic BER expression (14). We can invert (14) to express $k(\gamma)$ as a function of the power control $S(\gamma)$ and the fixed bit-error-rate $\overline{\mathrm{BER}}$ as

$$k(\gamma) = \begin{cases} \dfrac{1}{c_3}\log_2\left[c_4 - \dfrac{c_2\gamma}{\ln\left(\frac{\overline{\mathrm{BER}}}{c_1}\right)}\dfrac{S(\gamma)}{\overline{S}}\right], & S(\gamma) \geq 0,\ k(\gamma) \geq 0 \\ 0, & \text{else.} \end{cases} \tag{21}$$

To maximize spectral efficiency (1) we create the Lagrangian

$$J\big(S(\gamma)\big) = \int_0^\infty k(\gamma)p(\gamma)d\gamma + \lambda\left[\int_0^\infty S(\gamma)p(\gamma)d\gamma - \overline{S}\right]. \tag{22}$$

The optimal power adaptation must be nonnegative and satisfy

$$\frac{\partial J}{\partial S(\gamma)} = 0, \quad S(\gamma) \geq 0, \quad k(\gamma) \geq 0. \tag{23}$$

Solving (23) for $S(\gamma)$ with (21) for $k(\gamma)$ yields the optimal power adaptation

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} -\dfrac{1}{c_3(\ln 2)\lambda\overline{S}} - \dfrac{1}{\gamma K} & S(\gamma)\geq 0, k(\gamma)\geq 0, \\ 0 & \text{else} \end{cases} \tag{24}$$

where

$$K = -\frac{c_2}{c_4 \ln\left(\frac{\overline{\mathrm{BER}}}{c_1}\right)}. \tag{25}$$

The power adaptation (24) can be written in the more simplified form

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \mu - \dfrac{1}{\gamma K}, & S(\gamma)\geq 0,\ k(\gamma)\geq 0 \\ 0, & \text{else.} \end{cases} \tag{26}$$

The constant μ in (26) is obtained from the average power constraint (3). Although the analytical expression for the optimal power adaptation (26) looks simple, its behavior is highly dependent on the c_4 values in the BER approximation (14)–(15). In the BER approximations of Section III, c_4 takes values 1, -1 or 0. We now investigate the behavior of the optimal power and rate adaptation scheme for each of these values.

1) BER Approximations (9) and (11): $c_4 = 1$*:* When $c_4 = 1$, K is positive[3]. Thus μ must be positive for $S(\gamma)/\overline{S} = \mu - (1/\gamma K)$ to be nonnegative. Moreover, for K positive $k(\gamma)\geq 0$ for any $S(\gamma)\geq 0$. With $\mu\geq 0$ (26) can be expressed as

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \dfrac{1}{\gamma_0 K} - \dfrac{1}{\gamma K}, & S(\gamma)\geq 0 \\ 0, & \text{else} \end{cases} \tag{27}$$

where $\gamma_0 \geq 0$ is a cutoff fade depth below which no signal is transmitted. Note that the cutoff is dictated by the positivity constraint on power $\big(S(\gamma)\geq 0\big)$. The cutoff value γ_0 must satisfy the average power constraint (3) as

$$\int_{\gamma_0}^{\infty} \frac{1}{K}\left[\frac{1}{\gamma_0} - \frac{1}{\gamma}\right] p(\gamma)d\gamma = 1. \tag{28}$$

The optimal power adaptation (27) is a *waterfilling* in power: more power is used as the channel quality increases above the optimized cutoff fade depth γ_0. The same form of optimal power adaptation was previously derived in [7], since the BER approximation used in [7] is of the same form as (9) and (11). The optimal rate adaptation, obtained by substituting (27) into (21), is

$$k(\gamma) = \begin{cases} \dfrac{1}{c_3}\log_2\left(\dfrac{\gamma}{\gamma_0}\right), & \gamma\geq\gamma_0 \\ 0, & \text{else.} \end{cases} \tag{29}$$

2) BER Approximation (12): $c_4 = -1$*:* When $c_4 = -1$, K is negative. From (21), with K negative we must have $\mu\geq 0$ in (26) to make $k(\gamma)\geq 0$. Then the optimal power adaptation such that $S(\gamma)\geq 0$ and $k(\gamma)\geq 0$ becomes

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \mu - \dfrac{1}{\gamma K}, & k(\gamma)\geq 0 \\ 0, & \text{else.} \end{cases} \tag{30}$$

From (21) the optimal rate adaptation then becomes

$$k(\gamma) = \begin{cases} \dfrac{1}{c_3}\log_2\left(\frac{\gamma}{\gamma_0}\right), & \gamma\geq\gamma_0 \\ 0, & \text{else} \end{cases} \tag{31}$$

where $\gamma_0 = -(1/K\mu)$ is a cutoff fade depth below which the channel is not used. Note that in the previous section it was the positivity constraint on power $\big(S(\gamma)\geq 0\big)$ that dictated the cutoff fade depth, whereas it is the positivity constraint on rate $\big(k(\gamma)\geq 0\big)$ that determines this cutoff. We can rewrite (30) in terms of γ_0 as

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \dfrac{1}{\gamma_0(-K)} + \dfrac{1}{\gamma(-K)}, & \gamma\geq\gamma_0 \\ 0, & \text{else.} \end{cases} \tag{32}$$

