DEGREES OF FREEDOM OF WEAK GRAVITATIONAL FIELD ON A SPHERICALLY SYMMETRIC BACKGROUND*

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We propose a description of linearised vacuum perturbation of a Kottler metric in terms of four unconstrained scalar functions, invariant with respect to the infinitesimal coordinate change gauge. We present a derivation of the generalised Regge–Wheeler and Zerilli equations in this scheme.

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1. Introduction

This communication summarizes results from [1] — a way of reducing the perturbation ADM data for the linearised Cauchy problem for the vacuum Einstein equation on a spherically symmetric background to a set of four unconstrained functions, invariant with respect to the coordinate gauge.

2. Preliminaries

We choose the following convention for the Einstein equation:

$$2R_{\mu\nu} - Rg_{\mu\nu} + 2\Lambda g_{\mu\nu} = 16\pi T_{\mu\nu} \,. \tag{1}$$

We restrict ourselves to vacuum solutions ($T_{\mu\nu} = 0$) and assume $g_{\mu\nu}$ to be a perturbed Kottler metric, which is a general [2] spherically symmetric

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vacuum solution of (1), that encompasses Minkowski, Schwarzschild and (Anti)de Sitter spacetimes as special cases [3]:

$$\eta_{\mu\nu} = -f\mathrm{d}t^2 + \frac{1}{f}\mathrm{d}r^2 + r^2\left[\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right], \qquad f(r) = 1 - \frac{2m}{r} - \frac{r^2}{3}\Lambda\,.$$

We adopt a coordinate system: $(x^0, x^1, x^2, x^3) = (t, \vartheta, \varphi, r)$ and ask that the structure of spacetime corresponds to the symmetry of the background:

$$\Sigma_{\mathrm{s}} = \{x^0 = s, r_0 \le x^3 \le r_\infty\} = \bigcup_{r\in[r_0,r_\infty[} S_{\mathrm{s}}(r)\,, \quad S_{\mathrm{s}}(r) = \{x \in \Sigma_{\mathrm{s}} : x^3 = r\}\,,$$

where we will use space-like hypersurfaces Σ_{s} as our initial surfaces. We choose the ADM formulation of the Cauchy problem and perform a standard procedure of passing to the linear approximation. As the ADM momentum for $\eta_{\mu\nu}$ on Σ_{s} vanishes, the linearised momentum equals in value that of the full solution. Thus, our set of linearised initial data has the form of

$$\left(h_{kl}, P^{kl}\right)\,, \qquad h_{kl} = g_{kl} - \eta_{kl}\,, \qquad P^{kl} = P^{kl}(g_{\mu\nu})\,.$$

We use the following convention of indices: greek letters (α, β, γ) and covariant derivative symbol ; correspond to the four-dimensional geometry of the whole spacetime. Small latin letters (a, b, c) — to the three-dimensional geometry of the hypersurfaces Σ_{s}. Big latin letters (A, B, C) correspond to spherical coordinates and $||$ denotes the covariant derivative defined by internal geometry of spheres $S_{\mathrm{s}}(r)$. Note that covariant derivatives in the linearised theory are defined by the background metric.

3. Construction of invariants

Raw ADM data consists of twelve functions. We expect that four of these can be eliminated by gauge freedom of the linearised theory: $h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$ and further four should be fixed by the Gauss–Codazzi constraints. This leaves four true degrees of freedom. To extract them, we first split ADM data into scalar (and pseudo-scalar) functions, using the two-dimensional geometry of the spheres foliating our Cauchy surface:

$$\begin{aligned} H &:= \eta^{AB}h_{AB}\,, \qquad \chi_{AB} := h_{AB} - \tfrac{1}{2}\eta_{AB}H\,, \\ S &:= \eta^{AB}P_{AB}\,, \qquad S_{AB} := P_{AB} - \tfrac{1}{2}\eta_{AB}S\,. \end{aligned}$$

	Polar (Even)	Axial (Odd)
Scalar	${h^3}_3\,, H\,, {P^3}_3\,, S$	—
Vector ${h^3}_A,\ {P^3}_A$	${{h_3}^A}_{\|\|A}\,,\ {P^{3A}}_{\|\|A}$	$h_{3A\|\|B}\varepsilon^{AB},\ P^{3A\|\|B}\varepsilon_{AB}$
Traceless tensor $\chi_{AB},\ S_{AB}$	${\chi^{AB}}_{\|\|AB}\,,\ {S^{AB}}_{\|\|AB}$	${\chi^C}_{A\|\|CB}\varepsilon^{AB}\,,\ {S^C}_{A\|\|CB}\varepsilon^{AB}$

Axial and polar degrees of freedom decouple in the linear theory, allowing us to consider each set separately. Axial scalars reduce to the following pair of gauge-invariant combinations:

$$\begin{aligned} \boldsymbol{y} &:= 2\mathit{\Pi}^{-1}r^2P^{3A||B}\varepsilon_{AB}\,, \\ \boldsymbol{Y} &:= \mathit{\Pi}\left(\overset{\circ}{\Delta}+2\right)h_{3A||B}\varepsilon^{AB} - \mathit{\Pi}\left(r^2{\chi^C}_{A||CB}\varepsilon^{AB}\right)_{,3}\,. \end{aligned}$$