This power adaptation is an *inverse-waterfilling*: since K is negative, less power is used as the channel quality increases above the optimized cutoff fade depth γ_0. The value of γ_0 must satisfy the average power constraint (3):

$$\int_{\gamma_0}^{\infty} -\frac{1}{K}\left[\frac{1}{\gamma_0} + \frac{1}{\gamma}\right] p(\gamma)d\gamma = 1. \tag{33}$$

3) BER Approximation (13): $c_4 = 0$*:* When $c_4 = 0$, $K = \infty$, so from (26)

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \mu & k(\gamma)\geq 0,\ S(\gamma)\geq 0, \\ 0 & \text{else.} \end{cases} \tag{34}$$

This is *on–off* power transmission: either power is zero or a constant nonzero value[4]. From (21) the optimal rate adaptation $k(\gamma)$ with this power adaptation is

$$k(\gamma) = \begin{cases} \dfrac{1}{c_3}\log_2\left(\dfrac{\gamma}{\gamma_0}\right), & \gamma\geq\gamma_0 \\ 0, & \text{else} \end{cases} \tag{35}$$

where $\gamma_0 = -\big(\ln(\overline{\mathrm{BER}}/c_1)/c_2\mu\big)$ is a cutoff fade depth below which the channel is not used. As in Section IV-B-II, it is the rate positivity constraint that determines the cutoff fade depth γ_0. The optimal power adaptation as a function of γ_0 is

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \dfrac{K_0}{\gamma_0}, & \gamma\geq\gamma_0 \\ 0, & \text{else} \end{cases} \tag{36}$$

where $K_0 = -\big(\ln(\overline{\mathrm{BER}}/c_1)/c_2\big)$. The value of γ_0 is determined from the average power constraint to satisfy

$$\frac{K_0}{\gamma_0}\int_{\gamma_0}^{\infty} p(\gamma)d\gamma = 1. \tag{37}$$

So for all our BER approximations, the optimal adaptive rate schemes (29), (31) and (35) have the same form while the optimal adaptive power schemes (27), (32) and (36) have different forms. In other words, although all three BER approximations for MPSK (11)–(13) are tight, they lead to the same optimal adaptive rate policy but very different optimal adaptive power

[3]From (25) the sign of $K = -c_2/\big(c_4\ln(\overline{\mathrm{BER}}/c_1)\big)$ depends only on c_4, since c_1 and c_2 are positive constants and $0 < \overline{\mathrm{BER}} \leq c_1$.

[4]In [5] a similar formula for $S(\gamma)$ was obtained based on a similar BER approximation with $c_4 = 0$. However, the power adaptation in [5] did not specify a cutoff value: power is constant for *all* fading values, but the rate gets asymptotically small in bad channels. Therefore, this strategy leads to a suboptimal spectral efficiency as power is allocated below the optimal cutoff γ_0 defined by (37).

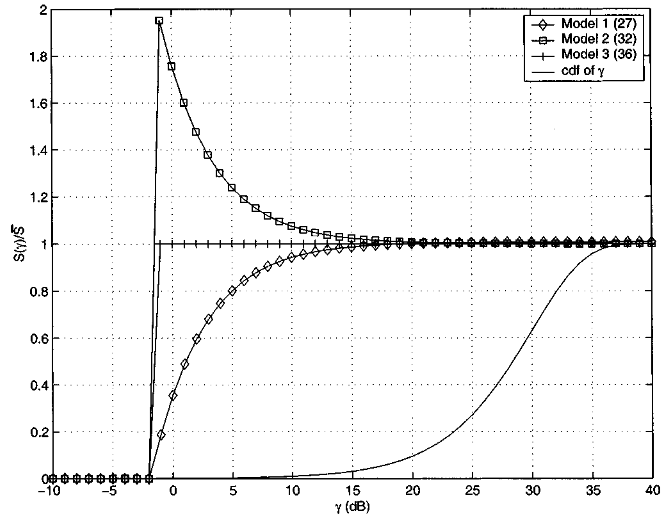

Fig. 4. $S(\gamma)/\overline{S}$ in Model 1, 2, 3 for MPSK ($\overline{\mathrm{BER}} = 10^{-3}$, $\overline{\gamma} = 30$ dB).

policies. The optimal power adaptations (27), (32), and (36) are plotted in Fig. 4 for Rayleigh fading with $\overline{\mathrm{BER}} = 10^{-3}$ and $\overline{\gamma} = 30$ dB. This figure clearly shows the water-filling, inverse water-filling, and on–off behavior of the different schemes. Note that the cutoff γ_0 for all these schemes is roughly the same. We also see from this figure that even though the power adaptation schemes are different at low SNRs, they are almost the same at high SNRs. Specifically we see that for $\gamma < 10$, the optimal transmit power adaptations are dramatically different, while for $\gamma \geq 10$ they rapidly converge to the same constant value. From the cumulative density function of γ also shown in Fig. 4, the probability that γ is less than 10 is 0.01. Thus, although the optimal power adaptation corresponding to low SNRs is very different for the different techniques, this behavior has little impact on spectral efficiency since the probability of being at those low SNRs is quite small.

C. *Discrete Rate and Average BER (D-Rate A-BER)*

In the discrete rate case, the rate is varied within a fixed set $\{k_i\}_{i=0}^{N-1}$, and we assign rate k_i to the rate region $[\gamma_i, \gamma_{i+1})$. Under this fixed rate assignment we wish to maximize spectral efficiency through optimal rate, power, and BER adaptation subject to an average power and BER constraint. Since the set of possible rates and their corresponding rate region assignments are fixed, the optimal rate adaptation corresponds to finding the optimal rate region boundaries $\gamma_i,\ i = 0, \ldots, N-1$. The Lagrangian for this constrained optimization problem is

$$\begin{aligned} &J\big(\gamma_1, \gamma_2, \ldots, \gamma_N, S(\gamma)\big) \\ &= \sum_{0 \leq i \leq N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma) d\gamma \\ &+ \lambda_1 \left[\sum_{0 \leq i \leq N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} \big(\mathrm{BER}(\gamma) - \overline{\mathrm{BER}}\big) p(\gamma) d\gamma \right] \\ &+ \lambda_2 \left[\int_{\gamma_0}^{\infty} S(\gamma) p(\gamma) d\gamma - \overline{S} \right]. \end{aligned} \tag{38}$$

The optimal power adaptation is obtained by solving the following equation for $S(\gamma)$:

$$\frac{\partial J}{\partial S(\gamma)} = 0. \tag{39}$$

Similarly, the optimal rate region boundaries are obtained by solving the following set of equations for γ_i:

$$\frac{\partial J}{\partial \gamma_i} = 0, \quad 0 \leq i \leq N-1. \tag{40}$$

From (39) we see that the optimal power and BER adaptation must satisfy

$$\frac{\partial \mathrm{BER}(\gamma)}{\partial S(\gamma)} = \frac{-\lambda_2}{k_i \lambda_1}, \quad \gamma_i \leq \gamma \leq \gamma_{i+1}. \tag{41}$$

Substituting (14) into (41) we get that

$$\mathrm{BER}(\gamma) = \lambda \frac{f(k_i)}{\gamma k_i} \tag{42}$$

where $\lambda = \overline{S}\lambda_2 / c_2 \lambda_1$. This form of BER adaptation is similar to the *waterfilling* power adaptation, since the BER decreases as the channel quality improves. Now setting the BER in (14) equal to (42) and solving for $S(\gamma)$ yields

$$S(\gamma) = S_i(\gamma), \quad \gamma_i \leq \gamma \leq \gamma_{i+1} \tag{43}$$

where

$$\frac{S_i(\gamma)}{\overline{S}} = \ln \left[\frac{\lambda f(k_i)}{c_1 \gamma k_i} \right] \frac{f(k_i)}{-\gamma c_2}, \quad 0 \leq i \leq N-1 \tag{44}$$

and $S(\gamma) = 0$ for $\gamma < \gamma_0$. We see from (44) that $S(\gamma)$ is discontinuous at the γ_i boundaries.

We now consider the optimal rate region boundaries. From (40) we get that

$$\begin{aligned} \mathrm{BER}(\gamma_i) =& \overline{\mathrm{BER}} - \frac{1}{\lambda_1} - \frac{\lambda_2}{\lambda_1} \frac{S_i(\gamma_i) - S_{i-1}(\gamma_i)}{k_i - k_{i-1}}, \\ 0 \leq& i \leq N-1 \end{aligned} \tag{45}$$

where $k_{-1} = 0$ and $S_{-1}(\gamma) = 0$. Unfortunately, this set of equations is very difficult to solve for the optimal boundary points $\{\gamma_i\}$. However, if we assume that $S(\gamma)$ is continuous at each boundary then we get that

$$\mathrm{BER}(\gamma_i) = \overline{\mathrm{BER}} - \frac{1}{\lambda}, \quad 0 \leq i \leq N-1 \tag{46}$$

for some constant λ. Under this assumption we can solve for the suboptimal rate region boundaries as

$$\gamma_i = \frac{f(k_i)}{k_i} \rho, \quad 0 \leq i \leq N-1 \tag{47}$$

for some constant ρ. The constants λ and ρ are found numerically such that the average power (3) and BER (7) constraints are satisfied. Note that the region boundaries (47) are suboptimal since $S(\gamma)$ is not necessarily continuous at the boundary regions, and therefore these boundaries yield a suboptimal spectral efficiency. However, we will see in Section V that these suboptimal boundaries yield a spectral efficiency close to that of continuous rate adaptation, so they impose little penalty on the discrete rate policy.

D. Discrete Rate and Instantaneous BER (D-Rate I-BER)

With the same discrete rate set $\{k_i\}_{i=0}^{N-1}$ and the same rate assignment k_i to region $[\gamma_i, \gamma_{i+1})$ as in the previous section, we now assume an instantaneous BER constraint, so that $\mathrm{BER}(\gamma) = \overline{\mathrm{BER}}$. Under these constraints the optimal power adaptation is given as

$$\frac{S(\gamma)}{\overline{S}} = \frac{h(k_i)}{\gamma} \tag{48}$$

where $h(k_i) = -(1/c_2)\ln\left(\overline{\mathrm{BER}}/c_1\right) f(k_i)$. We find the optimal rate region boundaries that maximize spectral efficiency using the Lagrangian method. The Lagrange equation is given as

$$J(\gamma_1, \gamma_2, \ldots, \gamma_N) = \sum_{0 \le i \le N-1} k_i \int_{\gamma_i}^{\gamma_{i+1}} p(\gamma) d\gamma + \lambda \left[\sum_{0 \le i \le N-1} \int_{\gamma_i}^{\gamma_{i+1}} \frac{h(k_i)}{\gamma} p(\gamma) d\gamma - 1 \right]. \tag{49}$$

The optimal rate region boundaries are obtained by solving the following equation for γ_i.

$$\frac{\partial J}{\partial \gamma_i} = 0, \quad 0 \le i \le N-1. \tag{50}$$

This yields

$$\gamma_0 = \frac{h(k_0)}{k_0}\rho \tag{51}$$

and

$$\gamma_i = \frac{h(k_i) - h(k_{i-1})}{k_i - k_{i-1}}\rho \quad 1 \le i \le N-1 \tag{52}$$

where ρ is determined by the average power constraint (3).