The symbol $\overset{\circ}{\Delta}$ denotes the Laplace–Beltrami operator on a unit sphere and $\mathit{\Pi} := \sqrt{f}\sqrt{\det\eta_{kl}} = r^2\sin\vartheta$. From the polar set, we obtain

$$\begin{aligned} \boldsymbol{x} &:= r^2{\chi^{AB}}_{||AB} - \frac{1}{2}\left(\overset{\circ}{\Delta}+2\right)H + \mathcal{B}\left[2h^{33} + 2r{h^{3C}}_{||C} - rfH_{,3}\right]\,, \\ \boldsymbol{X} &:= 2r^2{S^{AB}}_{||AB} + \mathcal{B}\left[2r{P^{3A}}_{||A} + \overset{\circ}{\Delta}{P^3}_3\right]\,, \\ \mathcal{B} &:= \left(\overset{\circ}{\Delta}+2\right)\left(\overset{\circ}{\Delta}+2-\frac{6m}{r}\right)^{-1}\,. \end{aligned}$$

The operator $\mathcal{B}$ is easily seen to be non-local. However, this non-locality concerns only individual spheres, *i.e.* compact sets. $\mathcal{B}$ is local with respect to the radial and temporal variable. We, therefore, call this operator quasi-local.

4. Dynamics

We chose this particular set of invariants because it constitutes of two conjugate pairs — the xs and ys — corresponding to reduction of the symplectic form of the system and bound together by equations of motion

$$\dot{\boldsymbol{x}} = \frac{f}{\mathit{\Pi}}\boldsymbol{X}\,, \qquad \dot{\boldsymbol{X}} = \frac{\mathit{\Pi}}{r^2}\left\{\left(fr^2\boldsymbol{x}_{,3}\right)_{,3} + \left[\overset{\circ}{\Delta} + f(1-2\mathcal{B}) + 1 - r^2\mathit{\Lambda}\right]\mathcal{B}\boldsymbol{x}\right\}\,, \tag{2}$$

$$\dot{\boldsymbol{y}} = \frac{f}{\mathit{\Pi}}\boldsymbol{Y}\,, \qquad \dot{\boldsymbol{Y}} = \mathit{\Pi}\left\{\partial_3\left[\frac{f}{r^2}\left(r^2\boldsymbol{y}\right)_{,3}\right] + \frac{1}{r^2}\left(\overset{\circ}{\Delta}+2\right)\boldsymbol{y}\right\}\,. \tag{3}$$

It can also be easily related to the works of Regge and Wheeler [4] and Zerilli [5], whose well-known equations can be rediscovered, in a generalised and gauge invariant form, in our scheme. Equations of motion induce deformed wave equations on $\boldsymbol{x}$ and $\boldsymbol{y}$. By splitting the invariants into spherical harmonics and making an appropriate change in the radial variable

$$\boldsymbol{x} = \exp(i\sigma t) Y_l(\theta,\phi) \frac{Z_l^{(+)}(r)}{r}\,, \quad \boldsymbol{y} = \exp(i\sigma t) Y_l(\theta,\phi) \frac{Z_l^{(-)}(r)}{r}\,, \quad \frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{1}{f}\,,$$
$$V^{(+)} := -\frac{f}{r^2}\left[\left(\mathring{\Delta} - \frac{6m}{r}\right) + \frac{36m^2}{r^2}\left(\mathring{\Delta} + 2\right)^{-2}\left(\mathring{\Delta} + 2 - \frac{2m}{r} + \frac{2}{3}r^2 \Lambda\right)\right]\mathcal{B}^2\,,$$
$$V^{(-)} := -\frac{f}{r^2}\left(\mathring{\Delta} + \frac{6m}{r}\right)\,,$$

we may cast these wave equations into a form resembling the Zerilli (polar) and Regge–Wheeler (axial) equations

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r*^2} + \sigma^2\right) Z_l^{(\pm)}\, Y_l = V^{(\pm)} Z_l^{(\pm)}\, Y_l\,.$$

5. Conserved charges

In the splitting above, the mono-dipole part distinguishes itself from higher multipoles. The following coefficients vanish by construction:

$$\mathrm{dip}(\boldsymbol{x}) = \mathrm{mon}(\boldsymbol{y}) = \mathrm{mon}(\boldsymbol{Y}) = \mathrm{dip}(\boldsymbol{Y}) = \mathrm{mon}(\boldsymbol{X}) = \mathrm{dip}(\boldsymbol{X}) = 0\,.$$

This implies, by (2) and (3), that $\mathrm{mon}(\boldsymbol{x})$ and $\mathrm{dip}(\boldsymbol{y})$ are constant in time. Their general form can be integrated from the constraint equations

$$\mathrm{mon}(\boldsymbol{x}) = \frac{\alpha}{r-3m}\,, \qquad \mathrm{dip}(\boldsymbol{y}) = \frac{\beta}{r^2}\,.$$

Integration constants α and β are conserved charges — the mass and angular momentum of the perturbation. Note that $\mathrm{mon}(\boldsymbol{x})$ is divergent if we work in a spacetime region containing $r = 3m$. We then need to choose the mass parameter of the background so that the perturbation mass $\alpha = 0$.

6. Final remarks

Our set of four unconstrained invariant functions encompasses full information about the linearised perturbation of a Kottler metric (see [6] for a way to reconstruct perturbation data from the invariants). We believe this scheme can find application in such areas of study as analysis of stability, gravitational wave research and attempts at describing quasilocal energy of gravitation.

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