V. Numerical Results

Although our derivations are for general fading distributions, modulations, and BER approximations, we compute our numerical results for adaptive MQAM in Rayleigh fading based on the BER approximation (9). We assume a BER requirement of either 10^{-3} or 10^{-7}. For the discrete rate cases we assume that 6 different MQAM signal constellations are available, corresponding to 2 (4 QAM), 4 (16 QAM), 6 (64 QAM), 8 (256 QAM), 10 (1024 QAM), and 12 (4096 QAM) bits/symbol.

The average spectral efficiencies for the four adaptation policies (C-Rate A-BER, C-Rate I-BER, D-Rate A-BER, and D-Rate I-BER) are plotted in Fig. 5. The spectral efficiencies of all four policies under the same BER constraints are very close to each other. The spectral efficiency of D-Rate I-BER is slightly higher than that of D-Rate A-BER since the latter

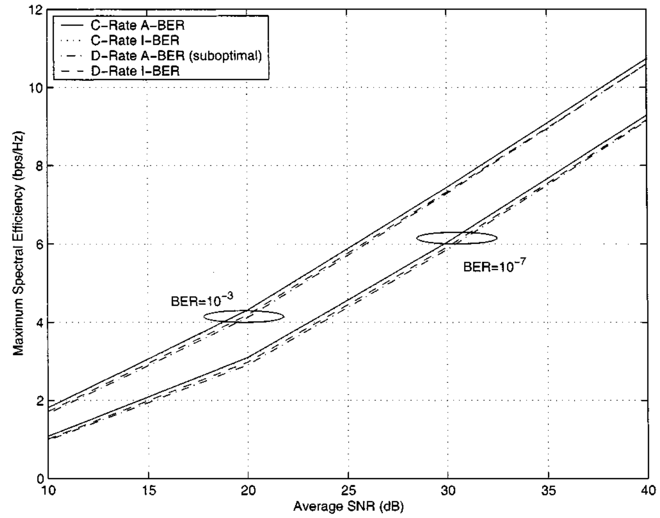

Fig. 5. Spectral efficiency for MQAM.

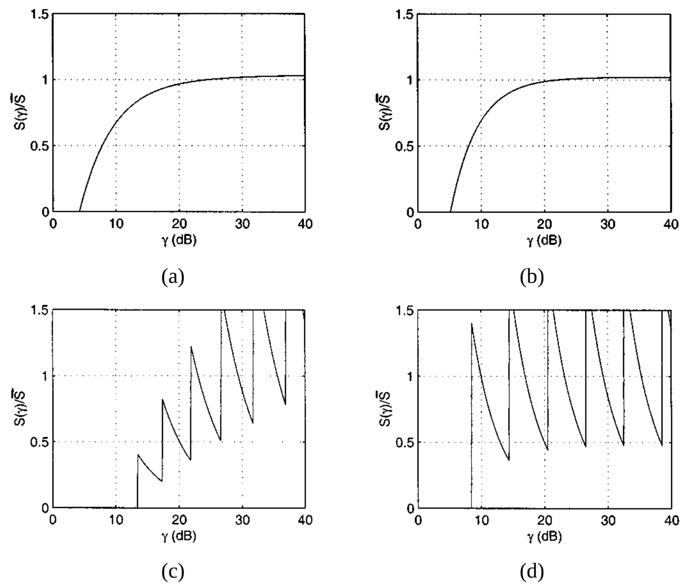

Fig. 6. $S(\gamma)/\overline{S}$ for MQAM ($\overline{\mathrm{BER}} = 10^{-3}, \overline{\gamma} = 30$ dB).

is calculated with suboptimal rate region boundaries. The optimal power control scheme, $S(\gamma)/\overline{S}$, for $\overline{\mathrm{BER}} = 10^{-3}$ is given in Fig. 6. We see from these figures that the optimal transmit power follows a smooth water-filling with respect to γ under the C-Rate A-BER and C-Rate I-BER policies while the optimal power adaptation curve is quite steep under the D-Rate A-BER and D-Rate I-BER policies. The optimal BER adaptation, $\mathrm{BER}(\gamma)$, for $\overline{\mathrm{BER}} = 10^{-3}$ is given in Fig. 7. For the C-Rate A-BER policy we see that the BER decreases monotonically, is within an order of magnitude of its target value, and is above this target at SNRs below $\overline{\gamma}$. The fluctuation of BER in the D-Rate A-BER policy is smaller than that in C-Rate case and goes above and below this target often. The optimal rate adaptation, $k(\gamma)$, for $\overline{\mathrm{BER}} = 10^{-3}$ is given in Fig. 8. All four rate adaptation schemes show that more bits are transmitted as γ increases. Although Figs. 6– 8 show $S(\gamma)$, $\mathrm{BER}(\gamma)$, and $k(\gamma)$ for $\overline{\mathrm{BER}} = 10^{-3}$, plots for these functions at $\overline{\mathrm{BER}} = 10^{-7}$ indicate similar trends.

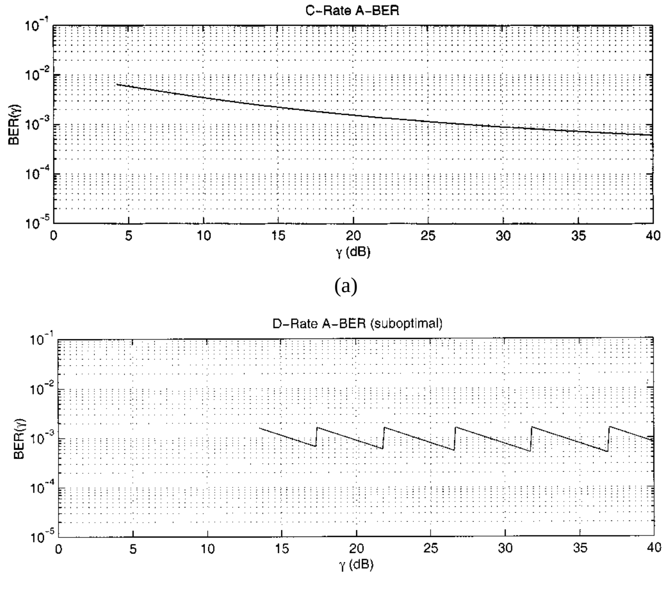

Fig. 7. BER (γ) for MQAM ($\overline{\mathrm{BER}} = 10^{-3}, \overline{\gamma} = 30$ dB).

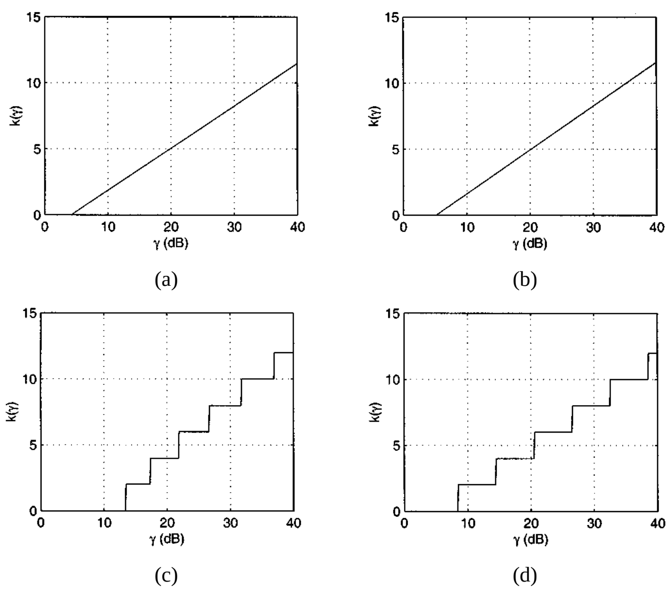

Fig. 8. $k(\gamma)$ for MQAM ($\overline{\mathrm{BER}} = 10^{-3}, \overline{\gamma} = 30$ dB).

VI. Constant Power and Rate

We now consider further restriction on the degrees of freedom in our adaptive modulation policies. Specifically, we now restrict our system to have either constant transmit power or constant rate. Restricting our adaptive policies to maintain a constant transmit power or rate significantly simplifies the hardware complexity of the system. In addition, a constant transmit power is desirable in multiuser systems to reduce variations in interference power, and constant rate transmission is desirable for applications with simple hardware and constant throughput. These restrictions will result in some loss of spectral efficiency relative to the nonrestricted policies described in Section IV, but we will see that this penalty is not very large.

For our numerical calculations we use MQAM constellations with a target BER of either 10^{-3} or 10^{-7} under Rayleigh fading, as in Section V.

A. Constant Power

In this section we maximize spectral efficiency assuming a constant transmit power with a cutoff threshold below which the channel is not used. We first consider the C-Rate I-BER policy. The average power constraint (3) dictates that with threshold γ_0, the constant transmit power $S(\gamma) = S$ satisfies

$$\frac{S}{\overline{S}} = \frac{1}{\int_{\gamma_0}^{\infty} p(\gamma)d\gamma}. \tag{53}$$

The rate $k(\gamma)$ should be adapted as follows. From (21)

$$k(\gamma) = \begin{cases} \frac{1}{c_3}\log_2\left[c_4 - \frac{c_2\gamma}{\ln\left(\frac{\overline{\mathrm{BER}}}{c_1}\right)}\frac{1}{\int_{\gamma_0}^{\infty} p(\gamma)d\gamma}\right], & \gamma \geq \gamma_0 \\ 0, & 0 \leq \gamma < \gamma_0. \end{cases} \tag{54}$$

Then the spectral efficiency is given by

$$\frac{R}{B} = \int_{\gamma_0}^{\infty} \frac{1}{c_3}\log_2\left[c_4 - \frac{c_2\gamma}{\ln\left(\frac{\overline{\mathrm{BER}}}{c_1}\right)}\frac{1}{\int_{\gamma_0}^{\infty} p(\gamma)d\gamma}\right] p(\gamma)d\gamma. \tag{55}$$

By optimizing (55) with respect to γ_0, we can find the maximum value of spectral efficiency. Numerical values of this maximum spectral efficiency are given in Fig. 9. The spectral efficiency loss is less than 1% when compared with the maximum possible spectral efficiency obtained using the adaptive power C-Rate A-BER policy (Section IV-A), so the two curves are indistinguishable in Fig. 9.

If we assume the threshold $\gamma_0 = 0$, then the constant transmit power is $S(\gamma) = \overline{S}$. The data rate is then given by

$$k(\gamma) = \frac{1}{c_3}\log_2\left[c_4 - \frac{c_2\gamma}{\ln\left(\frac{\overline{\mathrm{BER}}}{c_1}\right)}\right] \tag{56}$$

with a spectral efficiency of

$$\frac{R}{B} = \int_{0}^{\infty} \frac{1}{c_3}\log_2\left[c_4 - \frac{c_2\gamma}{\ln\left(\frac{\overline{\mathrm{BER}}}{c_1}\right)}\gamma\right] p(\gamma)d\gamma. \tag{57}$$

Numerical values of this spectral efficiency are also given in Fig. 9. We see that optimizing the threshold γ_0 results in little performance improvement relative to the zero threshold ($\gamma_0 = 0$) case, especially at high SNR's. We do not analyze the spectral efficiency of the C-Rate A-BER policy with constant transmit

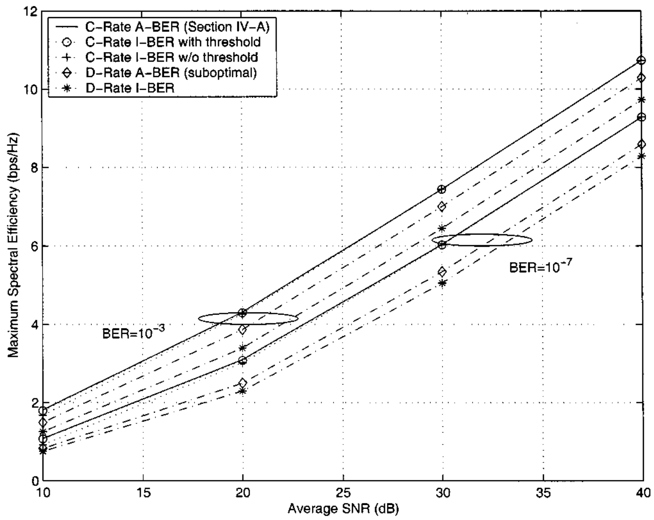

Fig. 9. Spectral efficiency for adaptive MQAM with constant transmit power.

power, since its efficiency will lie between the efficiency of the C-Rate I-BER policy with constant power and that of the adaptive power C-Rate A-BER policy.

We now consider the D-Rate I-BER policy with constant transmit power. We use the same set of signal constellations as were used for the D-Rate policies in Section V ($k_i = 2$, 4, 6, 8, 10, and 12 bits/symbol) and we let $\{\gamma_i\}$ denote the rate region boundaries, as in Section IV-C. The average power constraint (3) dictates that with threshold γ_0, the constant transmit power $S(\gamma) = S$ satisfies

$$\frac{S}{\overline{S}} = \frac{1}{\int_{\gamma_0}^{\infty} p(\gamma) d\gamma}. \tag{58}$$

To satisfy the instantaneous BER constraint $\mathrm{BER}(\gamma) \leq \overline{\mathrm{BER}}$ for all γ and to maximize spectral efficiency we must satisfy the BER constraint at each boundary point γ_i:

$$\mathrm{BER}(\gamma_i) \approx c_1 \exp\left[\frac{-c_2\gamma_i \frac{1}{\int_{\gamma_0}^{\infty} p(\gamma)d\gamma}}{2^{c_3 k_i} - c_4}\right] \leq \overline{\mathrm{BER}}, \quad 0 \leq i \leq N-1 \tag{59}$$

since for a constant transmit power $\mathrm{BER}(\gamma) \leq \mathrm{BER}(\gamma_i)$ for $\gamma_i \leq \gamma \leq \gamma_{i+1}$. Therefore, we find the optimal rate region boundaries γ_i $(0 \leq i \leq N-1)$ by solving (59). Spectral efficiency is then given by

$$\frac{R}{B} = \sum_{i=0}^{N-1} k_i p\left(\gamma_i \leq \gamma \leq \gamma_{i+1}\right). \tag{60}$$

Numerical values for (60) are given in Fig. 9. The spectral efficiency of this scheme is between 70% and 90% of the C-Rate A-BER policy with optimal power adaptation. This penalty is predictable since we have removed two degrees of freedom with respect to the adaptive power C-Rate A-BER policy: power and BER adaptation. For a constant transmit power with discrete rate adaptation, the instantaneous BER is often lower than our target BER. This is due to the rate discretization and constant power restriction. In particular, for the spectral efficiency given by (60) and shown in Fig. 9, the average BER ranges from $0.073 * 10^{-7}$ to $0.051 * 10^{-7}$, while our target instantaneous BER is 10^{-7}. Thus, in this case we are below our target BER by more than an order of magnitude.

We now consider the D-Rate A-BER policy with constant transmit power. The optimal solution is hard to find as was also the case in Section IV-C. As a suboptimal solution, we will use the discrete rate region boundaries of the D-Rate I-BER policy discussed in the previous paragraph. We scale each value of these discrete rate regions equally such that the A-BER constraint (7) is satisfied exactly. Spectral efficiency then follows the same formula as (60) with the scaled discrete rate region boundaries. Numerical values are given in Fig. 9. Even though we obtain the spectral efficiency of this D-Rate A-BER policy using suboptimal rate regions, the spectral efficiency is between 75% and 95% that of the optimal power C-Rate A-BER policy. We also investigated D-Rate A-BER policies with piecewise constant power, where the power is constant within each D-Rate region, but can be different for different regions (constellations). We found that allowing piecewise constant power led to negligible spectral efficiency gain over just constant power for the D-Rate A-BER policies, as shown in [5].

B. Constant Rate

We now consider constant rate policies, so k can take on only one optimized value. Let us first assume an I-BER constraint. From (48), the optimal power control scheme to maintain the BER target over all γ for rate k is

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{h(k)}{\gamma}, & \gamma \geq \gamma_0 \\ 0, & \text{else} \end{cases} \tag{61}$$

where the threshold value γ_0 is obtained from the power constraint (3):

$$\int_{\gamma_0}^{\infty} \frac{h(k)}{\gamma} p(\gamma) d\gamma = 1 \tag{62}$$

and the spectral efficiency is

$$\frac{R}{B} = k \int_{\gamma_0}^{\infty} p(\gamma) d\gamma. \tag{63}$$

We obtain the optimum value of k to maximize spectral efficiency at each average SNR value by numerical search. The resulting spectral efficiency values are given in Fig. 10. Here the penalty in spectral efficiency is about 10% with respect to the adaptive power C-Rate A-BER policy (Section IV-A) if we don't restrict the fixed rate k to be an integer. If we restrict the value of k to be an integer (as would be needed in practice) then the spectral efficiency decreases. However, as we see in Fig. 10, the restriction of k to integer values does not significantly decrease spectral efficiency.

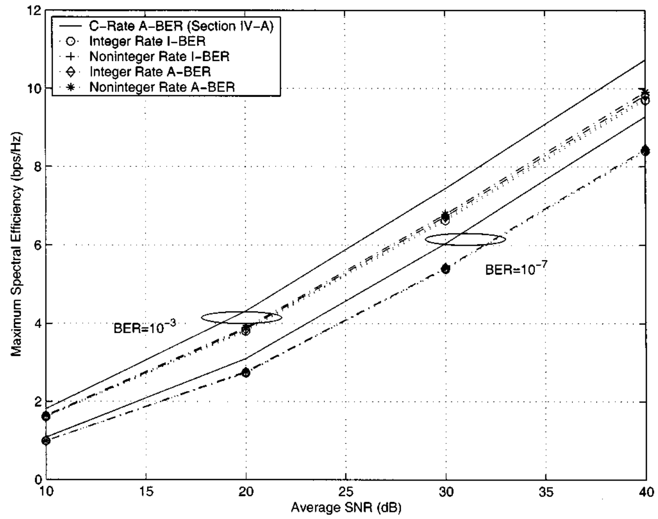

Fig. 10. Spectral efficiency for adaptive MQAM with constant rate.

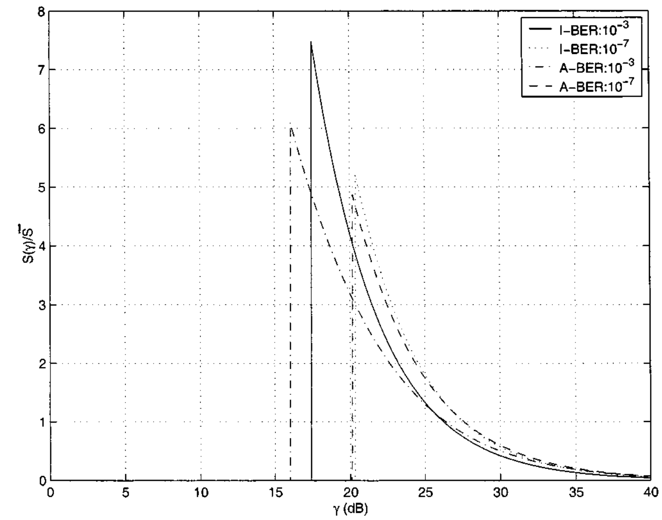

Fig. 11. $S(\gamma)/\overline{S}$ for MQAM ($\overline{\gamma} = 30$ dB) for constant rate.

Now consider the A-BER constraint. Assume a threshold γ_0 below which no data is sent. The Lagrangian as a function of power is

$$\begin{aligned} J\big(S(\gamma)\big) =& k\int_{\gamma_0}^{\infty} p(\gamma)d\gamma \\ &+ \lambda_1\left[k\int_{\gamma_0}^{\infty}\big(\mathrm{BER}(\gamma)-\overline{\mathrm{BER}}\big)p(\gamma)d\gamma\right] \\ &+ \lambda_2\left[\int_{\gamma_0}^{\infty} S(\gamma)p(\gamma)d\gamma - \overline{S}\right]. \end{aligned} \tag{64}$$

The optimal power adaptation is obtained by solving the following equation.

$$\frac{\partial J}{\partial S(\gamma)} = 0. \tag{65}$$

From (64), this becomes

$$\frac{\partial \mathrm{BER}(\gamma)}{\partial S(\gamma)} = \frac{-\lambda_2}{k\lambda_1}. \tag{66}$$

Then the optimal power control is

$$\frac{S(\gamma)}{\overline{S}} = \ln\left[\frac{\frac{\overline{S}\lambda_2}{c_2\lambda_1}f(k)}{c_1\gamma k}\right]\frac{f(k)}{-\gamma c_2}. \tag{67}$$

The optimal BER adaptation is also derived from (67) and (14) as

$$\mathrm{BER}(\gamma) = \frac{\overline{S}\lambda_2}{c_2\lambda_1}\frac{f(k)}{\gamma k}. \tag{68}$$

Using numerical search techniques, the optimal rate k and threshold γ_0 are found. Details are described in Appendix B. The corresponding spectral efficiency $\left(k\int_{\gamma_0}^{\infty} p(\gamma)d\gamma\right)$ is plotted in Fig. 10. As the figure shows, the spectral efficiency in this case is quite close to that of a fixed-rate policy with an I-BER constraint, so we do not get much gain by relaxing the I-BER constraint. Fig. 10 also shows the case when the constant rate is restricted to integer values. We see that all constant rate policies yield almost the same spectral efficiency regardless of differences in the rate restriction (integer or noninteger) and BER constraint (average or instantaneous). All of these policies show a penalty of about 2 dB relative to the optimal adaptive power C-Rate A-BER policy of Section IV-A.

In the constant rate case, an optimal transmit power level could approach infinity when the channel gain is very small, which results in a bad peak-to-average-power ratio (PAPR). The optimal transmit power levels for the constant rate transmission schemes are shown in Fig. 11. By comparing Figs. 6 and 11, we see that the constant rate schemes have much worse PAPR than the adaptive rate schemes.

VII. Conclusion

We have shown that the maximum spectral efficiency of adaptive modulation is nearly the same under continuous and discrete rate adaptation as well as under an instantaneous or average BER constraint. We have also derived the optimal power, rate, and BER adaptation for these schemes for a large class of modulation techniques and general fading distributions. Restricting the power or rate of the adaptive modulation to be constant achieves near optimal performance in most cases. Therefore using just one or two degrees of freedom in adaptive modulation yields close to the maximum possible spectral efficiency obtained by utilizing all degrees of freedom. Therefore, the parameters to adapt should be chosen based on implementation considerations.

Appendix A

In Section IV-A, in order to solve (20) under $k(\gamma) \geq 0$ and $S(\gamma) \geq 0$ while satisfying the average power constraint (3) and the average BER constraint (6), we use *Mathematica* [17], a numerical math package. Specifically, for a fixed λ_1 and $\lambda_2\overline{S}$, we use *Mathematica* to find the function $k(\gamma)$ over all γ that satisfies (20) (using the "FindRoot" command for each γ). This function $k(\gamma)$ was also defined to be zero if $k(\gamma) \geq 0$ and $S(\gamma) \geq 0$ were not satisfied. Once $k(\gamma)$ is known, the BER $\big(BER(\gamma)\big)$ and power $\big(S(\gamma)\big)$ can be found as a function of γ using (18) and (19). We use a *bisection method* [18] to find λ_1

and $\lambda_2 \overline{S}$ that satisfy the average power constraint (3) and the average BER constraint (6).

APPENDIX B

In Section VI-B, $\widetilde{\lambda} = \big(f(k)/k\big)(\lambda_2 \overline{S}/\lambda_1 c_2)$ can be expressed as a function of the cutoff fade depth γ_0 from the average BER constraint (6) and (68) as

$$\widetilde{\lambda} = \overline{\mathrm{BER}} \frac{\int_{\gamma_0}^{\infty} p(\gamma) d\gamma}{\int_{\gamma_0}^{\infty} \frac{1}{\gamma} p(\gamma) d\gamma}. \tag{69}$$

Then from (67) the transmit power $S(\gamma)/\overline{S}$ can be expressed in terms of $\widetilde{\lambda}$ and γ as

$$\frac{S(\gamma)}{\overline{S}} = -\ln \left[\frac{\widetilde{\lambda}}{c_1 \gamma} \right] \frac{f(k)}{\gamma c_2}. \tag{70}$$

From the average power constraint (3) k is expressed using (70) as

$$k = \frac{1}{c_3} \log_2 \left[c_4 + \frac{1}{\int_{\gamma_0}^{\infty} -\frac{\ln \frac{\widetilde{\lambda}}{c_1 \gamma}}{\gamma c_2} p(\gamma) d\gamma} \right]. \tag{71}$$

The spectral efficiency is $k \int_{\gamma_0}^{\infty} p(\gamma) d\gamma$ from (71). We find γ_0 that maximizes this spectral efficiency using a *bisection method* [18].

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Seong Taek Chung (S'01) received the B.S. degree with honors in electrical engineering from Seoul National University, Seoul, Korea, in 1998, and the M.S. degree in electrical engineering from Stanford University, Stanford, CA, in 2000. He is currently pursuing the Ph.D. degree in electrical engineering at Stanford University.

During the summer of 2000, he was an intern with Bell Laboratories, Lucent Technologies, where he worked on multiple antenna systems. His current research interest includes work in communication theory.

Andrea J. Goldsmith (S'94–M'95–SM'01) received the B.S., M.S., and Ph.D. degrees in electrical engineering from the University of California, Berkeley, in 1986, 1991, and 1994, respectively.

From 1986 to 1990, she was with Maxim Technologies, where she worked on packet radio and satellite communication systems, and from 1991 to 1992, she was affiliated with AT&T Bell Laboratories, where she worked on microcell modeling and channel estimation. From 1994 to 1998, she was an Assistant Professor of electrical engineering at the California Institute of Technology, Pasadena. She is currently an Assistant Professor of electrical engineering at Stanford University, Stanford, CA. Her research includes work in capacity of wireless channels, wireless communication theory, adaptive modulation and coding, joint source and channel coding, and resource allocation in cellular and packet radio networks.

Dr. Goldsmith is a Terman Faculty Fellow at Stanford University and is the receipient of a National Science Foundation CAREER Development Award, an ONR Young Investigator Award, two National Semiconductor Faculty Development Awards, and IBM Graduate Fellowship, and the David Griep Memorial Prize from the University of California at Berkeley. She is an Editor for the IEEE TRANSACTIONS ON COMMUNICATIONS and the *IEEE Personal Communications Magazine.*